Positive Solutions for a System of Fractional Differential Equations with Parameters and Coupled Multi-point Boundary Conditions

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Abstract: In this paper, we study a system of nonlinear Riemann-Liouville fractional ordinary differential equations with parameters, subject to coupled multi-point boundary conditions which contain fractional derivatives. By using some properties of the associated Green's functions and the Guo-Krasnosel'skii fixed point theorem, we prove the existence of positive solutions for this problem when the parameters belong to various intervals. Then, we present sufficient conditions for the nonexistence of positive solutions.

Key words: Fractional differential equations, multi-point boundary conditions, positive solutions, existence, nonexistence.

1. Introduction

We consider the system of nonlinear ordinary fractional differential equations

$$(S)\qquad \begin{cases} D_{0+}^{\alpha}u(t)+\lambda f\big(t,u(t),v(t)\big)=0, & t\in(0,1),\\ D_{0+}^{\beta}v(t)+\mu g\big(t,u(t),v(t)\big)=0, & t\in(0,1),\end{cases}$$

with the coupled multi-point boundary conditions

$$(BC)\qquad \begin{cases} u^{(j)}(0)=0,\ \ j=0,\dots,n-2;\ \ D_{0+}^{p_1}u(t)|_{t=1}=\sum_{i=1}^{N}a_iD_{0+}^{q_1}v(t)|_{t=\xi_i},\\ v^{(j)}(0)=0,\ \ j=0,\dots,m-2;\ \ D_{0+}^{p_2}v(t)|_{t=1}=\sum_{i=1}^{M}b_iD_{0+}^{q_2}u(t)|_{t=\eta_i},\end{cases}$$

where $\alpha,\beta\in\mathbb{R}$, $\alpha\in(n-1,n]$, $\beta\in(m-1,m]$, $n,m\in\mathbb{N}$, $n,m\geq 3, p_1,p_2,q_1,q_2\in\mathbb{R}, p_1\in[1,n-2], p_2\in[1,m-2]$, $q_1\in[0,p_2], q_2\in[0,p_1]$, $\xi_i,a_i\in\mathbb{R}$ for all $i=1,\dots,N$ $(N\in\mathbb{N})$, $0<\xi_1<\dots<\xi_N\leq 1, \eta_i,b_i\in\mathbb{R}$ for all $i=1,\dots,M$ $(M\in\mathbb{N}), 0<\eta_1<\dots<\eta_M\leq 1$, $\lambda,\mu>0$, and D_{0+}^{k} denotes the Riemann-Liouville derivative of order k (for $k=\alpha,\beta,p_1,p_2,q_1,q_2$).

Under some assumptions on the nonnegative functions f and g, we present intervals for the parameters λ and μ such that positive solutions of *(S)-(BC)* exist. By a positive solution of problem *(S)-(BC)* we mean a pair of functions $(u,v)\in\big(C([0,1]),[0,\infty)\big)^2$ satisfying (S) and (BC), with $u(t)>0$ for all

$t \in (0,1]$ or $v(t) > 0$ for all $t \in (0,1]$. The nonexistence of positive solutions for *(S)-(BC)* is also investigated. The system *(S)* without parameters λ, μ with the boundary conditions *(BC)* was investigated in [1], where f and g are non-singular or singular functions. The existence of positive solutions of the system *(S)* without parameters subject to the uncoupled multi-point boundary conditions

$$(\widetilde{BC}) \qquad \begin{cases} u^{(j)}(0) = 0, \ \ j = 0, \dots, n-2; \ \ D_{0+}^{p_1} u(t)|_{t=1} = \sum_{i=1}^{N} a_i D_{0+}^{q_1} u(t)|_{t=\xi_i}, \\ v^{(j)}(0) = 0, \ \ j = 0, \dots, m-2; \ \ D_{0+}^{p_2} v(t)|_{t=1} = \sum_{i=1}^{M} b_i D_{0+}^{q_2} v(t)|_{t=\eta_i}, \end{cases}$$

was studied in [2], [3]. For other papers which investigate the existence, nonexistence and multiplicity of positive solutions for systems of fractional differential equations with nonnegative or sign-changing nonlinearities, subject to various nonlocal boundary conditions we mention [4]-[12].

Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (for example, the primary infection with HIV), economics, control theory, signal and image processing, thermoelasticity, aerodynamics, viscoelasticity, electromagnetics and rheology (see [13-28]). Fractional differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations.

The paper is organized as follows. In Section 2, we present some auxiliary results which investigate a nonlocal boundary value problem for fractional differential equations, and we give the properties of the Green functions associated to our problem. Section 3 contains the existence and nonexistence results for the positive solutions of problem (S)-(BC). In the proof of our main existence theorems we use the Guo-Krasnosel'skii fixed point theorem (see [29]). Finally, in Section 4, an example is given to support the new results.

2. Auxiliary Results

In this section we present some auxiliary results from [1] that will be used to prove our main theorems.

We consider the fractional differential system

$$\begin{cases} D_{0+}^{\alpha} u(t) + x(t) = 0, \ \ t \in (0,1), \\ D_{0+}^{\beta} v(t) + y(t) = 0, \ \ t \in (0,1), \end{cases} \tag{1}$$

with the coupled multi-point boundary conditions (BC), where $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \geq 3, p_1, p_2, q_1, q_2 \in \mathbb{R}, p_1 \in [1, n-2], p_2 \in [1, m-2]$, $q_1 \in [0, p_2], q_2 \in [0, p_1]$, $\xi_i, a_i \in \mathbb{R}$ for all $i = 1, \dots, N$ $(N \in \mathbb{N})$, $0 < \xi_1 < \dots < \xi_N \leq 1, \eta_i, b_i \in \mathbb{R}$ for all $i = 1, \dots, M$ $(M \in \mathbb{N}), 0 < \eta_1 < \dots < \eta_M \leq 1$, and $x, y\colon (0,1) \to \mathbb{R}$ are continuous functions.

We denote by Δ the constant

$$\Delta = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha-p_1)\Gamma(\beta-p_2)} - \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha-q_2)\Gamma(\beta-q_1)} \left(\sum_{i=1}^{N} a_i \xi_i^{\beta-q_1-1}\right) \left(\sum_{i=1}^{M} b_i \eta_i^{\alpha-q_2-1}\right). \tag{2}$$

Lemma 2.1 If $\Delta \neq 0$ and $x, y \in C(0,1) \cap L^1(0,1)$, then the pair of functions $(u, v) \in C[0,1] \times C[0,1]$ given by

$$\begin{cases} u(t) = \int_0^1 G_1(t,s)x(s)ds + \int_0^1 G_2(t,s)y(s)ds, & t \in [0,1], \\ v(t) = \int_0^1 G_3(t,s)y(s)ds + \int_0^1 G_4(t,s)x(s)ds, & t \in [0,1], \end{cases} \tag{3}$$

is solution of problem (1)-(BC), where

$$\begin{aligned} G_1(t,s) &= g_1(t,s) + \frac{t^{\alpha-1}\Gamma(\beta)}{\Delta\Gamma(\beta-q_1)}\left(\sum_{i=1}^N a_i \xi_i^{\beta-q_1-1}\right)\left(\sum_{i=1}^M b_i g_2(\eta_i,s)\right), \\ G_2(t,s) &= \frac{t^{\alpha-1}\Gamma(\beta)}{\Delta\Gamma(\beta-p_2)}\left(\sum_{i=1}^N a_i\, g_3(\xi_i,s)\right), \\ G_3(t,s) &= g_4(t,s) + \frac{t^{\beta-1}\Gamma(\alpha)}{\Delta\Gamma(\alpha-q_2)}\left(\sum_{i=1}^M b_i \eta_i^{\alpha-q_2-1}\right)\left(\sum_{i=1}^N a_i g_3(\xi_i,s)\right), \\ G_4(t,s) &= \frac{t^{\beta-1}\Gamma(\alpha)}{\Delta\Gamma(\alpha-p_1)}\left(\sum_{i=1}^M b_i\, g_2(\eta_i,s)\right), \quad \forall\, t,s \in [0,1], \end{aligned} \tag{4}$$

And

$$\begin{aligned} g_1(t,s) &= \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-p_1-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-p_1-1}, & 0 \le t \le s \le 1, \end{cases} \\ g_2(t,s) &= \frac{1}{\Gamma(\alpha-q_2)}\begin{cases} t^{\alpha-q_2-1}(1-s)^{\alpha-p_1-1} - (t-s)^{\alpha-q_2-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-q_2-1}(1-s)^{\alpha-p_1-1}, & 0 \le t \le s \le 1. \end{cases} \\ g_3(t,s) &= \frac{1}{\Gamma(\beta-q_1)}\begin{cases} t^{\beta-q_1-1}(1-s)^{\beta-p_2-1} - (t-s)^{\beta-q_1-1}, & 0 \le s \le t \le 1, \\ t^{\beta-q_1-1}(1-s)^{\beta-p_2-1}, & 0 \le t \le s \le 1. \end{cases} \\ g_4(t,s) &= \frac{1}{\Gamma(\beta)}\begin{cases} t^{\beta-1}(1-s)^{\beta-p_2-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ t^{\beta-1}(1-s)^{\beta-p_2-1}, & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{5}$$

Lemma 2.2 The functions $g_i, 1 = 1, \ldots, 4$ given by (5) have the properties:

$a_1)$ $g_1(t,s) \le h_1(s)$ *for all* $t,s \in [0,1]$, *where* $h_1(s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-p_1-1}(1-(1-s)^{p_1})$, $s \in [0,1]$;

$a_2)$ $g_1(t,s) \ge t^{\alpha-1}h_1(s)$ *for all* $t,s \in [0,1]$;

$a_3)$ $g_1(t,s) \le \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$ *for all* $t,s \in [0,1]$;

$b_1)$ $g_2(t,s) \ge t^{\alpha-q_2-1}h_2(s)$ *for all* $t,s \in [0,1]$, *where*

$$h_2(s) = \frac{1}{\Gamma(\alpha-q_2)}(1-s)^{\alpha-p_1-1}(1-(1-s)^{p_1-q_2}),\ s \in [0,1];$$

$b_2)$ $g_2(t,s) \le \frac{1}{\Gamma(\alpha-q_2)}t^{\alpha-q_2-1}(1-s)^{\alpha-p_1-1}$ *for all* $t,s \in [0,1]$;

$b_3)$ $g_2(t,s) \le \frac{1}{\Gamma(\alpha-q_2)}t^{\alpha-q_2-1}$ *for all* $t,s \in [0,1]$;

$c_1)$ $g_1(t,s) \ge t^{\beta-q_1-1}h_3(s)$ *for all* $t,s \in [0,1]$, *where*

$$h_3(s) = \frac{1}{\Gamma(\beta-q_1)}(1-s)^{\beta-p_2-1}(1-(1-s)^{p_2-q_1}),\ s \in [0,1];$$

$c_2)$ $g_3(t,s) \le \frac{1}{\Gamma(\beta-q_1)}t^{\beta-q_1-1}(1-s)^{\beta-p_2-1}$ *for all* $t,s \in [0,1]$;

$c_3)$ $g_3(t,s) \le \frac{1}{\Gamma(\beta-q_1)}t^{\beta-q_1-1}$ *for all* $t,s \in [0,1]$;

$d_1)$ $g_4(t,s) \le h_4(s)$ *for all* $t,s \in [0,1]$, *where* $h_4(s) = \frac{1}{\Gamma(\beta)}(1-s)^{\beta-p_2-1}(1-(1-s)^{p_2})$, $s \in [0,1]$;

$d_2)$ $g_4(t,s) \ge t^{\beta-1}h_4(s)$ *for all* $t,s \in [0,1]$;

$d_3)$ $g_4(t,s) \le \frac{1}{\Gamma(\beta)}t^{\beta-1}$ *for all* $t,s \in [0,1]$;

e) The functions $g_i, 1 = 1, \dots, 4$ *are continuous on* $[0,1] \times [0,1]$; $g_i(t,s) \geq 0$, *for all* $t,s \in [0,1]$; $g_i(t,s) > 0$, *for all* $t,s \in (0,1)$, $i = 1, \dots, 4$.

Lemma 2.3 If $\Delta > 0$, $a_i \geq 0$ for all $i = 1, \dots, N$, and $b_i \geq 0$ for all $i = 1, \dots, M$, then the functions $G_i, i = 1, \dots, 4$ given by (4) have the properties

a_1) $G_1(t,s) \leq J_1(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$J_1(s) = h_1(s) + \frac{\Gamma(\beta)}{\Delta\Gamma(\beta-q_1)}\left(\sum_{i=1}^{N} a_i \xi_i^{\beta-q_1-1}\right)\left(\sum_{i=1}^{M} b_i g_2(\eta_i, s)\right), \forall s \in [0,1];$$

a_2) $G_1(t,s) \geq t^{\alpha-1} J_1(s)$, $\forall (t,s) \in [0,1] \times [0,1]$;

a_3) $G_1(t,s) \leq \delta_1 t^{\alpha-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\delta_1 = \frac{1}{\Gamma(\alpha)} + \frac{\Gamma(\beta)}{\Delta\Gamma(\beta-q_1)\Gamma(\alpha-q_2)}\left(\sum_{i=1}^{N} a_i \xi_i^{\beta-q_1-1}\right)\left(\sum_{i=1}^{M} b_i \eta_i^{\alpha-q_2-1}\right);$$

b_1) $G_2(t,s) \leq J_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $J_2(s) = \frac{\Gamma(\beta)}{\Delta\Gamma(\beta-p_2)}\left(\sum_{i=1}^{N} a_i\, g_3(\xi_i, s)\right), \forall s \in [0,1]$;

b_2) $G_2(t,s) = t^{\alpha-1} J_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$;

b_3) $G_2(t,s) \leq \delta_2 t^{\alpha-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\delta_2 = \frac{\Gamma(\beta)}{\Delta\Gamma(\beta-p_2)\Gamma(\beta-q_1)}\sum_{i=1}^{N} a_i \xi_i^{\beta-q_1-1}$;

c_1) $G_3(t,s) \leq J_3(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$J_3(s) = h_4(s) + \frac{\Gamma(\alpha)}{\Delta\Gamma(\alpha-q_2)}\left(\sum_{i=1}^{M} b_i \eta_i^{\alpha-q_2-1}\right)\left(\sum_{i=1}^{N} a_i g_3(\xi_i, s)\right), \forall s \in [0,1];$$

c_2) $G_3(t,s) \geq t^{\beta-1} J_3(s)$, $\forall (t,s) \in [0,1] \times [0,1]$;

c_3) $G_3(t,s) \leq \delta_3 t^{\beta-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\delta_3 = \frac{1}{\Gamma(\beta)} + \frac{\Gamma(\alpha)}{\Delta\Gamma(\alpha-q_2)\Gamma(\beta-q_1)}\left(\sum_{i=1}^{M} b_i \eta_i^{\alpha-q_2-1}\right)\left(\sum_{i=1}^{N} a_i \xi_i^{\beta-q_1-1}\right);$$

d_1) $G_4(t,s) \leq J_4(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $J_4(s) = \frac{\Gamma(\alpha)}{\Delta\Gamma(\alpha-p_1)}\left(\sum_{i=1}^{M} b_i\, g_2(\eta_i, s)\right), \forall s \in [0,1]$;

d_2) $G_4(t,s) = t^{\beta-1} J_4(s)$, $\forall (t,s) \in [0,1] \times [0,1]$;

d_3) $G_4(t,s) \leq \delta_4 t^{\beta-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\delta_4 = \frac{\Gamma(\alpha)}{\Delta\Gamma(\alpha-p_1)\Gamma(\alpha-q_2)}\sum_{i=1}^{M} b_i \eta_i^{\alpha-q_2-1}$.

e) The functions $G_i, 1 = 1, \dots, 4$ *are continuous on* $[0,1] \times [0,1]$, *and* $G_i(t,s) \geq 0$, *for all* $t,s \in [0,1]$, $i = 1, \dots, 4$.

Lemma 2.4 If $\Delta > 0$, $a_i \geq 0$ for all $i = 1, \dots, N$, $b_i \geq 0$ for all $i = 1, \dots, M$, and $x, y \in C(0,1) \cap L^1(0,1)$ with $x(t) \geq 0, y(t) \geq 0$ for all $t \in (0,1)$, then the solution (u,v) of problem (1)-(BC) given by (3) satisfies the inequalities $u(t) \geq 0, v(t) \geq 0$ for all $t \in [0,1]$. Moreover, we have the inequalities $u(t) \geq t^{\alpha-1} u(t')$ and $v(t) \geq t^{\beta-1} v(t')$ for all $t, t' \in [0,1]$.

Remark 2.1 Under the assumptions of Lemma 2.4, for any interval $[c_1, c_2] \subset [0,1]$ with $0 < c_1 < c_2 \leq 1$, the solution of problem (1)-(BC) given by (3) satisfies the inequalities $\min_{t\in[c_1,c_2]} u(t) \geq c_1^{\alpha-1} \max_{t'\in[0,1]} u(t')$ and $\min_{t\in[c_1,c_2]} v(t) \geq c_1^{\beta-1} \max_{t'\in[0,1]} v(t')$.

In the proof of our main existence results we will use the Guo-Krasnosel'skii fixed point theorem presented below (see [29]).

Theorem 2.1 Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $A: C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator such that, either

1) $\|Au\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|Au\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$, *or*
2) $\|Au\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|Au\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.

Then A has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Main Results

In this section we give first some sufficient conditions on λ, μ, f and g such that positive solutions with respect to a cone for our problem *(S)-(BC)* exist.

We present now the assumptions that we will use in the sequel.

(H1) $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \geq 3, p_1, p_2, q_1, q_2 \in \mathbb{R}, p_1 \in [1, n-2], p_2 \in [1, m-2]$, $q_1 \in [0, p_2]$, $q_2 \in [0, p_1]$, $\xi_i \in \mathbb{R}$, $a_i \geq 0$ for all $i = 1, \dots, N$ $(N \in \mathbb{N}), 0 < \xi_1 < \dots < \xi_N \leq 1, \eta_i \in \mathbb{R}$, $b_i \geq 0$ for all $i = 1, \dots, M$ $(M \in \mathbb{N}), 0 < \eta_1 < \dots < \eta_M \leq 1$, $\lambda, \mu > 0$ and $\Delta > 0$ (given by (2)).

(H2) The functions $f, g: [0,1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous.

For $[c_1, c_2] \subset [0,1]$ with $0 < c_1 < c_2 \leq 1$, we introduce the following extreme limits

$$f_0^s = \limsup_{u+v \to 0^+} \max_{t \in [0,1]} \frac{f(t,u,v)}{u+v}, \quad g_0^s = \limsup_{u+v \to 0^+} \max_{t \in [0,1]} \frac{g(t,u,v)}{u+v},$$

$$f_0^i = \liminf_{u+v \to 0^+} \min_{t \in [c_1,c_2]} \frac{f(t,u,v)}{u+v}, \quad g_0^i = \liminf_{u+v \to 0^+} \min_{t \in [c_1,c_2]} \frac{g(t,u,v)}{u+v},$$

$$f_\infty^s = \limsup_{u+v \to \infty} \max_{t \in [0,1]} \frac{f(t,u,v)}{u+v}, \quad g_\infty^s = \limsup_{u+v \to \infty} \max_{t \in [0,1]} \frac{g(t,u,v)}{u+v},$$

$$f_\infty^i = \liminf_{u+v \to \infty} \min_{t \in [c_1,c_2]} \frac{f(t,u,v)}{u+v}, \quad g_\infty^i = \liminf_{u+v \to \infty} \min_{t \in [c_1,c_2]} \frac{g(t,u,v)}{u+v}.$$

In the definitions of the extreme limits above the variables u and v are nonnegative.

By using Lemma 2.1, a solution of the following nonlinear system of integral equations

$$\begin{cases} u(t) = \lambda \int_0^1 G_1(t,s) f(s, u(s), v(s)) ds + \mu \int_0^1 G_2(t,s) g\big(s, u(s), v(s)\big) ds, & t \in [0,1], \\ v(t) = \mu \int_0^1 G_3(t,s) g\big(s, u(s), v(s)\big) ds + \lambda \int_0^1 G_4(t,s) f(s, u(s), v(s)) ds, & t \in [0,1], \end{cases}$$

is solution of problem *(S)-(BC).*

We consider the Banach space $X = C[0,1]$ with the supremum norm $\|\cdot\|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cone $P \subset Y$ by

$$P = \{(u,v) \in Y;\ u(t) \geq t^{\alpha-1}\|u\|, \quad v(t) \geq t^{\beta-1}\|v\|, \quad \forall\, t \in [0,1]\}.$$

For $\lambda, \mu > 0$, we introduce the operators $Q_1, Q_2: Y \to X$ and $Q: Y \to Y$ defined by

$$Q_1(u,v)(t) = \lambda \int_0^1 G_1(t,s) f\big(s, u(s), v(s)\big) ds + \mu \int_0^1 G_2(t,s) g\big(s, u(s), v(s)\big) ds, \quad t \in [0,1],$$
$$Q_2(u,v)(t) = \mu \int_0^1 G_3(t,s) g\big(s, u(s), v(s)\big) ds + \lambda \int_0^1 G_4(t,s) f(s, u(s), v(s)) ds, \quad t \in [0,1],$$

and $Q(u,v) = (Q_1(u,v), Q_2(u,v))$, $(u,v) \in Y$. Then if (u,v) is a fixed point of operator Q, then (u,v) is a solution of problem *(S)-(BC).* Using Lemma 2.4 and similar arguments as those used in the proof of Lemma 3.1 from [5], we deduce that under assumptions (H1)-(H2), the operator $Q: P \to P$ is a completely continuous operator.

For $[c_1, c_2] \subset [0,1]$ with $0 < c_1 < c_2 \leq 1$, we denote by

$A = \int_0^1 J_1(s)ds,\ B = \int_0^1 J_2(s)ds,\ C = \int_0^1 J_3(s)ds,\ D = \int_0^1 J_4(s)ds,\ \tilde{A} = \int_{c_1}^{c_2} J_1(s)ds,\ \tilde{B} = \int_{c_1}^{c_2} J_2(s)ds,\ \tilde{C} = \int_{c_1}^{c_2} J_3(s)ds,\ \widetilde{D} = \int_{c_1}^{c_2} J_4(s)ds$, where $J_i,\ i = 1,\dots,4$ are defined in Section 2 (Lemma 2.3).

For $f_0^s,\ g_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$ and numbers $\alpha_1,\alpha_2 \in [0,1]$, $\alpha_3,\alpha_4 \in (0,1)$, $a \in [0,1]$ and $b \in (0,1)$, we define the numbers

$$L_1 = \max\left\{\frac{a\alpha_1}{\gamma\gamma_1 f_\infty^i \tilde{A}}, \frac{(1-a)\alpha_2}{\gamma\gamma_2 f_\infty^i \widetilde{D}}\right\},\quad L_2 = \min\left\{\frac{b\alpha_3}{f_0^s A}, \frac{(1-b)\alpha_4}{f_0^s D}\right\},$$

$$L_3 = \max\left\{\frac{a(1-\alpha_1)}{\gamma\gamma_1 g_\infty^i \tilde{B}}, \frac{(1-a)(1-\alpha_2)}{\gamma\gamma_2 g_\infty^i \tilde{C}}\right\},\quad L_4 = \min\left\{\frac{b(1-\alpha_3)}{g_0^s B}, \frac{(1-b)(1-\alpha_4)}{g_0^s C}\right\},$$

$$L_2' = \min\left\{\frac{b}{f_0^s A}, \frac{1-b}{f_0^s D}\right\},\qquad L_4' = \min\left\{\frac{b}{g_0^s B}, \frac{1-b}{g_0^s C}\right\},$$

where $\gamma_1 = c_1^{\alpha-1}$, $\gamma_2 = c_1^{\beta-1}$ and $\gamma = \min\{\gamma_1, \gamma_2\}$.

Theorem 3.1 Assume that (H1) and (H2) hold, $[c_1, c_2] \subset [0,1]$ with $0 < c_1 < c_2 \le 1$, $\alpha_1,\alpha_2 \in [0,1]$, $\alpha_3,\alpha_4 \in (0,1)$, $a \in [0,1]$ and $b \in (0,1)$.

1) If $f_0^s,\ g_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$, $L_1 < L_2$ and $L_3 < L_4$, then for each $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$ there exists a positive solution $(u(t), v(t)), t \in [0,1]$ for (S)-(BC).
2) If $f_0^s = 0,\ g_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$, and $L_3 < L_4'$, then for each $\lambda \in (L_1, \infty)$ and $\mu \in (L_3, L_4')$ there exists a positive solution $(u(t), v(t)), t \in [0,1]$ for (S)-(BC).
3) If $g_0^s = 0,\ f_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$, and $L_1 < L_2'$, then for each $\lambda \in (L_1, L_2')$ and $\mu \in (L_3, \infty)$ there exists a positive solution $(u(t), v(t)), t \in [0,1]$ for (S)-(BC).
4) If $f_0^s = g_0^s = 0,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$, then for each $\lambda \in (L_1, \infty)$ and $\mu \in (L_3, \infty)$ there exists a positive solution $(u(t), v(t)), t \in [0,1]$ for (S)-(BC).
5) If $f_0^s,\ g_0^s \in (0,\infty)$, and at least one of $f_\infty^i,\ g_\infty^i$ is ∞, then for each $\lambda \in (0, L_2)$ and $\mu \in (0, L_4)$ there exists a positive solution $(u(t), v(t)), t \in [0,1]$ for (S)-(BC).
6) If $f_0^s = 0,\ g_0^s \in (0,\infty)$, and at least one of $f_\infty^i,\ g_\infty^i$ is ∞, then for each $\lambda \in (0,\infty)$ and $\mu \in (0, L_4')$ there exists a positive solution $(u(t), v(t)), t \in [0,1]$ for (S)-(BC).
7) If $f_0^s \in (0,\infty)$, $g_0^s = 0$, and at least one of $f_\infty^i,\ g_\infty^i$ is ∞, then for each $\lambda \in (0, L_2')$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t)), t \in [0,1]$ for (S)-(BC).
8) If $f_0^s = g_0^s = 0$, and at least one of $f_\infty^i,\ g_\infty^i$ is ∞, then for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t)), t \in [0,1]$ for (S)-(BC).

Proof. We consider the above cone $P \subset Y$ and the operators Q_1, Q_2 and Q. Because the proofs of the above cases are similar, in what follows we will prove one of them, namely Case 3). So, we suppose $g_0^s = 0$ and $f_0^s,\ f_\infty^i,\ g_\infty^i \in (0,\infty)$. Let $\lambda \in (L_1, L_2')$ and $\mu \in (L_3, \infty)$. We choose $\tilde{\alpha}_3 \in \left(0, 1 - \frac{\lambda f_0^s A}{b}\right)$ and $\tilde{\alpha}_4 \in \left(0, 1 - \frac{\lambda f_0^s D}{1-b}\right)$. Let $\varepsilon > 0$ be a positive number such that $\varepsilon < f_\infty^i$, $\varepsilon < g_\infty^i$ and

$$\frac{a\alpha_1}{\gamma\gamma_1(f_\infty^i - \varepsilon)\tilde{A}} \le \lambda,\quad \frac{a(1-\alpha_1)}{\gamma\gamma_1(g_\infty^i - \varepsilon)\tilde{B}} \le \mu,\quad \frac{(1-a)\alpha_2}{\gamma\gamma_2(f_\infty^i - \varepsilon)\widetilde{D}} \le \lambda,\quad \frac{(1-a)(1-\alpha_2)}{\gamma\gamma_2(g_\infty^i - \varepsilon)\tilde{C}} \le \mu,$$
$$\frac{b(1-\tilde{\alpha}_3)}{(f_0^s + \varepsilon)A} \ge \lambda,\quad \frac{b\tilde{\alpha}_3}{\varepsilon B} \ge \mu,\quad \frac{(1-b)(1-\tilde{\alpha}_4)}{(f_0^s + \varepsilon)D} \ge \lambda,\quad \frac{(1-b)\tilde{\alpha}_4}{\varepsilon C} \ge \mu.$$

By using (H2) and the definitions of f_0^s and g_0^s, we deduce that there exists $R_1 > 0$ such that $f(t,u,v) \le (f_0^s + \varepsilon)(u+v)$ and $g(t,u,v) \le \varepsilon(u+v)$ for all $t \in [0,1]$, $u,v \in [0,\infty)$ with $0 \le u+v \le R_1$. We define the set $\Omega_1 = \{(u,v) \in Y,\ \|(u,v)\|_Y < R_1\}$. Now let $(u,v) \in P \cap \partial\Omega_1$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_1$ or equivalently $\|u\| + \|v\| = R_1$. Then $u(t) + v(t) \le R_1$ for all $t \in [0,1]$, and by Lemma 2.3, we obtain

$$\begin{aligned} Q_1(u,v)(t) &\le \lambda \int_0^1 J_1(s) f\big(s,u(s),v(s)\big) ds + \mu \int_0^1 J_2(s) g\big(s,u(s),v(s)\big) ds \\ &\le \lambda \int_0^1 J_1(s)(f_0^s + \varepsilon)(u(s)+v(s)) ds + \mu \int_0^1 J_2(s)\varepsilon(u(s)+v(s)) ds \\ &\le \lambda(f_0^s + \varepsilon) \int_0^1 J_1(s)(\|u\| + \|v\|) ds + \mu\varepsilon \int_0^1 J_2(s)(\|u\|+\|v\|) ds \\ &= [\,\lambda(f_0^s+\varepsilon)A + \mu\varepsilon B]\|(u,v)\|_Y \le [b(1-\tilde{\alpha}_3) + b\tilde{\alpha}_3]\|(u,v)\|_Y = b\|(u,v)\|_Y, \qquad \forall t \in [0,1]. \end{aligned}$$

Therefore $\|Q_1(u,v)\| \le b\|(u,v)\|_Y$.

In a similar manner, we conclude

$$\begin{aligned} Q_2(u,v)(t) &\le \mu \int_0^1 J_3(s) g\big(s,u(s),v(s)\big) ds + \lambda \int_0^1 J_4(s) f\big(s,u(s),v(s)\big) ds \\ &\le \mu \int_0^1 J_3(s)\varepsilon(u(s)+v(s)) ds + \lambda \int_0^1 J_4(s)(f_0^s+\varepsilon)(u(s)+v(s)) ds \\ &\le \mu\varepsilon \int_0^1 J_3(s)(\|u\|+\|v\|) ds + \lambda(f_0^s+\varepsilon) \int_0^1 J_4(s)(\|u\|+\|v\|) ds \\ &= [\,\mu\varepsilon C + \lambda(f_0^s+\varepsilon)D]\|(u,v)\|_Y \le [(1-b)\tilde{\alpha}_4 + (1-b)(1-\tilde{\alpha}_4)]\|(u,v)\|_Y \\ &= (1-b)\|(u,v)\|_Y, \qquad \forall t \in [0,1]. \end{aligned}$$

Hence $\|Q_2(u,v)\| \le (1-b)\|(u,v)\|_Y$.

Then, for $(u,v) \in P \cap \partial\Omega_1$, we deduce

$$\|Q(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| \le b\|(u,v)\|_Y + (1-b)\|(u,v)\|_Y = \|(u,v)\|_Y. \tag{6}$$

By the definitions of f_∞^i and g_∞^i, there exists $\bar{R}_2 > 0$ such that $f(t,u,v) \ge \big(f_\infty^i - \varepsilon\big)(u+v)$ and $g(t,u,v) \ge \big(g_\infty^i - \varepsilon\big)(u+v)$ for all $u,v \ge 0$ with $u+v \ge \bar{R}_2$ and $t \in [c_1,c_2]$. We consider $R_2 = \max\{2R_1, \bar{R}_2/\gamma\}$ and we define $\Omega_2 = \{(u,v) \in Y,\ \|(u,v)\|_Y < R_2\}$. Then for $(u,v) \in P$ with $\|(u,v)\|_Y = R_2$, we obtain $u(t) + v(t) \ge t^{\alpha-1}\|u\| + t^{\beta-1}\|v\| \ge \gamma\|(u,v)\|_Y = \gamma R_2 \ge \bar{R}_2$, for all $t \in [c_1,c_2]$.

Then by Lemma 2.3, we conclude

$$\begin{aligned} Q_1(u,v)(c_1) &\ge \lambda c_1^{\alpha-1} \int_{c_1}^{c_2} J_1(s) f\big(s,u(s),v(s)\big) ds + \mu c_1^{\alpha-1} \int_{c_1}^{c_2} J_2(s) g\big(s,u(s),v(s)\big) ds \\ &\ge \lambda\gamma_1 \int_{c_1}^{c_2} J_1(s)\big(f_\infty^i - \varepsilon\big)(u(s)+v(s)) ds + \mu\gamma_1 \int_{c_1}^{c_2} J_2(s)\big(g_\infty^i - \varepsilon\big)(u(s)+v(s)) ds \\ &\ge \lambda\gamma\gamma_1\big(f_\infty^i - \varepsilon\big) \int_{c_1}^{c_2} J_1(s)\|(u,v)\|_Y ds + \mu\gamma\gamma_1\big(g_\infty^i - \varepsilon\big) \int_{c_1}^{c_2} J_2(s)\|(u,v)\|_Y ds \\ &= \big[\,\lambda\gamma\gamma_1\big(f_\infty^i - \varepsilon\big)\tilde{A} + \mu\gamma\gamma_1\big(g_\infty^i - \varepsilon\big)\tilde{B}\big]\|(u,v)\|_Y \end{aligned}$$

$$\geq [a\alpha_1 + a(1-\alpha_1)]\|(u,v)\|_Y = a\|(u,v)\|_Y.$$

So, $\|Q_1(u,v)\| \geq Q_1(u,v)(c_1) \geq a\|(u,v)\|_Y$.

In a similar manner, we conclude

$$Q_2(u,v)(c_1) \geq \mu c_1^{\beta-1}\int_{c_1}^{c_2} J_3(s)g\big(s,u(s),v(s)\big)ds + \lambda c_1^{\beta-1}\int_{c_1}^{c_2} J_4(s)f\big(s,u(s),v(s)\big)ds$$
$$\geq \mu\gamma_2\int_{c_1}^{c_2} J_3(s)\big(g_\infty^i-\varepsilon\big)(u(s)+v(s))ds + \lambda\gamma_2\int_{c_1}^{c_2} J_4(s)\big(f_\infty^i-\varepsilon\big)(u(s)+v(s))ds$$
$$\geq \mu\gamma\gamma_2\big(g_\infty^i-\varepsilon\big)\int_{c_1}^{c_2} J_3(s)\|(u,v)\|_Y ds + \lambda\gamma\gamma_2\big(f_\infty^i-\varepsilon\big)\int_{c_1}^{c_2} J_4(s)\|(u,v)\|_Y ds$$
$$= \big[\,\mu\gamma\gamma_2\big(g_\infty^i-\varepsilon\big)\tilde{C} + \lambda\gamma\gamma_2\big(f_\infty^i-\varepsilon\big)\tilde{D}\big]\|(u,v)\|_Y$$
$$\geq [(1-a)(1-\alpha_2) + (1-a)\alpha_2]\|(u,v)\|_Y = (1-a)\|(u,v)\|_Y.$$

So, $\|Q_2(u,v)\| \geq Q_2(u,v)(c_1) \geq (1-a)\|(u,v)\|_Y$.

Hence, for $(u,v) \in P \cap \partial\Omega_2$, we obtain

$$\|Q(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| \geq a\|(u,v)\|_Y + (1-a)\|(u,v)\|_Y = \|(u,v)\|_Y. \qquad (7)$$

By using (6), (7) and Theorem 2.1, i), we conclude that Q has a fixed point $(u,v) \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$ such that $R_1 \leq \|u\| + \|v\| \leq R_2$, $u(t) \geq t^{\alpha-1}\|u\|$, $v(t) \geq t^{\beta-1}\|v\|$ for all $t \in [0,1]$. If $\|u\| > 0$ then $u(t) > 0$ for all $t \in (0,1]$, and if $\|v\| > 0$ then $v(t) > 0$ for all $t \in (0,1]$.

In what follows, for f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$ and numbers $\alpha_1, \alpha_2 \in [0,1]$, $\alpha_3, \alpha_4 \in (0,1)$, $a \in [0,1]$ and $b \in (0,1)$, we define the numbers

$$\tilde{L}_1 = \max\left\{\frac{a\alpha_1}{\gamma\gamma_1 f_0^i\tilde{A}}, \frac{(1-a)\alpha_2}{\gamma\gamma_2 f_0^i\tilde{D}}\right\},\quad \tilde{L}_2 = \min\left\{\frac{b\alpha_3}{f_\infty^s A}, \frac{(1-b)\alpha_4}{f_\infty^s D}\right\},$$
$$\tilde{L}_3 = \max\left\{\frac{a(1-\alpha_1)}{\gamma\gamma_1 g_0^i\tilde{B}}, \frac{(1-a)(1-\alpha_2)}{\gamma\gamma_2 g_0^i\tilde{C}}\right\},\quad \tilde{L}_4 = \min\left\{\frac{b(1-\alpha_3)}{g_\infty^s B}, \frac{(1-b)(1-\alpha_4)}{g_\infty^s C}\right\},$$
$$\tilde{L}_2' = \min\left\{\frac{b}{f_\infty^s A}, \frac{1-b}{f_\infty^s D}\right\},\qquad \tilde{L}_4' = \min\left\{\frac{b}{g_\infty^s B}, \frac{1-b}{g_\infty^s C}\right\}.$$

By using similar arguments as those used in the proof of Theorem 3.1 (see also [5]) we obtain the following result.

Theorem 3.2 Assume that (H1) and (H2) hold, $[c_1,c_2] \subset [0,1]$ with $0 < c_1 < c_2 \leq 1$, $\alpha_1, \alpha_2 \in [0,1]$, $\alpha_3, \alpha_4 \in (0,1)$, $a \in [0,1]$ and $b \in (0,1)$.

1) If f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$, $\tilde{L}_1 < \tilde{L}_2$ and $\tilde{L}_3 < \tilde{L}_4$ then for each $\lambda \in (\tilde{L}_1, \tilde{L}_2)$ and $\mu \in \big(\tilde{L}_3, \tilde{L}_4\big)$ there exists a positive solution $\big(u(t), v(t)\big), t \in [0,1]$ for (S)-(BC).
2) If f_0^i, g_0^i, $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$ and $\tilde{L}_1 < \tilde{L}_2'$, then for each $\lambda \in (\tilde{L}_1, \tilde{L}_2')$ and $\mu \in \big(\tilde{L}_3, \infty\big)$ there exists a positive solution $\big(u(t), v(t)\big), t \in [0,1]$ for (S)-(BC).
3) If f_0^i, g_0^i, $g_\infty^s \in (0,\infty)$, $f_\infty^s = 0$, and $\tilde{L}_3 < \tilde{L}_4'$, then for each $\lambda \in (\tilde{L}_1, \infty)$ and $\mu \in \big(\tilde{L}_3, \tilde{L}_4'\big)$ there exists a positive solution $\big(u(t), v(t)\big), t \in [0,1]$ for (S)-(BC).
4) If f_0^i, $g_0^i \in (0,\infty)$, $f_\infty^s = g_\infty^s = 0$, then for each $\lambda \in (\tilde{L}_1, \infty)$ and $\mu \in \big(\tilde{L}_3, \infty\big)$ there exists a positive solution $\big(u(t), v(t)\big), t \in [0,1]$ for (S)-(BC).

5) If f_∞^s, $g_\infty^s \in (0,\infty)$, and at least one of f_0^i, g_0^i is ∞, then for each $\lambda \in (0,\tilde{L}_2)$ and $\mu \in (0,\tilde{L}_4)$ there exists a positive solution $(u(t),v(t)), t \in [0,1]$ for (S)-(BC).
6) If $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$ and at least one of f_0^i, g_0^i is ∞, then for each $\lambda \in (0,\tilde{L}'_2)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t),v(t)), t \in [0,1]$ for (S)-(BC).
7) If $f_\infty^s = 0$, $g_\infty^s \in (0,\infty)$ and at least one of f_0^i, g_0^i is ∞, then for each $\lambda \in (0,\infty)$ and $\mu \in (0,\tilde{L}'_4)$ there exists a positive solution $(u(t),v(t)), t \in [0,1]$ for (S)-(BC).
8) If $f_\infty^s = g_\infty^s = 0$ and at least one of f_0^i, g_0^i is ∞, then for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t),v(t)), t \in [0,1]$ for (S)-(BC).

Next we present intervals for λ and μ for which there exists no positive solutions of problem *(S)-(BC)*, viewed as fixed points of operator *Q*.

Theorem 3.3 Assume that (H1) and (H2) hold. If there exist positive numbers M_1, M_2 such that

$$f(t,u,v) \le M_1(u+v),\ g(t,u,v) \le M_2(u+v),\ \forall\, t \in [0,1],\ u,v \ge 0, \tag{8}$$

then there exist positive constants λ_0 and μ_0 such that for every $\lambda \in (0,\lambda_0)$, $\mu \in (0,\mu_0)$ the boundary value problem (S)-(BC) has no positive solution.

In the proof of Theorem 3.3 we can show that $\lambda_0 = \min\left\{\frac{1}{4M_1A}, \frac{1}{4M_1D}\right\}$ and $\mu_0 = \min\left\{\frac{1}{4M_2B}, \frac{1}{4M_2C}\right\}$, where $A = \int_0^1 J_1(s)ds$, $B = \int_0^1 J_2(s)ds$, $C = \int_0^1 J_3(s)ds$, $D = \int_0^1 J_4(s)ds$, satisfy the conditions of our theorem. If f_0^s, g_0^s, f_∞^s, $g_\infty^s < \infty$, then there exist positive constants M_1 and M_2 such that (8) holds, and then we obtain the conclusion of Theorem 3.3.

Theorem 3.4 a) Assume that (H1) and (H2) hold. If there exist positive numbers c_1, c_2 with $0 < c_1 < c_2 \le 1$ and $m_1 > 0$ such that

$$f(t,u,v) \ge m_1(u+v),\ \forall\, t \in [c_1,c_2],\ u,v \ge 0, \tag{9}$$

then there exists a positive constant $\tilde{\lambda}_0$ such that for every $\lambda > \tilde{\lambda}_0$ and $\mu > 0$, the boundary value problem (S)-(BC) has no positive solution.

b) Assume that (H1) and (H2) hold. If there exist positive numbers c_1, c_2 with $0 < c_1 < c_2 \le 1$ and $m_2 > 0$ such that

$$g(t,u,v) \ge m_2(u+v),\ \forall\, t \in [c_1,c_2],\ u,v \ge 0, \tag{10}$$

then there exists a positive constant $\tilde{\mu}_0$ such that for every $\lambda > 0$ and $\mu > \tilde{\mu}_0$, the boundary value problem (S)-(BC) has no positive solution.

c) Assume that (H1) and (H2) hold. If there exist positive numbers c_1, c_2 with $0 < c_1 < c_2 \le 1$ and $m_1, m_2 > 0$ such that

$$f(t,u,v) \ge m_1(u+v),\ g(t,u,v) \ge m_2(u+v),\ \forall\, t \in [c_1,c_2],\ u,v \ge 0, \tag{11}$$

then there exists positive constants $\hat{\lambda}_0$ and $\hat{\mu}_0$ such that for every $\lambda > \hat{\lambda}_0$ and $\mu > \hat{\mu}_0$, the boundary value problem (S)-(BC) has no positive solution.

In the proof of Theorem 3.4 we define $\tilde{\lambda}_0 = \min\left\{\frac{1}{\gamma\gamma_1 m_1\tilde{A}}, \frac{1}{\gamma\gamma_2 m_1\tilde{D}}\right\}$, $\tilde{\mu}_0 = \min\left\{\frac{1}{\gamma\gamma_1 m_2\tilde{B}}, \frac{1}{\gamma\gamma_2 m_2\tilde{C}}\right\}$, where $\tilde{A} = \int_{c_1}^{c_2} J_1(s)ds$, $\tilde{B} = \int_{c_1}^{c_2} J_2(s)ds$, $\tilde{C} = \int_{c_1}^{c_2} J_3(s)ds$, $\tilde{D} = \int_{c_1}^{c_2} J_4(s)ds$, and $\hat{\lambda}_0 = \frac{1}{2\gamma\gamma_1 m_1\tilde{A}}$, $\hat{\mu}_0 = \frac{1}{2\gamma\gamma_2 m_2\tilde{C}}$.

If for c_1, c_2 with $0 < c_1 < c_2 \le 1$, f_0^i, $f_\infty^i > 0$ and $f(t,u,v) > 0$ for all $t \in [c_1,c_2]$ and $u,v \ge 0$ with

$u+v>0$, then relation (9) holds and we obtain the conclusion of Theorem 3.4 a). If for c_1, c_2 with $0<c_1<c_2\le 1$, g_0^i, $g_\infty^i>0$ and $g(t,u,v)>0$ for all $t\in[c_1,c_2]$ and $u,v\ge 0$ with $u+v>0$, then relation (10) holds and we obtain the conclusion of Theorem 3.4 b). If for c_1,c_2 with $0<c_1<c_2\le 1$, f_0^i, f_∞^i, g_0^i, $g_\infty^i>0$ and $f(t,u,v)>0, g(t,u,v)>0$ for all $t\in[c_1,c_2]$ and $u,v\ge 0$ with $u+v>0$, then relation (11) holds and we obtain the conclusion of Theorem 3.4 c).

4. An Example

Let $=\frac{9}{2}$ $(n=5)$, $\beta=\frac{8}{3}$ $(m=3)$, $p_1=4/3$, $p_2=1$, $q_1=1/2$, $q_2=2/3$, $N=1, M=2$, $\xi_1=1/2$, $a_1=2$, $\eta_1=1/3$, $\eta_2=2/3$, $b_1=1$ and $b_2=1/2$.

We consider the system of fractional differential equations

$$(S_0)\qquad \begin{cases} D_{0+}^{9/2}u(t)+\lambda(t+1)^{\tilde{a}}\left(u^2(t)+v^2(t)\right)=0, & t\in(0,1),\\ D_{0+}^{8/3}v(t)+\mu t^{\tilde{b}}(e^{u(t)+v(t)}-1)=0, & t\in(0,1), \end{cases}$$

with the coupled multi-point boundary conditions

$$(BC_0)\qquad \begin{cases} u(0)=u'(0)=u''(0)=u'''(0)=0,\ D_{0+}^{4/3}u(t)|_{t=1}=2D_{0+}^{1/2}v(t)|_{t=1/2},\\ v(0)=v'(0)=0,\ v'(1)=D_{0+}^{2/3}u(t)|_{t=1/3}+\frac{1}{2}D_{0+}^{2/3}u(t)|_{t=2/3}. \end{cases}$$

Here we have $f(t,u,v)=(t+1)^{\tilde{a}}(u^2+v^2)$, $g(t,u,v)=t^{\tilde{b}}(e^{u+v}-1)$ for $t\in[0,1]$, $u,v\ge 0$, where $\tilde{a},\tilde{b}>0$. Then we obtain $\Delta=\frac{5\Gamma(9/2)}{3\Gamma(19/6)}-\frac{\Gamma(9/2)\Gamma(8/3)(1+2^{11/6})}{2^{1/6}3^{17/6}\Gamma(13/6)\Gamma(23/6)}\approx 7.6683666$, and so the assumptions (H1) and (H2) are satisfied. In addition, we deduce (see [1])

$$g_1(t,s)=\frac{1}{\Gamma(9/2)}\begin{cases} t^{7/2}(1-s)^{13/6}-(t-s)^{7/2}, & 0\le s\le t\le 1,\\ t^{7/2}(1-s)^{13/6}, & 0\le t\le s\le 1, \end{cases}$$

$$g_2(t,s)=\frac{1}{\Gamma(23/6)}\begin{cases} t^{17/6}(1-s)^{13/6}-(t-s)^{17/6}, & 0\le s\le t\le 1,\\ t^{17/6}(1-s)^{13/6}, & 0\le t\le s\le 1, \end{cases}$$

$$g_3(t,s)=\frac{1}{\Gamma(13/6)}\begin{cases} t^{7/6}(1-s)^{2/3}-(t-s)^{7/6}, & 0\le s\le t\le 1,\\ t^{7/6}(1-s)^{2/3}, & 0\le t\le s\le 1, \end{cases}$$

$$g_4(t,s)=\frac{1}{\Gamma(8/3)}\begin{cases} t^{5/3}(1-s)^{2/3}-(t-s)^{5/3}, & 0\le s\le t\le 1,\\ t^{5/3}(1-s)^{2/3}, & 0\le t\le s\le 1, \end{cases}$$

$h_1(s)=\frac{1}{\Gamma(9/2)}(1-s)^{13/6}\left(1-(1-s)^{4/3}\right)$, $h_2(s)=\frac{1}{\Gamma(23/6)}(1-s)^{13/6}\left(1-(1-s)^{2/3}\right)$, $h_3(s)=\frac{1}{\Gamma(13/6)}(1-s)^{2/3}\left(1-(1-s)^{1/2}\right)$, $h_4(s)=\frac{1}{\Gamma(8/3)}s(1-s)^{2/3}$ for all $s\in[0,1]$. For the functions J_i, $i=1,\dots,4$, we obtain

$$J_1(s)=\begin{cases} \frac{1}{\Gamma(9/2)}(1-s)^{13/6}\left(1-(1-s)^{4/3}\right)+\frac{\Gamma(8/3)}{2^{7/6}3^{17/6}\Delta\Gamma(13/6)\Gamma(23/6)}[2(1-s)^{13/6}\\ \quad -2(1-3s)^{17/6}+2^{17/6}(1-s)^{13/6}-(2-3s)^{17/6}], \quad 0\le s<1/3,\\ \frac{1}{\Gamma(9/2)}(1-s)^{13/6}\left(1-(1-s)^{4/3}\right)+\frac{\Gamma(8/3)}{2^{7/6}3^{17/6}\Delta\Gamma(13/6)\Gamma(23/6)}[2(1-s)^{13/6}\\ \quad +2^{17/6}(1-s)^{13/6}-(2-3s)^{17/6}], \quad 1/3\le s<2/3,\\ \frac{1}{\Gamma(9/2)}(1-s)^{13/6}\left(1-(1-s)^{4/3}\right)+\frac{\Gamma(8/3)}{2^{1/6}3^{17/6}\Delta\Gamma(13/6)\Gamma(23/6)}[(1-s)^{13/6}\\ \quad +2^{11/6}(1-s)^{13/6}], \quad \frac{2}{3}\le s\le 1, \end{cases}$$

$$J_2(s) = \frac{5}{3 \cdot 2^{1/6}\Delta\Gamma(13/6)} \begin{cases} (1-s)^{2/3} - (1-2s)^{7/6}, & 0 \le s < 1/2, \\ (1-s)^{2/3}, & \frac{1}{2} \le s \le 1, \end{cases}$$

$$J_3(s) = \begin{cases} \frac{1}{\Gamma(8/3)} s(1-s)^{2/3} + \frac{(1+2^{11/6})\Gamma(9/2)}{2^{1/6}3^{17/6}\Delta\Gamma(13/6)\Gamma(23/6)}[(1-s)^{2/3} \\ \quad -(1-2s)^{7/6}], \quad 0 \le s < 1/2, \\ \frac{1}{\Gamma(8/3)} s(1-s)^{2/3} + \frac{(1+2^{11/6})\Gamma(9/2)}{2^{1/6}3^{17/6}\Delta\Gamma(13/6)\Gamma(23/6)}(1-s)^{2/3}, \quad 1/2 \le s \le 1, \end{cases}$$

$$J_4(s) = \frac{\Gamma(9/2)}{2 \cdot 3^{17/6}\Delta\Gamma(19/6)\Gamma(23/6)} \begin{cases} 2(1-s)^{13/6} - 2(1-3s)^{17/6} + 2^{17/6}(1-s)^{13/6} \\ \quad -(2-3s)^{17/6}, \quad 0 \le s < 1/3, \\ 2(1-s)^{13/6} + 2^{17/6}(1-s)^{13/6} \\ \quad -(2-3s)^{17/6}, \quad 1/3 \le s < 2/3 \\ 2(1-s)^{13/6} + 2^{17/6}(1-s)^{13/6}, \quad 2/3 \le s \le 1. \end{cases}$$

For $c_1 = 1/2$, $c_2 = 1$, we deduce $\gamma_1 = (1/2)^{7/2}$, $\gamma_2 = (1/2)^{5/3}$, $\gamma = \gamma_1$. In addition, we obtain $f_0^s = 0$, $g_0^s = 1$, $f_\infty^i = g_\infty^i = \infty$, $B = \int_0^1 J_2(s)ds \approx 0.06605546$, $C = \int_0^1 J_3(s)ds \approx 0.16869513$, $\tilde{B} = \int_{1/2}^1 J_2(s)ds \approx 0.03381002$, $\tilde{C} = \int_{1/2}^1 J_3(s)ds \approx 0.09615861$. Besides, for $b = 1/2$, we get $L_4' = \frac{1}{2C} \approx 2.96392$. Then, by Theorem 3.1, 6) we conclude that for each $\lambda \in (0,\infty)$ and $\mu \in (0, L_4')$, there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for problem $(S_0) - (BC_0)$. Because $g_0^i = 2^{-\tilde{b}}$ and $g_\infty^i = \infty$, we can also apply Theorem 3.4, b). Then there exists $\tilde{\mu}_0$ such that for every $\mu > \tilde{\mu}_0$ and $\lambda > 0$, problem $(S_0) - (BC_0)$ has no positive solution. For example, if $\tilde{b} = 1$, then we deduce $m_2 = \frac{1}{2}$ and $\tilde{\mu}_0 = \frac{1}{\gamma\gamma_2 m_2 \tilde{C}} \approx 747.074$.

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