Commom Solutions for a Class of Simultaneous Pell Equations

Ran Yinxia*
Longnan Teachers College, College of Mathematics and Information Science, Longnan, China.

* Corresponding author. Tel.: 18993930265; email: ranyinxia@163.com
Manuscript submitted January 10, 2019; accepted May 3, 2019.
doi: 10.17706/ijapm.2019.9.4.182-188

Abstract: In recent years, the common solution of pell equations is a hot field in indefinite equations. For example, the equations 1) mentioned in the paper. However, due to the diverse forms of such equations, many scholars have done more studies on the smaller values of k and m, and the main conclusions are focused on the estimation of solutions under some special forms of D_1 and the specific values of D. So there is a lot of room for studying these kinds of equations. In this paper, we studied the common solution of the system of indefinite equations 2) mentioned in this paper by using the elementary method and the recursive property of solution sequence. If *D* is the case in this paper, the common solution of the equations is given.

Key words: The system of indefinite equations, pell equation, integer solution, common solution, odd prime.

1. Introduction

The Diophantine equation is the oldest branch in number theory, whose content is extremely abundant, and it has close connections with the algebraic number theory, the algebraic geometry, the combinatorics and so on. In the recent 30 years, this field also has developed too much. In such fields as the information encoding theory, the algebraic number theory and the diophantine analysis theory, many types of the results of higher diophantine equation are used, which make it necessary for us to study some basic types of the solutions of higher diophantine equation. We are familiar to study some basic types of the simple diophantine equation and quadratic diophantine equation, while with the solution of higher diophantine equation, there is no general conclusion, so it needs further discussing.

The Diophantine equation not only developed actively itself, but also was apply to else fields of Discerete Mathematics. It plays an important role in people's study and research to solve the actual problems. So many researchers study the Diophantine equation extensively and highly in the domestic and abroad. Along with the development of the Diophantine equation, Algebraic Number Theory obtained the first formation and developments. Currently, Algebracic Number Theory has become a branch of mathematics with abundant contents, is also an important tool of studying of the Diophantine equation.

In recent years, the common solution of pell equations

$$\begin{cases} x^2 - D_1 y^2 = k \\ y^2 - Dz^2 = m \end{cases} \tag{1}$$

is a hot field in indefinite equations. The main conclusions are as follows:

1) When k=1 and m=1,

the research results of the system focus on the scope and estimation of the solution, and the main conclusions are shown in [1], [2].

2) When k=1 and m=4,

a) If D_1 =2, for the solution of the system, the main conclusion is shown in [3]-[10];

b) If D_1 =6, it is shown in [11]-[15];

c) If D_1 =10 , it is shown in the main conclusion [16].

d) If D_1 =12, it is shown in the main conclusion [17]-[19].

e) If D_1 =30, it is shown in the main conclusion [20].

3) When k=1 and m=25,

a) If D_1 =23, the situation of the system is discussed in [21].

However, the pell equations

$$\begin{cases} x^2 - k(k+1)y^2 = k \\ y^2 - Dz^2 = 4 \end{cases} (k \in \mathrm{N}^*),$$

is one of the kind of the equations (1). When k=2, it is shown in [11]-[15], when k=3, it is shown in the main conclusion [17]-[19]. In this paper, we deal with the case of k=4, namely

$$\begin{cases} x^2 - 20y^2 = 1 \\ y^2 - Dz^2 = 4 \end{cases} \tag{2}$$

And the following conclusions are obtained:

Theorem If $D = 2^t p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_4} p_4^{\alpha_3}$, where $\alpha_s = 0$ or $1, p_s\left(1 \le s \le 4\right)$ are distinct odd primes, t is a positive integer, and the solution of the indefinite system (1) is as follows:

a) D=2×7×23, the system (2) has non-trivial solutions (x, y, z)=(±2889, ±646, ±36);

b) D=2^3×7×23, the system (2) has non-trivial solutions (x, y, z)=(±2889, ±646, ±18);

c) D=2^5×7×23, the system (2) has non-trivial solutions (x, y, z)=(±2889, ±646, ±9).

d) When $t \neq 1, 3, 5$, the system (2) only has trivial solutions (x, y, z) =(±9, ±2,0).

2. Preliminaries

Lemma 1 [18] If p is an odd prime number, then the diophantine equation x^4-py^2=1 has no other positive integer solution except p=5, x=3, y=4 and p=29, x=99, y=1820.

Lemma 2 [18] If a is a square number and a >1, the equation $ax^4 - by^2 = 1$ has only one positive integer solution.

Lemma 3 [18] If D is a non-square positive integer, then $x^4 - Dy^4 = 1$ has at most two positive integer solutions. And the sufficient and necessary condition for the equation to have two groups of solutions is that D=1785 or D=28560, or that $2x_0$ and $2y_0$ are squares, where (x_0, y_0) is the fundamental solution of the equation.

Lemma 4 If x_n, y_n is any integer solution of Pell equation x^2-104y^2=1, then x_n, y_n has the following properties:

(I) $y_n^2 - 4 = y_{n-1}y_{n+1}$

(II) $y_{2n} = 2x_n y_n, x_{2n} = 2x_n^2 - 1;$

(III) $x_{n+1} = 9x_n + 40y_n, y_{n+1} = 2x_n + 9y_n,$

$x_{n+2} = 18x_{n+1} - x_n, y_{n+2} = 18y_{n+1} - y_n,$

$x_0 = 1, x_1 = 5, \quad y_0 = 0, y_1 = 2;$

(IV) $(x_n, y_n) = 1, (x_n, x_{n+1}) = 1, (y_n, y_{n+1}) = 2;$

（V） $x_{2n} \equiv \pm 1 (\bmod 9), \quad y_{2n} \equiv 0 (\bmod 9);$

$x_{2n+1} \equiv 0 (\bmod 9), y_{2n+1} \equiv \pm 2 (\bmod 9);$

$y_{2n} \equiv 0 (\bmod 4), \quad y_{2n+1} \equiv 2 (\bmod 4).$

Lemma 5 If (x_1, y_1) is the fundamental solution of Pell equation x^2-20y^2=1, and all integer solutions are (x_n, y_n),$n \in$Z. For any (x_n, y_n),it has the following properties:

a) x_n is square if and only if n=0;

b) $\frac{x_n}{5}$ is square if and only if n=$\pm$1;

c) $\frac{y_n}{2}$ is square if and only if n=0, 1.

3. Proof of Theorem

Proof: Since the fundamental solution of Pell equation x^2-20y^2=1 is $(x_1, y_1) = (9, 2)$, all integer solutions of pell equation are $x_n + y_n\sqrt{20} = \left(9 + 2\sqrt{20}\right)^n, n \in \mathrm{Z}$. Thus:

If $(x, y, z) = (x_n, y_n, z)$ is the integer solution to (2), then $\forall n \in \mathrm{Z}$,

$$y_n^2 - 4 = y_n^2 - 4\left(x_n^2 - 20y_n^2\right) = 81y_n^2 - 4x_n^2 = (9y_n + 2x_n)(9y_n - 2x_n) = y_{n+1}y_{n-1} \tag{3}$$

By (2) $Dz^2 = y_n^2 - 4$

Then

$$Dz^2 = y_{n+1}y_{n-1} \tag{4}$$

case1 Let n be odd, might as well $n = 2m-1, (m \in \mathrm{Z})$, At this point, equation (4) becomes:

$$Dz^2 = y_{n-1}y_{n+1} = y_{2m-2}y_{2m} = 4x_{m-1}y_{m-1}x_m y_m \tag{5}$$

case1. 1 Let m be odd, might as well $m = 2r, \left(r \in \mathrm{N}^*\right)$, At this point, equation (5) becomes:

$$Dz^2 = 4x_{2r-1}y_{2r-1}x_{2r}y_{2r} = 8x_{2r-1}y_{2r-1}x_{2r}x_r y_r \tag{6}$$

case 1.1.1 Let r be odd, might as well $r = 2u - 1, (u \in Z)$， At this point, equation (5) becomes：

$$Dz^2 = 8x_{4u-3}y_{4u-3}x_{4u-2}x_{2u-1}y_{2u-1} \tag{7}$$

From lemma 5, $\frac{x_{2u-1}}{9}, \frac{x_{4u-3}}{9}, \frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}, x_{4u-2}$ are two relatively prime, and $\frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}$ are odd, $x_{4u-2}, \frac{x_{2u-1}}{9}, \frac{x_{4u-3}}{9}$ are odd, namely $\frac{x_{2u-1}}{9}, \frac{x_{4u-3}}{9}, \frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}, x_{4u-2}$ are two relatively odd prime.

From lemma 5, that, if and only if u=0, 1, $\frac{x_{2u-1}}{9}$ is a square, and if and only if u=1, $\frac{x_{4u-3}}{9}$ is a square; For any l∈Z, x_{4u-2}, $\frac{y_{4u-1}}{2}$ it's not a square number. If and only if u=1, $\frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}$ all are a square number. So if u≠0,1, $\frac{x_{2u-1}}{9}, \frac{x_{4u-3}}{9}, \frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}, x_{4u-2}$ does not equal 0,1, it's not a square number.

When u=0, equation (7) is

$$Dz^2 = 2^5 \cdot 9^2 \cdot x_2 \cdot \frac{x_3}{9} \cdot \frac{y_3}{2} \tag{8}$$

However, $x_2 = 161 = 7 \times 23, \frac{x_3}{9} = \frac{2889}{9} = 3 \times 107, \frac{y_3}{2} = \frac{646}{2} = 17 \times 19$

Therefore, the right hand side of (8) contains six different odd prime Numbers, so formula (8) does not hold, and the system (2) has no solution.

When u = 1,

$$Dz^2 = 8x_1y_1x_2x_1y_1 = 2^3 \times 9^2 \times 2^2 \times 161 = 2^5 \times 3^4 \times 7 \times 23 = 2 \times 7 \times 23 \times \left(2^2 \times 3^2\right)^2 = 2^3 \times 7 \times 23 \times \left(2 \times 3^2\right)^2 = 2^5 \times 7 \times 23 \times 3^4.$$

So when D=2×7×23, the system (2) has a nontrivial solutions (x, y, z) = (±2889,±646,±36); D=2^3×7×23, (2) has a nontrivial solution (x, y, z) = (±2889,±646,±18), when D=2^3×7×23, (2) has a nontrivial solution (x, y, z) = (±2889,±646,±9).

case 1.1.2 If r is even, let $r = 2v, (v \in Z)$, then equation (5) can be written into

$$Dz^2 = 8x_{4v-1}y_{4v-1}x_{4v}x_{2v}y_{2v} = 16x_{4v-1}y_{4v-1}x_{4v}x_{2v}x_vy_v \tag{9}$$

From lemma 5, when v is even, $\frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}, x_{4v}, x_{2v}, x_v, \frac{y_v}{18}$ are two relatively prime, when v is odd $\frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}, x_{4v}, x_{2v}, \frac{x_v}{9}, \frac{y_v}{2}$ are two relatively prime. And when v is odd, $\frac{y_{4v-1}}{2}, \frac{x_{4v-1}}{9}, \frac{x_v}{9}, x_{4v}, x_{2v}, \frac{x_v}{9}$ all are odd; when v is even, $\frac{y_{4v-1}}{2}, \frac{x_{4v-1}}{9}, \frac{x_v}{2}, x_{4v}, x_{2v}$ all are odd;

From lemma 5, if and only if v=0, $\frac{x_{4v-1}}{5}, x_{4v}, x_{2v}, x_v \frac{x_{2u-1}}{51}$ are squares, and if and only if v=±1, $\frac{x_v}{5}$ is a square; For any v∈Z, x_{4u-2}, $\frac{y_{4v-1}}{2}$ is not square. If and only if v=0, 1, $\frac{y_v}{2}$ is a square. So if v ≠0 and v is even

$x_{4v}, x_{2v}, x_v, \frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}$ are not squares. At this point, they have at least five different odd prime Numbers, so formula (9) is not true, so when u≠0,1, the system (2) has no solution.

When v≠±1 and v is odd, $x_{4v}, x_{2v}, \frac{x_v}{9}, \frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}$ are not squares. At this point, they have at least five different odd prime Numbers, so formula (9) is not true, so when u≠0,1, the system (2) has no solution. So when v≠0, v≠±1 and v is even, the system (2) has no solution.

When v=0, (9) can be written into $Dz^2 = 16 \cdot x_0^{\ 3} \cdot y_0 \cdot x_{-1} \cdot y_{-1} = 0$, thus z=0, At this point, the system (2), only has ordinary solutions (x, y, z)= (±9,±2, 0).

When v=1, (9) can be written into

$$\begin{aligned} Dz^2 &= 16 \cdot x_3 \cdot y_3 \cdot x_4 \cdot x_2 \cdot x_1 \cdot y_1 = 16 \times 3^3 \times 107 \times 2 \times 17 \times 19 \times 47 \times 1103 \times 7 \times 23 \times 9 \times 2, \\ &= 2^6 \times 3^5 \times 7 \times 17 \times 19 \times 23 \times 47 \times 107 \times 1103 \end{aligned}$$

The right hand side of the above equation contains eight odd prime Numbers, so the above formula is impossible. Therefor when v=1, the system (2) has no common solution.

When v=-1,

$$\begin{aligned} Dz^2 &= 16 x_{-5} y_{-5} x_{-4} x_{-2} x_{-1} y_{-1} = 16 \cdot 9^2 \cdot 2^2 \cdot x_2 \cdot x_4 \cdot \frac{x_5}{9} \cdot \frac{y_5}{2} \\ &= 16 \times 9^2 \times 2^2 \times 161 \times 51841 \times \frac{930249}{9} \times \frac{208010}{2} \\ &= 2^6 \times 3^4 \times 7 \times 23 \times 47 \times 1103 \times 41 \times 2521 \times 5 \times 11 \times 31 \times 61 \\ &= 2^6 \times 3^4 \times 5 \times 7 \times 11 \times 23 \times 31 \times 41 \times 47 \times 61 \times 1103 \times 2521 \end{aligned}$$

Therefore, the right hand side of the above equation contains ten odd prime Numbers, so the above formula is impossible. Therefor when v=-1, the system (2) has no common solution.

case 1.2 If m is odd, modelled on the case 1.1, it can be proved that the equation (2) is only the common solution (x, y, z)= (±9,±2,0).

case2 If n is even, by lemma 4, $y_{n-1} \equiv y_{n+1} \equiv 1 \pmod 2$, the right-hand side of equation (4) is odd, while the left-hand side is even in the form of D, so the system (2) has no common solution.

4. Summary and Prospect

In this paper, we have gotten the solutions of the following equations

$$\begin{cases} x^2 - k(k+1)y^2 = k \\ y^2 - Dz^2 = 4 \end{cases} (k \in \mathrm{N}^*),$$

When k=4 and $D = 2^t p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_4} p_4^{\alpha_3}$, where $\alpha_s = 0$ or $1\ (1 \le s \le 4)$. And we can go on and talk about the solutions to this system when k > 4, and D is some other form. So there is a lot of room for studying these kinds of equations.

Due to the diverse forms of such equations, many scholars have done more studies on the smaller values of k and m, and the main conclusions are focused on the estimation of solutions under some special forms of D_1 and the specific values of D. We need more powerful ways of finding common solutions to more forms of equations.

Acknowledgement

Fund project: Education reform project of Longnan Teachers college (JXGG201736).

References

[1] Ljunggrenn, W. (1941). Litt om simuilane pellske ligninger. *Norsk Mat Tidsskr*, *23*, 132-138.

[2] Pan, J. Y., Zhang, Y. P., & Zou, R. (1999). The pell equations $x^2-ay^2=1$, $y^2-DZ^2=1$. *Chinese Quartly Journal of Mathematics*, *14(1)*, 73-77.

[3] Chen, Z. Y. (1998). On the diophantine equations $x^2-2y^2=1$, $y^2-DZ^2=4$. *Journal of Central China Normal University*, *32(2)*, 137-140.

[4] Hu, Y. Z., & Han, Q. (2002). Also talk about the indefinite equations equations $x^2-2y^2=1, y^2-DZ^2=4$. *Journal of Central China Normal University*, *36(1)*, 17-19.

[5] Dong, P., & Yang, S. C. (2003). On the Diophantine equations $x^2-2y^2=1, y^2-DZ^2=4$. Journal of North China University, *4(2)*, 98-100.

[6] Le, M. H. (2004). The common solution of simultaneous pell equations $x^2-2y^2=1, y^2-DZ^2=4$. *Journal of Yantai Normal University*, *20(1)*, 8-10.

[7] Chen, J. H. (1990). The common solution of simultaneous pell equations $x^2-2y^2=1$ and $y^2-DZ^2=4$. *Journal of Wuhan Normal University*, *(1)*, 8-12.

[8] Cao, Z. F. (1986). The common solution of simultaneous pell equations $x^2-2y^2=1$ and $y^2-DZ^2=4$. *Science Bulletin*, *31(6)*, 476.

[9] Chen, Y. G. (1994). The common solution of simultaneous pell equations $x^2-2y^2=1$ and $y^2-DZ^2=4$. *Journal of Peking University*, *30(3)*, 298-302

[10] Du, X. C., & Li, Y. L. (2015). The common solution of simultaneous pell equations $x^2-6y^2=1$ and $y^2-DZ^2=4$. *Journal of Anhui University*, *39(6)*, 19-22.

[11] Du, X. C., *et al*. (2014). The common solution of simultaneous pell equations $x^2-6y^2=1$ and $y^2-DZ^2=4$. *Journal of Central China Normal University*, *48(3)*, 310-313

[12] Ran, Y. X. (2009). The discussion and study of integer solutions for a class of indefinite systems. Master's thesis from Northwestern University.

[13] Su, X. Y. (2000). The common solution of simultaneous pell equations $x^2-6y^2=1$ and $y^2-DZ^2=4$. *Journal of Zhangzhou Normal University: Natural Science*, *13*, 35-38.

[14] He, l. R. The discussion and study of integer solutions for some classes of indefinite systems [D]. Master's thesis from northwestern university, 2012

[15] Ran, Y. P. (2012). The common solution of simultaneous pell equations $x^2-10y^2=1$ and $y^2-DZ^2=4$. *Journal of Yan'an University: Natural Science*, *3(31)*, 8-10.

[16] Guo, J., & Du, X. C. (2015). The common solution of simultaneous pell equations $x^2-12y^2=1$ and $y^2-DZ^2=4$. *Practice and Understanding of Mathematics*, *45(9)*, 289-293.

[17] Gao, L., & Li, G. R. (2016). The common solution of simultaneous pell equations $x^2-12y^2=1$ and $y^2-DZ^2=4$. *Journal of Yan'an University*, *35(3)*, 10-12.

[18] Ran, Y. X. (2017). On the Diophantine equations $x^2-12y^2=1$ and $y^2-DZ^2=4$. *Journal of Yan'an University*, *36(3)*, 68-71.

[19] Du, X. C., & Guo, J. (2015). The common solution of simultaneous pell equations $x^2-30y^2=1$ and $y^2-DZ^2=4$. *Practice and Understanding of Mathematics*, *45(9)*, 289-293.

[20] Zhao, J. H. (2018). On the diophantine equations $x^2-23y^2=1$ and $y^2-DZ^2=25$. *Practice and Understanding of Mathematics*, *45(2)*, 309-314.

[21] Ke, Z., & Sun, Q. (2011). *About the Indeterminate Equation*. Harbin: Harbin Institute of Technology Press.

Ran Yinxia was born in April 1983. She comes from Gansu province, China. She graduated from Northwestern University with a master of science degree in 2009, and studied on number theory research. After graduation, she has been taught in Longnan Teachers College.