Hochschild Homology Groups of System Quiver Algebras of Maximal Tame Type

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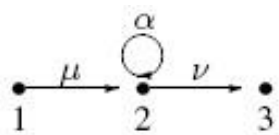
Manuscript submitted April 7, 2017; accepted September 12, 2017.
doi: 10.17706/ijapm.2017.7.4.259-267

Abstract: Let $\Lambda_i = kQ/I_i$ be system quiver algebras of maximal tame representation types. In this paper, based on the minimal projective bimodule resolutions, we calculate explicitly the dimensions of all Hochschild homology groups of the system quiver of maximal tame representation types.

Key words: System quiver algebra, aximal tame representation type, minimal projective bimodule resolution, hochschild cohomology group.

1. Introduction

The differential equation $\begin{cases} \dfrac{d(x(t))}{dt} = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$, Provides the linear time invariant dynamical system $\sum = (G;F;H)$, where $x(t) \in k^n, u(t) \in k^m, y(t) \in k^p$ are vector variables, and $G;F;H$ are $n \times m; n \times n; p \times n$ matrix respectively. Hazewinkel first obtained the one to one correspondence of every linear time invariant dynamical system $\sum = (G;F;H)$ and the dimensional vector $(m;n;p)$ of the following system quiver Q in [1].



The equivalence classes of a dynamical system are one-to-one correspondence with the isomorphism classes of the finite dimensional representations of the system quiver. Elements of the repreesentation theory had been explained in [2]. Longcai Li et al. classfied all the system quiver algebras according their representation type,and found the list of system quiver algebras of maximal tame representation type [3]. System quiver algebra of minimal wild type had been discussed in [4].

Let Λ is a system quiver algebra. In the isomorphism and dual sense, Λ is tame type if and only if Λ is (or degenerate) kQ/I, where I is correspond to T. The algebra is called the system quiver algebra of maximal tame type.

T

(1) $I_1 = (\alpha^2)$;

(2) $I_2 = (\alpha^3, \mu\upsilon, \mu\alpha\upsilon)$;

(3) $I_3 = (\mu\upsilon, \alpha^2\upsilon, \mu\alpha^2, \alpha^3)$;

(4) $I_4 = (\mu\alpha, \alpha^2\upsilon, \alpha^4)$;

(5) $I_5 = (\alpha\upsilon, \mu\alpha, \alpha^m), (m \geq 5)$.

Hochschild homology group was introduced by Hochschild in 1945 in [5]. Let Λ be the finite dimensional combination basic algebra (Including identical element 1) in number field K. Its enveloping algebra was defined as $\Lambda^e = \Lambda^{op} \otimes_k \Lambda$, Λ^{op} be the reflexive algebra of Λ. M be the finite dimensional $\Lambda - \Lambda -$ dual-mode in number field K, then coefficients in the M of algebraic Λ,which Homology group at the n-th order is defined as $HH_n(\Lambda, M) = Tor_n^{\Lambda^e}(M, \Lambda)$.

Especially, when $M = \Lambda$, then say $HH_n(\Lambda) = HH_n(\Lambda, \Lambda)$, known as the n-th order algebraic Hochschild homology group.

The arrow diagram and its development and application , provided a effective method for calculating the Hochschild homology of algebra group.

Bardzell gave the explicit description of construction of minimal projective bimodule resolutions of monomial algebras with directed paths. Using Bardzell's method, the minimal projective bimodule resolutions of the system quiver algebras of maximal tame representation type is constructed in [6]. In this thesis, based on the resolutions, we calculate explicitly the dimensions of all Hochschild homology groups of the system quiver algebras of maximal tame representation type by means of combinatorics. Thus we get a further understanding of the homology property of those algebras.

2. Minimal Projective Bimodule Resolution

In the section, we first introduce the minimal projective bimodule resolutions of the system quiver algebras of maximal tame representation type constructed by Dandan Zhang in [7].

By the correspondence of the list in introduction, it can obtain the following ideals corresponding to the system quiver algebras of maximal tame type $\Lambda_i = kQ/I_i$: $I_1 = (\alpha^2)$, $I_2 = (\alpha^3, \mu\upsilon, \mu\alpha\upsilon)$, $I_3 = (\mu\upsilon, \alpha^2\upsilon, \mu\alpha^2, \alpha^3)$, $I_4 = (\mu\alpha, \alpha^2\upsilon, \alpha^4)$, $I_5 = (\alpha\upsilon, \mu\alpha, \alpha^m)$, $(m \geq 5)$.

Let B_i be a series of $k-$ bases of Λ_i ,and e_1, e_2, e_3 be the primitive idempotent element corresponding to the vertexes 1, 2, 3 respectively, we have

$$B_1 = \{e_1, e_2, e_3, \mu, \alpha, \upsilon, \mu\upsilon, \alpha\upsilon, \mu\alpha, \mu\alpha\upsilon\}, \dim_k \Lambda_1 = 10;$$

$$B_2 = \{e_1, e_2, e_3, \mu, \alpha, \upsilon, \mu\alpha, \alpha\upsilon, \alpha^2, \mu\alpha^2, \alpha^2\upsilon, \mu\alpha^2\upsilon\}, \dim_k \Lambda_{12} = 12;$$

$$B_3 = \{e_1, e_2, e_3, \mu, \alpha, \upsilon, \mu\alpha, \alpha\upsilon, \alpha^2, \mu\alpha\upsilon\}, \dim_k \Lambda_3 = 10;$$

$$B_4 = \{e_1, e_2, e_3, \mu, \alpha, \upsilon, \mu\upsilon, \alpha\upsilon, \alpha^2, \alpha^3\}, \dim_k \Lambda_4 = 10;$$

$$B_5 = \{e_1, e_2, e_3, \mu, \alpha, \upsilon, \mu\upsilon, \alpha^2, \alpha^3, \cdots, \alpha^{m-1}\}, \dim_k \Lambda_5 = m+5$$

And then, we construct the minimal projective bimodule resolutions of Λ_i. Let $\Lambda_i^e = \Lambda_i^{op} \otimes_k \Lambda_i$ be the enveloping algebra of Λ_i, writed for $\otimes := \otimes_k$. $P_n^i := \coprod_{p \in AP_i(n)} \Lambda_i o(p) \otimes t(p) \Lambda_i$, where the constructions of $AP_i(n)$ are as follows: $AP_i(0) = \{e_1, e_2, e_3\}$, $AP_i(1) = \{\mu, \alpha, \upsilon\}$.

For Λ_1, $AP_1(n) = \{\alpha^n\}$, When $n \geq 2$.

For Λ_2, $AP_2(2) = \{\alpha^3, \mu\upsilon, \mu\alpha\upsilon\}$. When $n \geq 3$,

$$AP_2(n) \begin{cases} \left\{\alpha^{\frac{3n-1}{2}}\right\} & \text{when n is an odd number;} \\ \left\{\alpha^{\frac{3n}{2}}\right\} & \text{when n is an even number.} \end{cases}$$

For Λ_3, $AP_3(2) = \{\mu\upsilon, \alpha^2\upsilon, \mu\alpha^2, \alpha^3\}$. When $n \geq 3$,

$$AP_3(n) = \begin{cases} \left\{\mu\alpha^{\frac{3n-5}{2}}\upsilon, \mu\alpha^{\frac{3n-3}{2}}, \alpha^{\frac{3n-3}{2}}\upsilon, \alpha^{\frac{3n-1}{2}}\right\} & \text{when n is an odd number;} \\ \left\{\mu\alpha^{\frac{3n-4}{2}}\upsilon, \mu\alpha^{\frac{3n-2}{2}}, \alpha^{\frac{3n-3}{2}}\upsilon, \alpha^{\frac{3n}{2}}\right\} & \text{when n is an even number.} \end{cases}$$

For Λ_4, $AP_4(2) = \{\mu\alpha, \alpha^2\upsilon,, \alpha^4\}$. When $n \geq 3$

,

$$AP_4(n) = \begin{cases} \{\mu\alpha^{2n-2}, \mu\alpha^{2n-4}\upsilon, \alpha^{2n-1}, \alpha^{2n-2}\upsilon\} & \text{when n is an odd number;} \\ \{\mu\alpha^{2n-3}, \mu\alpha^{2n-4}\upsilon, \alpha^{2n}, \alpha^{2n-2}\upsilon\} & \text{when n is an even number.} \end{cases}$$

For Λ_5, $AP_5(2) = \{\mu\alpha, \alpha\upsilon, \alpha^m\}$, $(m \geq 5)$. When $n \geq 3$,

$$AP_5(n) = \begin{cases} \left\{\mu\alpha^{\frac{(n-1)m}{2}}, \mu\alpha^{\frac{(n-3)m}{2}+1}\upsilon, \alpha^{\frac{(n-1)m}{2}+1}, \alpha^{\frac{(n-1)m}{2}}\upsilon\right\} & \text{when n is an odd number;} \\ \left\{\mu\alpha^{\frac{(n-2)m}{2}+1}, \mu\alpha^{\frac{(n-2)m}{2}}\upsilon, \alpha^{\frac{mn}{2}}, \alpha^{\frac{(n-1)m}{2}+1}\upsilon\right\} & \text{when n is an even number.} \end{cases}$$

For all $p^n \in AP_i(n)$, define $\sup(p^n) = \{p^{n-1} \in AP_i(n-1) : p^{n-1} \text{is a subpath of } p^n\}$

Let n be an odd number, $p^n \in AP_i(n)$, then $\sup(p^n) = \{p_1^{n-1},\ p_2^{n-1}\}$, where $p_l^{n-1} \in AP_i(n-1)$, $(l = 1,2)$, and there exists only one subpath R_2^1 and L_1^2, such that p^n exists the following decomposition: $p^n = p_1^{n-1}R_2^1 = L_1^2 p_2^{n-1}$,and then $p^{n+1} \in AP_i(n+1)$, $\sup(p^{n+1}) = \{p_1^n,\ p_2^n, \cdots, p_s^n\}$, $(i = 1, \cdots, s)$,where $p^{n+1} \in \theta_i p_i^n \mu_i$, θ_i and μ_i are the supplemental subpaths of p_i^n with respect to p^{n+1}.

Define $\delta_n^i : P_n^i \to P_{n-1}^i$ the following form:

$$\delta_n^i(o(p^n) \otimes t(p^n)) = L_1^2 \otimes t(p^n) - o(p^n) \otimes R_2^1$$

$$\delta_{n+1}^i(o(p^{n+1}) \otimes t(p^{n+1})) = \sum_{i=1}^{s} \theta_i \otimes \mu_i$$

And then it follows the following theorem according to [6].

Theorem 1 For the above the system quiver algebras Λ_i of maximal tame representation type, the complex

$$(P^i, \delta^i) \cdots \to P_{n+1}^i \xrightarrow{\delta_{n+1}^i} P_n^i \xrightarrow{\delta_n^i} \cdots \xrightarrow{\delta_3^i} P_2^i \xrightarrow{\delta_2^i} P_1^i \xrightarrow{\delta_1^i} P_0^i \to 0$$

is the minimal projective $\Lambda_i^e -$ resolution of Λ_i.

3. Hochschild Homology Group

Let X, Y is a set that consists of all consistent elements of kQ, and $X \odot Y = \{(x, y) \in X \times Y \mid t(x) = o(y),\quad t(y) = o(x)\}$.Then $k(X \odot Y)$ is a linear space whose base is $X \odot Y$. In the section, we calculate the dimensions of all Hochschild homology groups of the system quiver algebras of maximal tame representation type by means of combinatorics.

Acting on the minimal projection $\Lambda_i^e -$ resolution (P^i, δ^i) of Λ_i by the functor $- \otimes_{\Lambda_i^e} \Lambda_i$, it follows

$$\cdots \to P_n^i \otimes_{\Lambda_i^e} \Lambda_i \xrightarrow{\delta_n^i \otimes 1} P_{n+1}^i \otimes_{\Lambda_i^e} \Lambda_i \to \cdots P_1^i \otimes_{\Lambda e_i} \Lambda_i \xrightarrow{\delta_1^i \otimes 1} P_0^i \otimes_{\Lambda_i^e} \Lambda_i \to 0$$

Note that

$$P_n^i \otimes_{\Lambda_i^e} \Lambda_i \cong \coprod_{p \in AP_i(n)} (o(p) \otimes t(p)) \otimes_{kQ_0} \Lambda^i \cong \coprod_{p \in AP_i(n)} (o(p) \otimes t(p)) \otimes_{kQ_0} (\oplus_{k,l=1}^3 e_k \Lambda^i e_l)$$
$$\cong k(B_i \otimes AP_i(n)).$$

It follows Bardzell chain complex

$$\cdots \to k(B_i \odot AP_i(n)) \xrightarrow{\tau_n^i} k(B_i \odot AP_i(n-1)) \cdots$$

$$\to k(B_i \odot AP_i(1)) \xrightarrow{\tau_1^i} k(B_i \odot AP_i(0) \to 0$$

where, for all $(c, p^n) \in (B_i \odot AP_i(n))$, if n is an odd number, then

$$\tau_n^i(c, p^n) = (c\lambda, p_1^{n-1}) - (\theta c, p_2^{n-1}) \tag{1}$$

where $p^n = \lambda p_1^{n-1} = p_2^{n-1}\theta$, $p_1^{n-1}, p_2^{n-1} \in AP_i(n-1)$ aresubpaths of pn. And if is an even number, then

$$\tau_n^i(c, p^n) = \sum_k (\theta_k c\lambda_k, p^{n-1}) \tag{2}$$

where p_k^{n-1} is any one of $\sup(p^n)$ such that $p^n = \lambda_k p_k^{n-1}\theta_k$.

Theorem 2 Let $\Lambda_1 = kQ/I_1 = kQ/(\alpha^2)$, then

$$|B_1 \odot \mathrm{AP}_1(n)| = \begin{cases} 4 & when \quad n = 0; \\ 2 & when \quad n \geq 1. \end{cases}$$

and

$$\dim_k HH_n(\Lambda_1) = \begin{cases} 4 & when \quad n = 0; \\ 1 & when \quad n \geq 1. \end{cases}$$

Let $\Lambda_2 = kQ/I_2 = kQ/(\alpha^3, \mu\upsilon, \mu\alpha\upsilon)$, then

$$|B_2 \odot \mathrm{AP}_2(n)| = \begin{cases} 5 & when \quad n = 0; \\ 3 & when \quad n \geq 1. \end{cases}$$

and

$$\dim_k HH_n(\Lambda_2) = \begin{cases} 5 & when \quad n = 0; \\ 2 & when \quad n \geq 1. \end{cases}$$

Let $\Lambda_3 = kQ/I_3 = kQ/(\mu\upsilon, \alpha^2\upsilon, \mu\alpha^2, \alpha^3)$, then

$$|B_3 \odot \mathrm{AP}_3(n)| = \begin{cases} 5 & when \quad n = 0; \\ 3 & when \quad n \geq 1. \end{cases}$$

and

$$\dim_k HH_n(\Lambda_3) = \begin{cases} 5 & when \quad n=0; \\ 2 & when \quad n \geq 1. \end{cases}$$

Let $\Lambda_4 = kQ/I_4 = kQ/(\mu\upsilon, \alpha^2\upsilon, \alpha^4)$, then

$$|B_4 \odot AP_4(n)| = \begin{cases} 6 & when \quad n=0; \\ 4 & when \quad n \geq 1. \end{cases}$$

and

$$\dim_k HH_n(\Lambda_4) = \begin{cases} 6 & when \quad n=0; \\ 3 & when \quad n \geq 1. \end{cases}$$

Let $\Lambda_5 = kQ/I_5 = kQ/(\alpha\upsilon, \mu\alpha, \alpha^m), (m \geq 5)$, then

$$|B_5 \odot AP_5(n)| = \begin{cases} m+2 & when \quad n=0; \\ m & when \quad n \geq 1. \end{cases}$$

$$\dim_k HH_n(\Lambda_5) = \begin{cases} m+2 & when \quad n=0; \\ m-1 & when \quad n \geq 1. \end{cases}$$

Proof: Here, the conclusion of Λ_3 is only proved and the others can similarly be proved. From the second section, it follows

$B_3 = \{e_1, e_2, e_3, \mu, \alpha, \upsilon, \mu\alpha, \alpha\upsilon, \alpha^2, \mu\alpha\upsilon\}$, $\dim_k \Lambda_3 = 10$, $AP_3(0) = \{e_1, e_2, e_3\}$, $AP_3(1) = \{\mu, \alpha, \upsilon\}$.

For Λ_3, $AP_3(2) = \{\mu\upsilon, \alpha^2\upsilon, \mu\alpha^2, \alpha^3\}$, when $n \geq 3$, and n is an odd number,

$$AP_3(n) = \left\{ \mu\alpha^{\frac{3n-5}{2}}\upsilon, \mu\alpha^{\frac{3n-3}{2}}, \alpha^{\frac{3n-3}{2}}\upsilon, \alpha^{\frac{3n-1}{2}} \right\}$$

when $n \geq 3$, and *n* is an even number,

$$AP_3(n) = \left\{ \mu\alpha^{\frac{3n-4}{2}}\upsilon, \mu\alpha^{\frac{3n-2}{2}}, \alpha^{\frac{3n-2}{2}}\upsilon, \alpha^{\frac{3n}{2}} \right\}$$

so

$$B_3 \odot AP_3(0) = \{(e_1, e_1), (e_2, e_2), (\alpha, e_2), (\alpha^2, e_2)(e_3, e_3)\},$$

$$B_3 \odot AP_3(1) = \{(e_2,\alpha),(\alpha,\alpha),(\alpha^2,\alpha)\},$$

$$B_3 \odot AP_3(2) = \{(e_2,\alpha^3),(\alpha,\alpha^3),(\alpha^2,\alpha^3)\},$$

when $n \geq 3$, and n is an odd number,

$$B_3 \odot AP_3(n) = \left\{(e_2,\alpha^{\frac{3n-1}{2}}),(\alpha,\alpha^{\frac{3n-1}{2}}),(\alpha^2,\alpha^{\frac{3n-1}{2}})\right\},$$

when $n \geq 3$, and n is an even number,

$$B_3 \odot AP_3(n) = \left\{(e_2,\alpha^{\frac{3n}{2}}),(\alpha,\alpha^{\frac{3n}{2}}),(\alpha^2,\alpha^{\frac{3n}{2}})\right\}.$$

so

$$\left|B_3 \odot AP_3(n)\right| = \begin{cases} 5 & when \quad n = 0; \\ 3 & when \quad n \geq 1. \end{cases}$$

For proving the next conclusion, we can sequence the base of $B_3 \odot AP_3(n)$ in accordance with the order in the previous proving, and still denote its corresponding matrix by τ_n^3.

From (1) and (2), it follows:

$$\tau_1^3(e_2,\alpha) = (e_2\alpha,e_2) - (\alpha e_2,e_2) = 0,$$

$$\tau_1^3(\alpha,\alpha) = (\alpha^2,e_2) - (\alpha^2,e_2) = 0,$$

$$\tau_1^3(\alpha^2,\alpha) = (\alpha^3,e_2) - (\alpha^3,e_2) = 0,$$

therefore $\tau_1^3 = 0$, and then $rank\tau_1^3 = 0$.

When $n = 2$, for the same reason, it follows:

$$\tau_2^3(e_2,\alpha^3) = (\alpha^2,\alpha) + (\alpha^2,\alpha) + (\alpha^2,\alpha) = 3(\alpha^2,\alpha),$$

$$\tau_2^3(\alpha,\alpha^3) = (\alpha^3,\alpha) + (\alpha^3,\alpha) + (\alpha^3,\alpha) = 3(\alpha^3,\alpha) = 0,$$

$$\tau_2^3(\alpha^2,\alpha^3) = (\alpha^4,\alpha) + (\alpha^4,\alpha) + (\alpha^4,\alpha) = 3(\alpha^4,\alpha) = 0,$$

therefore the matrix τ_2^3 is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix},$$

then $rank\tau_2^3 = 1$.

When $n \geq 3$,and n is an odd number,

$$\tau_n^3(e_2, \alpha^{\frac{3n-1}{2}}) = (e_2\alpha, \alpha^{\frac{3(n-1)}{2}}) - (\alpha e_2, \alpha^{\frac{3(n-1)}{2}}) = 0,$$

$$\tau_n^3(\alpha, \alpha^{\frac{3n-1}{2}}) = (\alpha^2, \alpha^{\frac{3(n-1)}{2}}) - (\alpha^2, \alpha^{\frac{3(n-1)}{2}}) = 0,$$

$$\tau_n^3(\alpha^2, \alpha^{\frac{3n-1}{2}}) = (\alpha^3, \alpha^{\frac{3(n-1)}{2}}) - (\alpha^3, \alpha^{\frac{3(n-1)}{2}}) = 0.$$

Therefore the matrix $\tau_n^3 = 0$, then $rank\tau_n^3 = 0$.

When $n \geq 4$, and n is an odd number,

$$\tau_n^3(e_2, \alpha^{\frac{3n}{2}}) = (\alpha^2, \alpha^{\frac{3n-4}{2}}) + (\alpha^2, \alpha^{\frac{3n-4}{2}}) + (\alpha^2, \alpha^{\frac{3n-4}{2}}) = 3(\alpha^2, \alpha^{\frac{3n-4}{2}}),$$

$$\tau_n^3(\alpha, \alpha^{\frac{3n}{2}}) = (\alpha^3, \alpha^{\frac{3n-4}{2}}) + (\alpha^3, \alpha^{\frac{3n-4}{2}}) + (\alpha^3, \alpha^{\frac{3n-4}{2}}) = 3(\alpha^3, \alpha^{\frac{3n-4}{2}}) = 0,$$

$$\tau_n^3(\alpha, \alpha^{\frac{3n}{2}}) = (\alpha^4, \alpha^{\frac{3n-4}{2}}) + (\alpha^4, \alpha^{\frac{3n-4}{2}}) + (\alpha^4, \alpha^{\frac{3n-4}{2}}) = 3(\alpha^4, \alpha^{\frac{3n-4}{2}}) = 0.$$

Therefore the matrix τ_n^3 is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix},$$

then $rank\tau_n^3 = 1$.

In conclusion, $rank\tau_n^3 = \begin{cases} 0, & \text{n is an odd number;} \\ 1, & \text{n is an even number.} \end{cases}$

And because

$$\dim_k HH_n(\Lambda_3) = \dim\ker\tau_n^3 - \dim\operatorname{Im}\tau_{n+1}^3 = |B_3\Theta AP_3(n)| - rank\tau_n^3 - rank\tau_{n+1}^3,$$

Thus

$$\dim_k HH_n(\Lambda_3) = \begin{cases} 5 & when \quad n = 0; \\ 2 & when \quad n \geq 1. \end{cases}$$

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