On π-nilpotency of Finite Groups

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Abstract: A group G is called л-nilpotent, л a set of primes, if G has a normal л'-subgroup N with G/N a nilpotent л-group. Let H be a nilpotent л-Hall subgroup of G, $1<Z_1(H)<Z_2(H) <\cdots<Z_n(H)=H$ be the upper central series of H. If every Z_i(H) is weakly closed in H (about G). Then we say that the upper central series of H is weakly closed in H (about G). Let H be a subgroup of a finite group G. We call H weakly s-normal in G if there exists a Sylow p-subgroup S_p which is permutable with H for every prime $p \mid |G|$. In this paper, with the conception above, several determine theorems for G to be a л—nilpotent group are given and some properties about л—nilpotent groups are considered. Several results about nilpotent groups are generalized.

Key words: л—nilpotent groups, minimal normal subgroups, л-normal groups, weakly s-normal subgroups.

1. Introduction

All groups considered in this paper will be finite. We use conventional notions and notations, as in [1]. Let л be any set of primes and л′ the complementary set of primes. We denote M ⋖ G to indicate that M is a maximal subgroup of G. Also, $|G:M|_\pi$ denotes the л-part of |G:M|.

Definition 1.1 A group G is called л-nilpotent, л a set of primes, if G has a normal л'-subgroup N with G/N a nilpotent л-group.

It is very easy to prove that every subgroup and every image of a л-nilpotent group are likewise л-nilpotent.

Definition 1.2 Let H be a subgroup of a finite group G. We call H weakly s-normal in G if there exists a Sylow p-subgroup S_p which is permutable with H for every prime p | |G|.

Definition 1.3 Let G be a finite group, H be a л-Hall subgroup of G. We call G л-normal if $Z(H)^g \leq H \Rightarrow Z(H)^g = Z(H)$, for every g∈G.

Definition 1.4 Let H be a nilpotent л-Hall subgroup of G, $1<Z_1(H)<Z_2(H)<\cdots<Z_n(H)=H$ be the upper central series of H. If every Z_i(H) is weakly closed in H (about G). Then we say that the upper central series of H is weakly closed in H (about G).

Definition 1.5 $\Phi_\pi(G)=\cap\{M|M ⋖ G \text{ with } [G: M]^\pi=1\}$.

2. Preliminarie

We will give some lemmas that are useful to the proofs of the theorems.

Lemma 2.1 Let H be a nilpotent л-Hall subgroup of G, N $\trianglelefteq$ G. Then $N_{G/N}$ (HN/N)= N_G (H)N/N.

Proof (1) $N_{G/N}$ (HN/N)= N_G (HN)/N

Since HN $\trianglelefteq$ N_G (HN). Hence HN/N $\trianglelefteq$ N_G (HN)/N. Therefore (HN)/N $\leq$ $N_{G/N}$ (HN/N). Let $N_{G/N}$ (HN/N)=M/N. Then HN/N $\trianglelefteq$ $N_{G/N}$ (HN/N)=M/N. Hence HN $\trianglelefteq$ M. It implies that M $\leq$ N_G (HN). Hence M/N $\leq$ N_G (HN)/N. Therefore $N_{G/N}$ (HN/N)= N_G (HN)/N.

(2) N_G (HN)= N_G (H)N

Since N $\trianglelefteq$ G. Hence N_G (H) $\leq$ N_G (HN). Again N $\leq$ N_G (HN). Therefore N_G (H)N $\leq$ N_G (HN). Conversely, picking arbitrarily an element x in N_G (HN). Then H^xN $\leq$ HN. It implies that $H^x \leq$ HN. Now both H^x and H are л-Hall subgroups of HN. Again, H is a nilpotent л-Hall subgroup of HN. By [2, Theorem 9.1.10], there exist an element hn in HN, where h $\in$ H and n $\in$ N, such that $H^x = H^{hn} = H^n$, It implies that H= $H^{xn^{-1}}$. Hence x$n^{-1} \in N_G$ (H). It implies x $\in N_G$ (H)N. Therefore N_G (HN) $\leq$ N_G (H)N. Since N_G (H)N $\leq$ N_G (HN). Hence N_G (HN)= N_G (H)N.

By (1) and (2), we have that $N_{G/N}$ (HN/N) = N_G (H)N/N.

Lemma 2.2 Let G be a finite group. Then G is л—nilpotent if and only if G/Z(G) is л-nilpotent.

Proof By introduction, we need only prove the "if" part. Let G/Z(G) be л-nilpotent. It implies that G is л-solvable. Hence there are π'-Hall subgroups in G. Let N be a π'-Hall subgroup of G. Then NZ(G)/Z(G) is a π'-Hall subgroup of G/Z(G). Hence NZ(G)/Z(G) $\trianglelefteq$ G/Z(G). It yields that NZ(G) $\trianglelefteq$ G. Obviously N $\trianglelefteq$ NZ(G). Since N is a π'-Hall subgroup of NZ(G). Hence N char NZ(G) $\trianglelefteq$ G. It yields that N $\trianglelefteq$ G. Thus G has normal л-complements. Let H be a л-Hall subgroup of G. Then HZ(G)/Z(G) is a л-Hall subgroup of G/Z(G). By assumption, HZ(G)/Z(G) $\cong$ H/H $\cap$ Z(G) is nilpotent. Again H $\cap$ Z(G) $\leq$ Z(H). Hence H/Z(H) is nilpotent. By [2], we get that H is nilpotent. Therefore G is л—nilpotent.

Lemma 2.3 Let H be a nilpotent л-Hall subgroup of G. Then G is л—nilpotent if and only if G is л-normal and N_G (Z(H)) is л—nilpotent.

Proof By [3, Th 3], we need only prove the "if"part. Let H_1 be a subgroup of H with Z(H)$\leq H_1$.We consider $N_G(H_1)$. $\forall$ x $\in N_G(H_1)$. Since $[Z(H)]^x \leq H_1 \leq H$ and G is л-normal. We have that $[Z(H)]^x = Z(H)$. Hence Z(H) $\trianglelefteq$ $N_G(H_1)$ It implies that $N_G(H_1) \leq N_G$ (Z(H)). Since N_G (Z(H)) is л—nilpotent. Hence $N_G(H_1)$ is also л—nilpotent. By [4, Th1], we get that $N_G(H_1)/C_G(H_1)$ is a л-group. Again, By [4, Th2], we have that G is л—nilpotent.

Lemma 2.4 Let N be a π'-nilpotent normal Hall-subgroup of G and let N $\cap$ Φ_π (G) be a nilpotent subgroup of G. Then Φ_π (N)=N $\cap$ Φ_π (G).

Proof The same argument as that of corresponding theorem in [5].

Lemma 2.5 Let G be a soluble group and let M $\triangleleft$ N $\triangleleft$ G. Suppose that N is a π'-nilpotent Hall-subgroup of G and N $\cap$ Φ_π (G) is a л-nilpotent group. We have that if N/M(N $\cap$ Φ_π (G)) is π_1-closed, then N/M is π_1-closed, where π_1 is a set of some primes with л $\subseteq \pi_1$.

Proof Let L= M(N∩ Φ_π (G)) and let H/L be Hall π_1-subgroup of N/L. Since N/L is π_1-closed. We have that H/L ⊲ N/L. Since N∩ Φ_π (G) is a nilpotent. We have that L/M≌ $\frac{N\cap\Phi_\pi(G)}{M\cap\Phi_\pi(G)}$ is a nilpotent group. Hence there exists normal Hall π_1'-subgroup K/M in L/M. It follows that $L/M\big/K/M$ ≌K/L is a π_1-group. Since K/M char L/M ⊲ H/M. We have that K/M ⊲ H/M. Since [H/M : K/M]=[H : K]=[H : L][L : K]. Again [H : L] and [L : K]. are π_1-numbers. Therefore [H/M : K/M] is a π_1-number. Since K/M is a π_1'-group. It follows that K/M is a Hall π_1'-subgroup of H/M. By Schur throrem, we get that H/M has π_1'-complement A/M. That is H/M=(K/M)(A/M), with K∩A=M. By the generalized Frattin argument, we have that N/M=($N_{N/M}$ (A/M))(H/M)=(N ^{N} (A)H)/M. It follows that N=N ^{N} (A)H= N ^{N} (A)AK= N ^{N} (A)K= N ^{N} (A)L= N ^{N} (A)M(N∩ Φ_π (G)). By lemma 2.4, we have that N∩ Φ_π (G)= Φ_π (N). Again M≤A. Therefore we have that N= N ^{N} (A)M(N∩ Φ_π (G))= N ^{N} (A) Φ_π (N). We can prove that A/M is a Hall π_1-subgroup of N/M. In fact, [N/M : A/M]=[N : A]=[N : H][H : A]=[N/L : H/L][H/M : A/M] is a π'-number. Hence [N : N ^{N} (A)] $^\pi$ =1. By [6, Theorem3.1], we get that N= N ^{N} (A). It implies that A ⊲ N. Hence A/M is a normal Hall π_1-subgroup of N/M. That is to say that N/M is π_1-closed.

3. Main Results

Theorem 3.1 Let |G| |= $p^\alpha q^\beta$, P∈syl ^{p} G. Then G is p-nilpotent if and only if

a) $p^\alpha q^\beta$ is a p-subgroup.

b) Every maximal subgroup of P is weakly s-normal in G..

Proof First we prove the “only if” part. Let G be a p-nilpotent group. Then G has a normal p-complement Q. By [4, theorem1], we have that a) holds. Since Q ⊲ G. We have that P_1Q=QP_1 for every maximal subgroup P_1 of P. That is b) holds.

Next we prove the “if” part. Assume that the hypothesis holds. Then we have

Every Sylow p-subgroup P^* of G satisfies the hypothesis a) and b)

In fact, by Sylow’s theorem, there exists y∈G such that $P^*=P^y$, This yield that $N_G(P^*)=[N_G(P)]^y$, Hence $|N_G(P^*)/C_G(P^*)|=|N_G(P)/C_G(P)|$ is a power of p. Picking arbitrarily a maximal subgroup P_1^* of P^*. Hence we have that $(P_1^*)^{y^{-1}}$ is a maximal subgroup of P. By assumption, there exists Q∈Syl ^{q} (G) such that (P_1^*) P_1^*Q=Q$(P_1^*)^{y^{-1}}$. This yield that $P_1^*Q^y=Q^y P_1^*$.

4. The Final Conclusion

We can easily prove that every quotient group of G satisfies the hypothesis. Assume N is a minimal normal subgroup in G.. By induction on |G|, we can assume that G/N has a normal p-complement H/N. If N is a q-subgroup, then H is a normal p-complement of G. Since G is solvable, next we can assume that N is a p-subgroup . Furthermore we can assume that only p-subgroup can be the minimal normal subgroup in G. If Φ(G)≠1, then G/Φ(G) is a p-nilpotent group by induction on |G|. Therefore G is a p-nilpotent group. Next we assume thatΦ(G)=1 and A is a maximal subgroup of G such that N$\nless$A. It yield that G=AN and A∩N=1. Let P^* be a Sylow p-subgroup of A. This yield that P^*A is a Sylow p-subgroup of G. By (1), with no loss, we can assume that P=P^*A. Picking P_1 is a maximal subgroup of P such that $P^* \angle P_1$. By assumption, there exists Q∈Syl^q (G) such that P_1Q=OP_1. It is easy to show that |N: P_1∩N|=p. Let P_1Q∩N=P_1∩N=D. We have that D$\lhd$G. By the minimal characteristics of N. We have that D=1. This yield that |N|=p. This implies that every minimal normal subgroup of G. is a cyclic group of order p. If q>p. Since N∈syl^p H and N is cyclic. We get that H has a normal p-complement L. It is easy to show that L is also a normal p-complement of G. If q<p. Picking arbitrarily a maximal subgroup M of G. If M$\ngeq$F(G). Since F(G) is a products of all minimal normal subgroup of G. Then there exists a minimal normal subgroup N_1 of G such that MN_1=G and M∩N_1=1. Since G/N_1≅M/M∩N_1. We have that |G:M|=|N_1|=p. By theorem 3.3 in [7], We get that G is supersoluble. It implies that P$\lhd$G. By hypothesis a), we get that Q≤C_G (P). Hence we have that Q$\lhd$G. Therefore G is a p-nilpotent group.

Theorem 3.2 Suppose that G is a л-nilpotent group, H is a nilpotent л-Hall subgroup of G. Suppose further that N is normal in H. Then N is weakly closed in H (about G).

Proof Since G is л-nilpotent and H is a nilpotent л-Hall subgroup of G. So we can assume that G=HM, where M$\trianglelefteq$G and H∩M=1. Suppose that N^g≤H, where g∈G. We prove that $N^g = N$. Since G=HM. So we can assume that g=hm, where h∈H and m∈M. Hence $N^g = N^{hm} = N^m$. ∀ n∈N, we have that $n^m = m^{-1}$nm=n$n^{-1}m^{-1}$nm∈NM. Hence n^m ∈H∩NM=N(M∩H)=N. It implies that N^m≤N. Therefore N^m=N. Thus completes the proof.

Theorem 3.3 Let H be a nilpotent л-Hall subgroup of G. Then G is л-nilpotent if and only if the upper central series of H is weakly closed in H (about G) and N_G (H) is л-nilpotent.

Proof By Theorem 3.2, we need only prove the "if" part. By induction on |G|, we prove that G is л-nilpotent. We consider the following cases.

(1) N_G(Z(H))=G

Since C_G(Z(H))≤N_G(Z(H))=G. We distinguish two cases again.

Case 1 C_G(Z(H))=N_G(Z(H))=G

C_G (Z(H))=G implies that Z(H)≤Z(G). Let $\overline{G}$=G/Z(H) and $\overline{H}$ be a л-Hall subgroup of $\overline{G}$. If Z(H)=H. Then H≤Z(G). By Schur Theorem, G is л-nilpotent. If Z(H)≠H. Then we can prove easily that $Z_i(\overline{H}) = \overline{Z_{i+1}(H)}$. ∀ g ∈ G, if $\overline{g}^{-1}Z_i(\overline{H})\overline{g} \le \overline{H}$, then $\overline{g^{-1}Z_{i+1}(H)g} \le \overline{H}$. It implies that $g^{-1}Z_{i+1}(H)g \le H$. Since $Z_{i+1}(H)$ is weakly closed in H (about G). We have that $g^{-1}Z_{i+1}(H)g = Z_{i+1}(H)$. It follows that $Z_i(\overline{H}) = \overline{Z_{i+1}(H)} = \overline{g^{-1}Z_{i+1}(H)g} = \overline{g}^{-1}Z_i(\overline{H})\overline{g}$. That is that the upper central series of $\overline{H}$ is weakly closed in $\overline{H}$ (about $\overline{G}$). Moreover, by Lemma 2.1, we have that $N_{\overline{G}}(\overline{H}) = \overline{N_G(H)}$. We obtain that $N_{\overline{G}}(\overline{H})$ is л-nilpotent. Now we conclude that $\overline{G}$=G/Z(H) is л-nilpotent by induction on |G|. Since Z(H) ≤Z(G). By lemma 2.2, we obtain that G is л-nilpotent.

Case 2 C_G (Z(H))< N_G (Z(H))= G

Let $G_1=C_G$ (Z(H)). Then $G_1 \trianglelefteq N_G$ (Z(H))= G. Since H≤G_1. Hence the upper central series of H is weakly closed in H (about G_1). Since $N_{G_1}(H)$ is a subgroup of $N_G(H)$ which is л-nilpotent. So $N_{G_1}(H)$ is also л-nilpotent. By induction on |G|, we get that G_1 is л-nilpotent. So we can assume that G_1 =HK, where K$\trianglelefteq G_1$ and K is a normal л-complement of G_1. If K=1. Then H=$G_1 \trianglelefteq$ G. It follows that G=$N_G(H)$. By assumption, G is л-nilpotent. If K ≠1. Then K char $G_1 \trianglelefteq$ G. It follows that K$\trianglelefteq$ G. Since K is a π' -group. Hence $\overline{H}$ ≅H. Therefore $Z_i(\overline{H})=\overline{Z_i(H)}$. ∀ g∈G, if $\overline{g}^{-1}Z_i(\overline{H})\overline{g} \le \overline{H}$, then $\overline{g^{-1}Z_i(H)g} \le \overline{H}$. It implies that $g^{-1}Z_i(H)g \le HK = G$. By [3, Theorem 9.1.10], there exist some element k of K such that $g^{-1}Z_i(H)g \le H^k$. Hence k$g^{-1}Z_i(H)gk^{-1}$≤H. Since $Z_i(H)$ is weakly closed in H (about G). Therefore k $g^{-1}Z_i(H)gk^{-1}$ = $Z_i(H)$. It implies $g^{-1}Z_i(H)g = k^{-1}Z_i(H)k$. Hence we have $\overline{g}^{-1}Z_i(\overline{H})\overline{g} = \overline{g^{-1}Z_i(H)g} = \overline{k^{-1}Z_i(H)k} = \overline{Z_i(H)} = Z_i(\overline{H})$ It shows that the upper central series of $\overline{H}$ is weakly closed in $\overline{H}$ (about $\overline{G}$).Since $N_{\overline{G}}(\overline{H}) = N_{G/K}(HK/K) = N_G(H)K/K = \overline{N_G(H)}$. By assumption, we obtain that $N_{\overline{G}}(\overline{H})$ is л-nilpotent. So, by induction on |G|, we get that $\overline{G}$ =G/K is л-nilpotent. Therefore G is л-nilpotent.

(2) N_G(Z(H))<G

Let $G_1 = N_G$(Z(H)). Since $N_G(H) \le N_G$(Z(H)). We can conclude that $N_{G_1}(H) = N_G(H)$. By assumption, we obtain that $N_{G_1}(H)$ is л-nilpotent. Since H≤G_1. The assumption in Theorem 3.2 implies that the upper central series of H is also weakly closed in H (about G_1). That is that G_1 satisfies the conditions of Theorem 3.2. By induction on |G|, $G_1 = N_G$(Z(H)) is л-nilpotent. By Lemma 2.3, we get that G is л-nilpotent.

Theorem 3.4 Let G be a л-soluble group. Suppose that H is a Hall л-subgroup of G and H is a cyclic group. If H≤G'. Then G' is a л-nilpotent group.

Proof If $O_{\pi'}$ (G)=1. Since G is a л-soluble group. By [8, Theorem 6.12], we have that $C_G(O_\pi(G)) \le O_\pi$ (G). Since H is a Hall л-subgroup of G. Hence O_π(G)≤H. It follows that H≤$C_G(O_\pi(G)) \le O_\pi$(G). This implies that H=O_π(G). Applying N/C theorem to O_π(G), we conclude that G/$C_G(O_\pi(G))$ $\lesssim$ Aut(H). Since H is a cyclic group . We obtain that G/$C_G(O_\pi(G))$ is an abelian group. It yield that $G' \le C_G(O_\pi(G))$=H. By the assumption that H≤G'. We get that G'=H. Therefore G' is a л-nilpotent group.

If $O_{\pi'}$(G)≠1. Then we consider G/$O_{\pi'}$(G). It is easy to show that H$O_{\pi'}$(G)/ $O_{\pi'}$(G) is a Hall л-subgroup of G/$O_{\pi'}$(G) and H$O_{\pi'}$(G)/$O_{\pi'}$(G) ≅ H/H∩$O_{\pi'}$(G) is a cyclic group. It is easy to prove that G/$O_{\pi'}$(G) satisfies the hypothesis of the theorem. By induction on |G|, we get that $G'O_{\pi'}$(G)/ $O_{\pi'}$(G) is a л-nilpotent group. Assume that N/$O_{\pi'}$(G) is a normal л-complement 0f $G'O_{\pi'}$(G)/$O_{\pi'}$(G). Then we have that $G'O_{\pi'}$(G)=HN. Hence $G' = G'$ ∩HN=H(N∩G'). Since N∩G' is a normal л-complement 0f G'. It follows that G' is a л-nilpotent group.

Theorem 3.5 Let G be a soluble group and let M ⊲ N ⊲ G. Suppose that N is a л-nilpotent Hall-subgroup of

G and N∩ Φ_{π} (G) is a nilpotent group. If N/ M(N∩ $\Phi_{\pi'}$ (G)) is a л-nilpotent group, then N/M is also a л-nilpotent group.

Proof Let L= M(N∩ $\Phi_{\pi'}$ (G)). Since N/M is a л-nilpotent group. It follows that N/L is π'-closed. By lemma 2.5, we have that N/M is π'-closed. It implies that $N^{\pi'}M/M \lhd N/M$. Picking arbitrarily q | |N/ $N^{\pi'}$ M|. Then we have that q∈л. Let $\pi_1 = \pi' \cup \{q\}$. Since $N/N^{\pi'}L \cong {}^{N/L}\!/\!_{N_{\pi'}L/L}$ is a nilpotent group. It yield that N/ $N^{\pi'}$L is π_1-closed. Since $N^{\pi'}M \lhd N \lhd G$. By lemma 2.5, we have that N/ $N^{\pi'}$M is π_1-closed. It implies that the Sylow q-subgroup of N/ $N^{\pi'}$M is normal in N/ $N^{\pi'}$M, By the arbitrariness of q, we obtain that $N/N^{\pi'}M \cong {}^{N/M}\!/\!_{N_{\pi'}M/M}$ is a nilpotent group., Therefore N/M is a л-nilpotent group.

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