Abelian Homegeneous Factorisations of Graphs

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Abstract: A homogeneous factorisation of a graph Γ is a partition of its arc set such that there exist vertex transitive subgroups M<G≤Aut(Γ) such that M fixes each part setwise of the partition and Gpreserves the partition and transitive permuting the parts. In the present paper, we study homogeneous factorisation with M abelian. We give some interesting characterizations and constructions of such factorisations.

Key words: Cayley graph, homogeneous factorisations, abelian group.

1. Introduction

Let Γ be a graph with vertex set VΓ and arc set AΓ. If there exist a partition $P=\{P_1,P_2,\cdots,P_k\}$ of the arc set AΓ and two subgroups $M<G$ of $Aut\Gamma$, such that

1) M is transitive on the vertex set $V\Gamma$ and fixes each Pi setwise;
2) P is a G-invariant partition and the induced action GP of G on P is transitive, we call that (Γ,P) is a (M,G)-homogeneous factorisation of index k.

If M is regular on VΓ and $M \lhd G$, then we call the factorisation an M-Cayley homogeneous factorisation; if, in particular, M is an abelian group, we occasionally simply call that Γ has a homogeneous factorisation, M-Cayley homogeneous factorisation, M-abelian homogeneous factorisation and M-circulant homogeneous factorisation. The purpose of this paper is to character M-abelian homogeneous factorisation.

The finite graphs homogeneous factorisation is initiated and researched by famous Algebra graphs theorem experts Praeger, Guralinck, and Saxl [1]. In 2003, Lim and Stringer gave characterizations for homogeneous factorisations of complete digraph and Edge-transitive homogeneous factorisations of complete graphs [2], [3]. In 2004, Cuaresma studied homogeneous factorisations of Johnson graph [4]. In 2007, Giudici, Li, Potocnik, and Praeger accomplished homogeneous factorisations of complete multipartite graphs [5]. Fang, Li, and Wang characterized transitive 1-factorizations of arc-transitive graphs [6]. The general theory of homogeneous factorisation was studied in [1], [7]. A necessary and sufficient condition for complete graphs having circulant homogeneous factorisations has been given by Praeger and Li, for complete graphs having (M,G) circulant homogeneous factorisations under the condition that G/M is a cyclic group [8]. This paper give a research in the base of the main work above.

2. Constructions

Let R be a group and $M=R:H<G=M:L$ with L a subgroup of $Aut(M)$. Let $\Gamma=Cay(R,S)$ and $\Gamma_i=Cay(R,S_i)$ for $i=1,2,\cdots,k$ are Cayley graphs of M. Finally let $P_i=A\Gamma_i$, $\mathrm{P}=\{P_1,P_2,\cdots,P_k\}$ and $\Sigma=\{S_1,S_2,\cdots,S_k\}$.

Lemma 2.1. Using notation defined above, (Γ,P) is a (M,G) homogeneous factorisation if and only if the following conditions hold:

1) H fixes each S_i.
2) $S=\bigcup_{1\le i\le k} S_i$ and setwise $S_i \cap S_j=\varnothing$ for all $i\neq j$.
3) Σ is a L–invariant partition and the induced action L^{Σ} of L on Σ is transitive. In particular, if we choose $M=R$, then (Γ,P) is a M-Cayley homogeneous factorisation if and only if the above conditions 2 and 3 hold.

The following construction of M-Cayley homogeneous factorisation is given in [5].

Construction 2.2. Let M be a group and let S be a subset of $M\setminus\{1\}$ preserved by some subgroup $H\le Aut(M)$. Let $\Gamma=Cay(M,S)$, $O=\{O_1,O_2,\cdots,O_r\}$ be the set of H-orbits in S. For each $i\in\{1,2,\cdots,r\}$, choose $x_i\in O_i$. Suppose that H has a proper subgroup R containing R. For each $i\in\{1,2,\cdots,r\}$, let $B_i\le O_i$ be the R-orbit of the element x_i, and let $S_1=B_1\cup B_2\cup\cdots\cup B_r$. Choose a set $\{h_1,h_2,\cdots,h_k\}$ of coset representations of R in H such that $h_1=1$. Define $S_i=S_1^{h_i}$, let P_i be the arc set of the Cayley graph $\Gamma_i=Cay(M,S_i)$, $\mathrm{P}=\{P_1,P_2,\cdots,P_k\}$ and $G=M:H$. Then (Γ,P) is a M-Cayey homogeneous factorisation.

Lemma 2.3. Let R be a group and $\Gamma=Cay(R,S)$ be a Cayley graph. Let $H\lhd L$ be subgroups of $Aut(R,S)$. Suppose L is transitive on S and H is intransitive on S, say $\Sigma=\{S_1,S_2,\cdots S_k\}$ is the set of orbits of H on S. Then there exists (Γ,P) to be a (M,G)-homogeneous factorisation of index k, where $R\le M\le R:H$ and $G=R:L$.

Proof: Because L is transitive on S, H is intransitive on S, and $H\lhd L$. We have

$$S=\alpha_1^H\dot{\bigcup}\alpha_2^H\dot{\bigcup}\cdots\dot{\bigcup}\alpha_k^H=S_1\dot{\bigcup}S_2\dot{\bigcup}\cdots\dot{\bigcup}S_k.$$

Therefore, $S_i=\alpha_i^H \quad (i=1,2,\cdots,k)$ is a block for L. Let $\Sigma=\{S_i^l \mid l\in L\}$, then Σ is a L-invariant partition. As $\forall S_i\in\Sigma, l\in L$, we have $S_i^l=(\alpha_i^H)^l=\alpha_i^{lH}=(\alpha_i^l)^H=\alpha_j^H=S_j$ for some j. Further, L preserves Σ and L is transitive on Σ, $\Sigma=\{S_1,S_2,\cdots,S_k\}$. Let $\Gamma=Cay(R,S)$. We conclude (Γ,P,M,G) satisfies conditions (i)-(iii) of Lemma 2.1. So we be sure that Γ has a factorisation whose factors are all of the same valency. Let $\Gamma_i=Cay(R,S_i) \quad (i=1,2,\cdots,k)$, $P=\{A\Gamma_1,A\Gamma_2,\cdots,A\Gamma_k\}$, then (Γ,P) is a $(\hat{R}:H,\hat{R}:L)$ homogeneous factorisation of index k.

Let M be a group with order bigger than 2. For a nonidentity automorphism α of M, the following construction gives a way to construct a M-Cayley cyclic (M,G) homogeneous factorisation (Γ,P) with $G=M:\langle\alpha\rangle$.

Construction 2.4. Let M and α are as above. Suppose $o(\alpha)=n\ge 2$. Then we may choose $l\ge 2$ such that $S:=\{x\in M: l \,\|\, |x^{\langle\alpha\rangle}|\}$ has at least 2 elements. That is, S is the union set of all $\langle\alpha\rangle$ orbits on M, say $O_1,O_2,\cdots,O_k$, whose length are multiple of l. Choose $x_i\in O_i$, and suppose $t_i=|O_i|/l$. Define $S_1=\bigcup_{1\le i\le k}\{x_i,x_i^{\alpha^l},\cdots,x_i^{\alpha^{(t_i-1)l}}\}$, and $S_j=S_1^{\alpha^{j-1}}$ for $j=2,3,\cdots,l$. Further let $\Gamma=Cay(M,S)$, and for $i=1,2,\cdots,l$, let P_i be the arc set of $\Gamma_i=Cay(M,S_i), \mathrm{P}=\{P_1,P_2,\cdots,P_l\}$ and $G=M:\langle\alpha\rangle$.

Lemma 2.5. (Γ,P) is a M-Cayley cyclic (M,G) homogeneous factorisation.

Proof: let $\Gamma = Cay(M,S), \alpha \in Aut(M), o(\alpha) = n \geq 2$.

Claim 1: $\forall x \in S$, we have $x^{\alpha^i} \in S$.

Proof: $x^{\alpha^i} \in S$. because $\alpha^i \in \langle \alpha \rangle$, so $\left|(x^{\alpha^i})^{\langle \alpha \rangle}\right| = \left|x^{\alpha^i \langle \alpha \rangle}\right| = \left|x^{\langle \alpha \rangle}\right|$, therefore $x^{\alpha^i} \in S$. According to Claim 1, we know S is the union set of some $\langle \alpha \rangle$ -orbits in M and $\langle \alpha \rangle \leq Aut(M,S)$, suppose $t_i = |O_i|/l$

Define $S_1 = \cup_{1 \leq i \leq k} \{x_i, x_i^{\alpha^l}, \cdots, x_i^{\alpha^{(t_i-1)l}}\}$ and $S_j = S_1^{\alpha^{j-1}}$ for $j = 2,3,\cdots,l$.

Suppose $\Sigma = \{S_1, S_2, \cdots, S_l\}$, obviously, we know that $\langle \alpha \rangle$ preserve Σ and transitive on Σ. Let $\Gamma_i = Cay(M,S_i)$, $\mathrm{P} = \{P_1, P_2, \cdots, P_l\}$ and $G = M:\langle \alpha \rangle$, that is $G/M \cong \langle \alpha \rangle$, we say that (Γ, P) is a M-Cayley cyclic (M,G) homogeneous factorisation. By construction 2.4, we have the following interesting proposition.

Proposition 2.6. For a given group M, there is a (M,G) homogeneous factorisation (Γ, P) for some Γ, P, G if and only if $M > 2$.

Proof: The sufficiency of the proposition follows directly by Construction 2.4. Since M is transitive on $V\Gamma$, it follows $|M| > 2$. Further, if $|M| = 2$, then $\Gamma = K_2$ obviously has no homogeneous factorisation, which proves the necessity of the proposition.

3. Abelian Homogeneous Factorisation of Complete Graphs

Let F_q be a field of order q with $q = p^n$ and p prime. Let $GL(n,p)$ be the group of all invertible transformation a vector space of dimension n over field F_q, and let $V = F_p^n \setminus \{0\}$. Then V can be viewed the set of non-zero vectors of n-dimension vector space over field F_q. It is known that $GL(n,k)$ has natural action on V, and containing a cyclic subgroup, say $\{\sigma\}$, which is regular on V. we call the cyclic group $\{\sigma\}$ a Singer subgroup of $GL(n,p)$. Then $\{\sigma\} \cong GL(1,q) \leq GL(n,p)$. Using the Singer subgroup, we may prove that complete graphs of prime power vertices have abelian homogeneous factorisations of certain index.

Lemma 3.1. Let $\Gamma = K_{p^n}$ be the complete graph. Then $Aut(\Gamma) = S_{p^n}$. For each $k \,|\, (p^n - 1)$ and $k \geq 2$, there exists a (M,G) homogeneous factorisation (Γ, P) of index k with M containing a regular cyclic subgroup on $V\Gamma$.

Proof: Let $R = Z_{p^n}$ be the additive group of finite field F_{p^n}. Then $\Gamma = Cay(R,S)$ with $S = R \setminus \{0\}$ and $Aut(M) = GL(n,p)$. Suppose $\{\sigma\}$ is a Singer subgroup of $GL(n,p)$. Since $\{\sigma\} \cong Z_{p^n - 1}$ is cyclic and $k \,|\, (p^n - 1)$, we may choose $L = \{\sigma\}$ such that $k|l$ and choose $H = \{\sigma^{l/k}\}$. Let $M = R:L$ and $G = R:H$, then $M \leq G \leq Aut(\Gamma)$ and $|H:L| = k$. Since $\{\sigma\}$ is regular on S, it follows L and H are semi-regular on S, and H has exactly t orbits, say $O_1, O_2, \cdots, O_t$, where $t := (p^n - 1)/|H|$. Choose $x_i \in O_i$ and let $X = \{h_1, h_2, \cdots, h_k\}$ be a set of coset representatives of L in H. Define $S_1 = \cup_{1 \leq i \leq t} (x_i)^L$ and $S_j = S_1^{h_j}$ for $2 \leq j \leq k$. Let P_i be the arc set of $\Gamma_i = Cay(M,S_i)$, $\mathrm{P} = \{P_1, P_2, \cdots, P_k\}$. Then (K_{p^n}, P) is a (M,G) homogeneous factorisation of the complete graph K_{p^n}, with M containing a regular subgroup R on $V\Gamma$.

In Lemma 3.1, if $L = 1$, then $M = R$ is regular on $V(K_{p^n})$. That is, (K_{p^n}, P) is a R-abelian homogeneous factorisation of the complete graph K_{p^n}. However, the next lemma proves that the complete graph K_{p^n} has a circulant homogeneous factorisation if and only if p is odd.

Lemma 3.2. Let $\Gamma = K_{p^d}$ be the complete graph with p a prime, and let M be a cyclic group of order n of order p^d acting regular on $V\Gamma$. Then there exist a partition of $A\Gamma$ and $G \le N_{Aut(\Gamma)}(M)$ such that (Γ, P) is a M-circulant homogeneous factorisation of index $k \ge 2$ if and only if $k|(p-1)$.

Proof: We first prove the necessity of the condition in the lemma. Suppose $G = M:H$ for some $H \le Aut(\Gamma)$. Since $Val(\Gamma) = p^d - 1$, it follows $k \big| p^d - 1$. Suppose $\mathrm{P} = \{P_1, P_2, \cdots, P_k\}$, $\Gamma_i = Cay(M, S_i)$ with $P_i = A\Gamma_i$ and $G = M:H$ for some $H \le Z_n^*$. Since the induce action G^{P} of G is transitive on P, H is transitive on $\{S_1, S_2, \cdots, S_k\}$, so k divides the order of H, thus $k \big| p^{d-1}(p-1)$ as $\left|Z_n^*\right| = p^{d-1}(p-1)$. Note, $k \big| p^d - 1$, it easily follows that $k|p-1$.

We now prove the sufficiency of the condition in the lemma. First, as $k|p-1$, p is odd. Identifying M with the additive group of the ring Z_{p^d}, then $Aut(M) = \mathrm{Z}_n^* = \langle r \rangle \cong \mathrm{Z}_{p^{d-1}(p-1)}$ acts on M by multiplication, where r is a primitive root of p^d. That is, $p^{d-1}(p-1)$ is the minimal positive solution of the equation $r^x \equiv 1 (\mathrm{mod}\, p^d)$. Let $S = M \setminus \{0\}$, and let $H = Aut(M)$. Then H has exactly d orbits: $1^H, p^H, \cdots, (p^d - 1)^H$, and their length equal $p^{d-1}(p-1), p^{d-2}(p-1), \cdots, p-1$, respectively. Let $X = \{1, p, \cdots, p^{d-1}\}$ be a set of the orbits representatives of H acting on S by multiplication. Obviously, the stabilizer $H_{p^i} \subseteq H_{p^j}$ for each $i < j$, so $\langle H_x : x \in X \rangle = H_{p^{d-1}} = \langle r^{p-1} \rangle \cong \mathrm{Z}_{p^{d-1}}$ is a subgroup of H with index $p-1$. Further, since H is abelian and $k|p-1$, there exists a subgroup R containing $\langle H_x : x \in X \rangle$ such that $|H:R| = k$. By Construction 2.2, K_{p^d} has a circulant homogeneous factorisation of index k.

In lemma 3.1. We prove that complete graph of prime power vertices has abelian homogeneous factorisation. This leads to the following question.

Question 3.3. For $n \ge 2$ not a prime power, whether the complete graph K_n has abelian homogeneous factorisation?

The following proposition give a sufficient condition for the existence of abelian homogeneous factorisation of the complete graphs with not prime power vertices, thus provides a partial answer of Question 3.3.

Proposition 3.4. Let $\Gamma = K_n$ be the complete graph and let $n = p_1^{d_1} p_2^{d_2} \cdots p_r^{d_r}$ be the prime power facterisation of n. If $(p_1^{d_1} - 1, p_2^{d_2} - 1, \cdots, p_r^{d_r} - 1) \ne 1$, then there exists a (M, G) abelian homogeneous factorisation (K_n, P).

Proof: Suppose $k | (p_1^{d_1} - 1, p_2^{d_2} - 1, \cdots, p_r^{d_r} - 1)$ with $k \ge 2$. Write $l_i = p_i^{d_i - 1} / k$. Let $M_i = \mathrm{Z}_{p_i}^{d_i}$ Mi = Zpdii for $i = 1, 2, \cdots, r$, and let $M = M_1 \times M_2 \times \cdots \times M_r$. Then, $Aut(M) = GL(d_1, p_1) \times GL(d_2, p_2) \times \cdots \times GL(d_r, p_r)$. Let $\langle \sigma \rangle$ be the Singer subgroup of $GL(d_i, p_i)$ for each i. Then $H : \langle (\sigma_1^{l_1}, \sigma_2^{l_2}, \cdots, \sigma_r^{l_r}) \rangle$ is a subgroup of $Aut(M)$ with order k. Write $\sigma = (\sigma_1^{l_1}, \sigma_2^{l_2}, \cdots, \sigma_r^{l_r})$. Since $\langle \sigma_i^{l_i} \rangle \cong \mathrm{Z}_k$ is semi-regular on $M_i \setminus \{1\}$, it follows H is semi-regular on $M \setminus \{1\}$,so $S := \{x \in M \| x^{\langle \sigma \rangle}| = o(\sigma)\} = M \setminus \{1\}$, thus by Construction 2.4, $\Gamma = Cay(M, S) = K_n \Gamma$ has a M-abelian homogeneous factorisation of index $o(\sigma) = k$.

4. Abelian homogeneous factorization of some complete multipartite graphs

Lemma 4.1. If M is a group and $L \le M$ a subgroup of index s and order t, then $Cay(M, M\setminus L)$ is isomorphic to the complete multipartite graph $K_{s[t]}$. Conversely, if $\Gamma = K_{s[t]}$ is a complete multipartite graph and M a regular group of automorphisms of Γ, then there exists a subgroup L of order t and index s in M such that Γ is isomorphic to $Cay(M, M\setminus L)$.

proof: See, for example,[9, P roposition2.2]

In this section, we let $M = P\times Q$, $P \cong Z_p^2$ P ~p and $Q \cong Z_q^2$ (p,q are primes), then

$$Aut(M) = Aut(P)\times Aut(Q) \cong GL(2,p)\times GL(2,q).$$

In $GL(2,p)$, the singer cyclic group is $GL(1,p^2) \cong Z_{p^2} -1 < GL(2,p)$, so there exists $\beta \in GL(2,p)$, such that $o(\beta) = p, p-1, p+1, p^2-1$, further $GL(1,p^2)$ is regular on $Z_p^2\setminus\{1\}$.

The same reason, in $GL(2,q,)$ there exists $\beta' \in GL(2,q)$ of order $o(\beta') = q, q-1, q+1, q^2-1. GL(1,q^2)$ is semiregular on $Z_q^2\setminus\{1\}$.

Take $\alpha = (x,y) \in Aut(M)$ where $x \in \langle\beta\rangle, y \in \langle\beta'\rangle$ such that $o(\alpha) = o(x) = o(y) \ne 1$.

Construction 4.2. $o(\beta) = p, o(\beta') = q-1, (p,q-1) \ne 1$ Claim: $(1)\Gamma \cong K_{pq^2[p]}$ or $(2)\Gamma \cong K_{pq[pq]}$.

proof:

Let $\alpha = (x,y), x\in\langle\beta\rangle, y\in\langle\beta'\rangle, o(x) = o(y) = o(\alpha) = (p,q-1) = p$, because $\langle\beta'\rangle$ acts on Z_p^2 semi-regular, let $((a_1,a_2),(1,1)) \in M$, $(a_1,a_2) \in Z_p^2$, $(1,1) \in Z_q^2$ and it is an identity, so we find fixed points of $\langle\alpha\rangle$ acts on M. That is, find fixed points of $\langle x\rangle$ acts on Z_p^2.

Let $\langle x\rangle = T = \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \middle| b\in F_p\right\} \cong Z_p < GL(2,p)$, $(a_1,a_2) \in Z_p^2$, we have $(a_1,a_2)^{x_i} = (a_1,a_2)\begin{pmatrix} 1 & ib \\ 0 & 1\end{pmatrix} = (a_1, iba_1 + a_2)$, $(1\le i\le p)$, Let $(a_1,a_2) = (a_1, a_2 + iba_1)$, then $a_2 = a_2 + iba_1$, and ib is arbitrary, so $a_1 = 0$, and $(0,a_2)$ is the fixed-points of $\langle x\rangle$ acts on Z_p^2 $(a_2 \in Z_p)$. That is, there is p fixed-points. These points form a cycle group L based on relation of Z_p, and $|L| = p$, $|M:L| = pq^2$, $|S| = |M\setminus L| = p^2q^2 - p = p(pq^2-1)$, by lemma 4.1, we conclude $\Gamma \cong K_{pq^2[p]}$.

The same reason, when $o(\beta) = p$, $o(\beta') = q+1$ or q^2-1; $(p,q+1) \ne 1$, $(p,q^2-1) \ne 1$, then $\Gamma \cong K_{pq^2[p]}$.

Obviously, $p|q-1$, let $\beta' \in GL(2,q)_u \cong Z_q : Z_{q-1}$. Let $\alpha = (x,y), x\in\langle\beta\rangle, y\in\langle\beta'\rangle$, $o(\alpha) = o(x) = o(y) = p$

$$\langle x\rangle = \langle\beta\rangle = \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \middle| b\in F_p\right\} \cong Z_p < GL(2,p)$$

$$\langle y\rangle = \langle\begin{pmatrix} 1 & 0 \\ & \\ 0 & d^{\frac{q-1}{p}}\end{pmatrix} | d\ne 0\rangle \cong Z_p < \langle\beta'\rangle = \left\{\begin{pmatrix} 1 & 0 \\ 0 & d\end{pmatrix} \middle| d\in F_q\setminus\{0\}\right\} \cong Z_{q-1}$$

Suppose $(a_1,a_2) \in Z_p^2$, We have

$$(a_1,a_2)^{x_i}=(a_1,a_2)\begin{pmatrix}1 & ib\\ 0 & 1\end{pmatrix}=(a_1,iba_1+a_2)\ ,(1\le i\le p)$$

Let $(a_1,a_2)=(a_1,a_2+iba_1)$, then $a_2=a_2+iba_1$, and ib is arbitrary. So $a_1=0$, and $(0,a_2)$ is the fixed-points of $\langle x\rangle$ acts on Z_p^2 $(a_2\in Z_p)$. That is, there is p fixed-points.

Let $(b_1,b_2)\in Z_q^2$

We have

$$(b_1,b_2)^{y_i}=(b_1,b_2)\begin{pmatrix}1 & 0\\ 0 & d^{i\frac{q-1}{p}}\end{pmatrix}=(b_1,b_2d^{i\frac{q-1}{p}}),\ (1\le i\le p)$$

Let $(b_1,b_2)=(b_1,b_2d^{i\frac{q-1}{p}})$. Then $b_2=b_2d^{i\frac{q-1}{p}}$ and $d^{i\frac{q-1}{p}}$ is arbitrary. So $b_2=0$, and $(b_1,0)$ is the fixed-points of $\langle y\rangle$ acts on Z_q^2 $(b_1\in Z_q)$. That is, there is q fixed-points. Therefore, there are pq fixed-points when $\langle\alpha\rangle$ acts on $Z_p^2\times Z_q^2$.

Let

$$\langle a_1\rangle=Z_p\ ,\ \langle b_1\rangle=Z_q$$

Then

$$\langle(a_1,0),(b_1,0)\rangle\cong Z_p\times Z_q$$

Obviously, this is an abelion group denoted by L and $|L|=pq$, $|M:L|=pq$, $|S|=|M\setminus L|=p^2q^2-pq=pq(pq-1)$. By lemma 4.1, we can conclude $\Gamma\cong K_{pq[pq]}$.

The same reason, we can draw the following conclusion

(i): $o(\beta)=p+1,\ p^2-1,\ o(\beta')=q,\ (p,q+1)\ne 1,\ (p,q^2-1)\ne 1$, then $\Gamma\cong K_{p^2q[q]}$;

(ii): $o(\beta)=p-1,\ o(\beta')=q,\ (p-1,q)\ne 1$, then $\Gamma\cong K_{p^2q[q]}$ or $\Gamma\cong K_{pq[pq]}$

Theorem 4.3. Let $M=P\times Q=Z_p^n\times Z_q^n$, p and q are prime, and n is positive integer, then $Aut(M)\cong GL(n,p)\times GL(n,q)$. Suppose $\beta\in GL(n,p)$, we have $o(\beta)=p,p-1,p^{n-1}+C_n^1p^{n-2}+C_n^2p^{n-3}+\cdots+1,p^n-1$. The same reason, when $\beta'\in GL(n,q)$, then $o(\beta')=q,q-1,q^{n-1}+C_n^1q^{n-2}+C_n^2q^{n-3}+\cdots+1,q^n-1$, According to the method of construction 4.2, we can construct Γ, such that Γ has homogeneous factorisation, we have these conclusion as the below:

(1)

(i): $o(\beta)=p-1,\ o(\beta')=q-1,q^{n-1}+C_n^1q^{n-2}+C_n^2q^{n-3}+\cdots+1,q^n-1$, and the orders of β and β' are not coprime, then $\Gamma\cong K_{p^nq^n}$;

(ii): $o(\beta)=p^{n-1}+C_n^1 p^{n-2}+C_n^2 p^{n-3}+\cdots+1, o(\beta')=q-1, q^{n-1}+C_n^1 q^{n-2}+C_n^2 q^{n-3}+\cdots+1, q^n-1$, and the orders of β and β' ' are not coprime, then $\Gamma \cong K_{p^n q^n}$.

$o(\beta)=p^n-1$, $o(\beta')=q-1, q^{n-1}+C_n^1 q^{n-2}+C_n^2 q^{n-3}+\cdots+1, q^n-1$, and the orders of β and β' ' are not coprime, then $\Gamma \cong K_{p^n q^n}$.

(2)

(i): $o(\beta)=p$, $o(\beta')=q-1$, $(p,q-1)\neq 1$, then $\Gamma \cong K_{p^{n-1}q^n[p]}$ or $\Gamma \cong K_{p^{n-1}q^{n-1}[pq]}$;

(ii): $o(\beta)=p-1$, $o(\beta')=q$, $(p-1,q)\neq 1$, then $\Gamma \cong K_{p^n q^{n-1}[q]}$ or $\Gamma \cong K_{p^{n-1}q^{n-1}[pq]}$];

(3)

(i): $o(\beta)=p$, $o(\beta')=q^{n-1}+C_n^1 q^{n-2}+C_n^2 q^{n-3}+\cdots+1, q^n-1$, and the orders of β and β' ' are not coprime, then $\Gamma \cong K_{p^{n-1}q^n[p]}$;

(ii): $o(\beta)=p^{n-1}+C_n^1 p^{n-2}+C_n^2 p^{n-3}+\cdots+1, p^n-1$, $o(\beta')=q$, and the orders of β and β' are not coprime, then $\Gamma \cong K_{p^n q^{n-1}[q]}$;

4.4. A method of construction connective subdigraph: In order to construct connective subdigraph, we need $M=\langle S_1\rangle$. Let

$$S_1=\{\delta_1,\delta_2,\cdots,\delta_m\},\ M\cong Z_p^2\times Z_q^2=\langle a,b\rangle\times\langle a',b'\rangle$$

We suppose

$$\delta_1=a;$$

$$\delta_i=a'b,\ i\neq 1;$$

$$\delta_j=b',\ j\neq 1,i;$$

$\delta_t\in O_t$, $t\in 1,2,\cdots,m$ and $t\neq 1,i,j$; We have $\langle S_1\rangle=\langle a,a',b,b'\rangle=M$, according to this method, we must be construct connective subdigraph.

In Theorem 4.3, we prove that the complete graph and the complete multipartite graph of product of two different primes power vertices has abeliant homogeneous factorisation. This leads to the following question.

Question 4.5: When

1) $M=Z_p^2\times Z_q^2\times Z_r^2$ (v are primes);

2) $M=Z_{p_1}^{r_1}\times Z_{p_2}^{r_2}\times Z_{p_3}^{r_3}$ (p_1,p_2,p_3 are primes);

Whether the complete graph and the complete multipartite graph have abeliant homogeneous factorization ?

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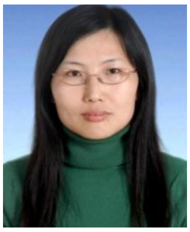

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