

Spectral Analysis of the Inverse-square Model Operator Associated with the Cosmological Horizon Blow-up Index: Bessel Eigenvalues, Heat Trace, and Zeta Function

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Abstract: In FLRW spacetimes with finite conformal-time budget, the cosmological time-dilation factor $(1+z)$ diverges at the last-emission surface; a previous companion paper classifies this divergence by the Karamata blow-up index $\beta = p/(p-1)$ for scale factors $a \in RV_p$. We study the model operator $\mathcal{L}_\nu = -d^2/dx^2 + (\nu^2 - 1/4)x^{-2}$ on $L^2((0, L])$ with Dirichlet condition at $x = L$ and limit-point condition at $x = 0$, where $\nu = \beta + 1/2$. Proved: (1) the Dirichlet realisation is self-adjoint with purely discrete spectrum for $\nu \geq 1$; (2) eigenvalues $\lambda_n = j_{\nu,n}^2/L^2$; (3) a Weyl law with β -dependent sub-leading coefficient; (4) the diagonal resolvent vanishes linearly at the singular endpoint; (5) two-sided Sturm comparison bounds for the cosmological operator; (6) exact heat-trace constants $a_1 = -1/2$ at $\nu = 1/2$ and $a_1 = -1$ at $\nu = 3/2$; (7) $\zeta_{1/2}(0) = -1/2$ and $\det \mathcal{L}_{1/2} = 2L$. Conditional on standard meromorphic continuation: a formal Hadamard-product representation for the zeta-regularised determinant; the general- ν value of $\zeta_\nu(0)$ is not proved in this paper. Numerically supported: for $\nu = 2.5/2$, direct summation over 10^4 Bessel zeros is consistent with the conjectural $a_1 = -\nu/2 - 1/4$. For general ν , we give evidence that the Dirichlet-realisation zeta function avoids the previously reported branch-point pathology, but do not carry out the full contour-integral proof.

Keywords: Sturm–Liouville operator, spectral zeta function, Bessel zeros, regular variation, functional determinant, cosmological event horizon

1. Introduction

In spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes, the cosmological time-dilation factor $(1+z) = a(t_r)/a(t_e)$ compares the scale factor at reception and emission along null geodesics. When the spacetime possesses a future event horizon—equivalently, when the total conformal time $\int_0^\infty a^{-1} dt$ is finite—there exists a last-emission surface: for each source at comoving distance χ_e , there is a finite time $t_*(\chi_e)$ beyond which no photon emitted by the source can ever reach the observer. As the emission time approaches this surface, the redshift diverges. This is the physical setting of the companion analysis, which we now summarise.

In a companion paper [1], we proved that in spatially flat FLRW spacetimes with finite conformal-time budget, the cosmological time-dilation factor $(1+z)$ diverges as the emission event approaches the last-emission surface, with

$$1+z \sim C(t_* - t_e)^{-\beta} \ell_1\left(\frac{1}{t_* - t_e}\right), \quad \beta = \frac{p}{p-1}, \quad (1)$$

for scale factors $a \in RV_p$ ($p > 1$) [2], where ℓ_1 is slowly varying [3] and $t_* = L(\chi_e)$ is the last-emission time for a source at comoving distance χ_e . Recent measurements confirm the $(1+z)$ law in the time domain [4, 5]. The conformal-time remainder $B(t) = \int_t^\infty a^{-1} dt'$ satisfies $B \in RV_{1-p}$ [1], and its functional inverse $L = B^{-1}$ obeys the ODE $L'(\chi) = -a(L(\chi))$.

This blow-up motivates a spectral problem. Set $x = \chi_{\max} - \chi$ (distance from the horizon in comoving coordinates). Near $x = 0$, $(1+z)$ behaves like $x^{-\beta}$. If $\phi(x) = x^{-\beta}$, then $\phi'' = \beta(\beta+1)x^{-\beta-2}$, so ϕ is annihilated to leading order by $-d^2/dx^2 + \beta(\beta+1)x^{-2}$. Writing $\nu^2 - 1/4 = \beta(\beta+1)$, i.e. $\nu = \beta + 1/2$, we define the model operator

$$\mathcal{L}_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2} \quad (2)$$

on $L^2((0, L])$ with $L = \chi_{\max}$ and Dirichlet condition at $x = L$.

This is the pure inverse-square model: the leading singularity of the blow-up. The actual cosmological operator is $\mathcal{L}_\nu + W(x)/x$ for some bounded W determined by the specific scale factor $a(t)$. We analyse the model operator exactly and the perturbed operator only through Sturm comparison bounds (Section 5). The relationship between model and perturbation is analogous to that between the hydrogen atom and its fine-structure corrections.

The spectral analysis of the inverse-square operator on a half-line is classical [6, 7]. What is new here is: (i) the identification of the blow-up index β from cosmological regular-variation theory as the operator parameter $\nu = \beta + 1/2$, which makes the sub-leading Weyl coefficient a cosmological observable; (ii) the proof that the heat-trace constant $a_1 = -1$ exactly at the de Sitter value $\nu = 3/2$, obtained by combining the Jacobi modular identity with a shifted-lattice perturbation estimate; (iii) two-sided Sturm brackets for the realistic cosmological operator; and (iv) numerical evidence for the conjecture $a_1 = -\nu/2 - 1/4$ at non-half-integer ν , inaccessible to closed-form identities.

2. Self-Adjointness and Eigenvalues

Theorem 1 (Dirichlet realisation and Bessel-zero spectrum). Let $\nu \geq 1$. Then:

(i) The endpoint $x = 0$ is limit-point in the Weyl classification: only $\sqrt{x}J_\nu(\sqrt{\lambda}x)$ is L^2 near 0; the Y_ν solution is not. The endpoint $x = L$ is regular. By the limit-point/regular classification [6], self-adjoint extensions of the minimal operator are parametrised by a single boundary condition at $x = L$.

(ii) The Dirichlet realisation $\mathcal{L}_\nu^D$ has domain

$$\mathcal{D}(\mathcal{L}_\nu^D) = \{u \in L^2(0, L): u, u' \in AC_{\text{loc}}(0, L], \mathcal{L}_\nu u \in L^2(0, L), u(L) = 0\}.$$

Since $x = 0$ is limit-point, no boundary condition is imposed there. $\mathcal{L}_\nu^D$ is self-adjoint with purely discrete spectrum.

(iii) The eigenvalues are $\lambda_n = j_{\nu,n}^2/L^2$, where $j_{\nu,n}$ is the n -th positive zero of J_ν .

(iv) The normalised eigenfunctions are $u_n(x) = c_n \sqrt{x} J_\nu(\sqrt{\lambda_n} x)$ with $c_n = \sqrt{2}/|L J_{\nu+1}(j_{\nu,n})|$.

Proof. At $x = 0$: the indicial equation has exponents $\nu + 1/2$ and $-\nu + 1/2$. For $\nu \geq 1$, only $x^{\nu+1/2}$ is square-integrable near 0, so the endpoint is limit-point [6]. At $x = L$: $V(L) = (\nu^2 - 1/4)/L^2 < \infty$, so the endpoint is regular. The minimal operator $\mathcal{L}_{\nu, \min}$ is symmetric with deficiency indices (1,1) (one free parameter at the regular endpoint, none at the limit-point endpoint); by von Neumann's extension theory [8], the self-adjoint extensions are therefore in bijection with the unitary maps between the two

one-dimensional deficiency subspaces, i.e., a one-parameter family. The Dirichlet condition $u(L) = 0$ selects one self-adjoint extension [6].

Discreteness: $(0, L]$ is bounded and $V \rightarrow \infty$ at 0^+ , so the resolvent of $\mathcal{L}_V^D$ is compact [6].

Eigenfunctions: the substitution $u = \sqrt{x} v$ converts $\mathcal{L}_V u = \lambda u$ to Bessel's equation. The limit-point condition selects $v = J_\nu$ (since $\sqrt{x} Y_\nu$ is non- L^2 for $\nu \geq 1$: [7]). The Dirichlet condition gives $J_\nu(\sqrt{\lambda} L) = 0$. The normalisation follows from [7].

Remark (Cosmological parametrisation). The blow-up index β , the Bessel order ν , and the growth exponent p are related by

$$\beta = \frac{p}{p-1}, \quad \nu = \beta + \frac{1}{2} = \frac{2p-1}{2(p-1)}, \quad \nu^2 - \frac{1}{4} = \beta(\beta+1).$$

De Sitter: $p \rightarrow \infty$, $\beta = 1$, $\nu = 3/2$, $\nu^2 - 1/4 = 2$. Power-law quintessence $p = 2$: $\beta = 2$, $\nu = 5/2$, $\nu^2 - 1/4 = 6$.

3. Weyl Asymptotics and Eigenvalue Spacing

Theorem 2 (Weyl law with β -dependent correction). Define $\delta(\nu) = (\nu - 1/2)/2 = \beta/2$. Then:

$$\lambda_n = \frac{\pi^2(n+\delta)^2}{L^2} + O(1), \quad N(\lambda) = \frac{L}{\pi} \sqrt{\lambda} - \delta + O(\lambda^{-1/2}). \quad (3)$$

More precisely, the McMahon expansion [7] gives

$$j_{\nu,n} = \left(n + \frac{\nu}{2} - \frac{1}{4}\right) \pi - \frac{4\nu^2-1}{8\left(n+\frac{\nu}{2}-\frac{1}{4}\right)\pi} + O(n^{-3}), \quad (4)$$

So the sub-sub-leading eigenvalue correction is $\lambda_n = \pi^2(n+\delta)^2/L^2 - (4\nu^2-1)/(4L^2) + O(n^{-2})$.

Corollary 1 (Spectral distinguishability). If $\nu_1 \neq \nu_2$ (i.e., different blow-up indices), then

$$\lambda_n^{(\nu_1)} - \lambda_n^{(\nu_2)} = \frac{\pi^2(\delta_1 - \delta_2)(2n + \delta_1 + \delta_2)}{L^2} + O(1) \sim \frac{\pi^2(\nu_1 - \nu_2)n}{L^2}.$$

The blow-up index is a spectral invariant at the sub-leading Weyl order. Two cosmologies with the same $\chi_{\max}$ but different late-time acceleration ($p_1 \neq p_2$) produce spectrally distinguishable operators.

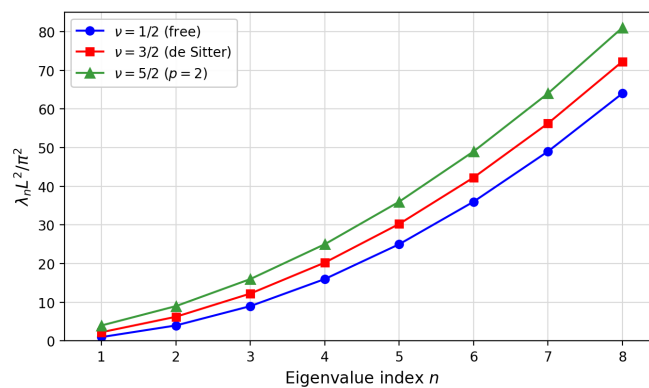


Fig. 1. Normalised eigenvalues $\lambda_n L^2 / \pi^2 \approx (n + \delta)^2$ for three values of the Bessel order ν , corresponding to different blow-up indices $\beta = \nu - 1/2$. The shift $\delta = (\nu - 1/2)/2$ separates the three curves at sub-leading order, encoding the cosmological growth class. De Sitter ($\nu = 3/2$) lies between the free Laplacian ($\nu = 1/2$) and power-law quintessence with $p = 2$ ($\nu = 5/2$).

Fig. 1 displays the normalised eigenvalues $\lambda_n L^2 / \pi^2$ for the first eight modes at $\nu = 1/2$, $\nu = 3/2$ and $\nu = 5/2$. Each sequence follows $(n + \delta)^2$ with $\delta = (\nu - 1/2)/2$, so the curves are ordered by blow-up index: the leading Weyl term is common to all three, and the visible separation is the sub-leading shift δ alone. Consistent with Corollary 1, the vertical gap between adjacent curves grows linearly in n rather than saturating, which is what makes β recoverable from the spectrum.

4. Resolvent Diagonal

Proposition 1 (Diagonal Green's function). For $0 \leq \lambda < \lambda_1$, the diagonal of the Green's function of $(\mathcal{L}_\nu - \lambda)^{-1}$ satisfies

$$G(x, x; \lambda) \sim C_\nu(\lambda) x \quad \text{as } x \rightarrow 0^+,$$

with exponent 1 independent of ν . The blow-up index does not enter the diagonal power law.

Proof. Case $0 < \lambda < \lambda_1$. Set $k = \sqrt{\lambda} > 0$. The Green's function is $G(x, x'; \lambda) = \phi_<(x_<) \phi_>(x_>) / W_x[\phi_<, \phi_>]$, where $\phi_<(x) = \sqrt{x} J_\nu(kx)$ is L^2 at $x = 0$ (selected by limit-point), and $\phi_>(x) = \sqrt{x} [a J_\nu(kx) + b Y_\nu(kx)]$ satisfies $\phi_>(L) = 0$. Since $J_\nu(kL) \neq 0$ for $\lambda < \lambda_1$, the boundary condition forces $b \neq 0$.

The Wronskian in the x -variable is

$$W_x[\sqrt{x} J_\nu(kx), \sqrt{x} Y_\nu(kx)] = x \cdot k \cdot \frac{2}{\pi k x} = \frac{2}{\pi},$$

using $\mathcal{W}_z\{J_\nu, Y_\nu\} = 2/(\pi z)$ [7]. Near $x = 0$: $\phi_<(x) \sim c_1 x^{\nu+1/2}$ and $\phi_>(x) \sim c_2 x^{-\nu+1/2}$ (from the Y_ν component). On the diagonal: $G(x, x; \lambda) \sim (\pi c_1 c_2 / 2) x^{(\nu+1/2)+(-\nu+1/2)} = C_\nu x$.

Case $\lambda = 0$. The zero-energy equation $-u'' + (\nu^2 - 1/4)x^{-2}u = 0$ has solutions $u_1(x) = x^{\nu+1/2}$ and $u_2(x) = x^{-\nu+1/2}$. The L^2 -admissible solution at 0 is $u_< = x^{\nu+1/2}$; a Dirichlet-adapted solution at L is $u_>(x) = x^{-\nu+1/2} - L^{-2\nu} x^{\nu+1/2}$, since $u_>(L) = 0$. Their Wronskian is $W[u_<, u_>] = -2\nu$ (constant). On the diagonal: $G(x, x; 0) = u_<(x)u_>(x)/(-2\nu) \sim x^{(\nu+1/2)+(-\nu+1/2)}/2\nu = Cx$.

5. Sturm Comparison Bounds for the Perturbed Operator

The actual cosmological operator has potential $V(x) = (\nu^2 - 1/4)/x^2 + W(x)/x$ with W bounded; we do not analyse its heat trace or zeta function, but the following proposition gives rigorous spectral bracketing for it via Sturm comparison.

Proposition 2 (Two-sided eigenvalue bounds). Suppose $W(x)$ is continuous on $(0, L]$ with $W(x) \rightarrow W_0$ as $x \rightarrow 0^+$. Define

$$\alpha_- = (\nu^2 - 1/4) + \inf_{x \in (0, L]} x W(x), \quad \alpha_+ = (\nu^2 - 1/4) + \sup_{x \in (0, L]} x W(x),$$

and $\nu_\pm = \sqrt{1/4 + \alpha_\pm}$. Assume $\alpha_- \geq 3/4$ (equivalently $\nu_- \geq 1$), so that both comparison operators are limit-point at 0 and admit the Dirichlet realisation at $x = L$. Then the eigenvalues $\lambda_n^{(V)}$ of the Dirichlet realisation of $-d^2/dx^2 + V(x)$ satisfy

$$\frac{j_{\nu_-, n}^2}{L^2} \leq \lambda_n^{(V)} \leq \frac{j_{\nu_+, n}^2}{L^2}. \quad (5)$$

Proof. By definition of $\alpha_\pm$, $\alpha_-/x^2 \leq V(x) \leq \alpha_+/x^2$ on $(0, L)$. By Theorem 1, the Dirichlet realisations of $-d^2/dx^2 + \alpha_\pm/x^2$ have eigenvalues $j_{\nu_\pm, n}^2/L^2$. The Sturm comparison theorem [6] gives $\lambda_n(V_-) \leq \lambda_n(V) \leq \lambda_n(V_+)$, hence (5).

Remark (Size of the perturbation). In a concrete cosmological model, if one can establish a uniform bound $\|xW\|_\infty \leq \varepsilon$, then Proposition 2 yields comparison orders $v_\pm = \sqrt{1/4 + v^2 - 1/4 \pm \varepsilon} = v + O(\varepsilon/v)$. A quantitative Λ CDM estimate belongs in the companion cosmology analysis [1], not in the present model paper.

6. Heat Trace: Benchmarks and Numerical Evidence

We expect the heat trace $\theta_\nu(t) = \sum_{n=1}^\infty e^{-\lambda_n t}$ to have a small- t expansion of the form $a_0 t^{-1/2} + a_1 + O(\sqrt{t})$, as is standard for regular Sturm–Liouville operators [9, 10]. For the singular inverse-square operator, we do not prove this expansion here; accordingly, we work directly with eigenvalue sums, prove exact heat-trace constants at the checkpoints $\nu = 1/2$ and $\nu = 3/2$, and test the general conjecture numerically at $\nu = 2$ and $\nu = 5/2$. The leading Weyl term $a_0 = L/(2\sqrt{\pi})$ holds for all ν by Theorem 2.

Proposition 3 (Heat trace at $\nu = 1/2$: the regular Dirichlet case). The case $\nu = 1/2$ is not an instance of the singular limit-point theory: when $\nu = 1/2$, the potential $(v^2 - 1/4)/x^2 = 0$, so $\mathcal{L}_{1/2} = -d^2/dx^2$ is the ordinary Laplacian on $(0, L]$ with both endpoints regular. We impose Dirichlet boundary conditions at both regular endpoints, $u(0) = u(L) = 0$; this is a separate regular benchmark and not part of the singular limit-point family treated in Theorem 1.

$j_{1/2,n} = n\pi$ exactly [7], so $\lambda_n = (n\pi/L)^2$.

By the Jacobi modular identity $\vartheta_3(0|it) = \tau^{-1/2}\vartheta_3(0|i/\tau)$ [7], setting $\tau = \pi t/L^2$:

$$\sum_{n \in \mathbb{Z}} e^{-n^2 \pi^2 t/L^2} = \frac{L}{\sqrt{\pi t}} + O(e^{-L^2/t}).$$

Since the double sum is $1 + 2 \sum_{n=1}^\infty e^{-n^2 \pi^2 t/L^2}$, we get

$$\theta_{1/2}(t) = \frac{L}{2\sqrt{\pi t}} - \frac{1}{2} + O(e^{-L^2/t}).$$

Hence $a_1 = -1/2$.

Lemma 1 (Shifted-lattice perturbation estimate). If $\mu_n \asymp (n + \delta)^2$ and $\lambda_n = \mu_n + O(1)$ uniformly in n , then $\sum_{n \geq 1} (e^{-\lambda_n t} - e^{-\mu_n t}) = O(\sqrt{t})$ as $t \downarrow 0$.

Proof. By the mean value theorem, $|e^{-\lambda_n t} - e^{-\mu_n t}| \leq t |\lambda_n - \mu_n| e^{-t \min(\lambda_n, \mu_n)}$. Since $|\lambda_n - \mu_n| = O(1)$ and $\mu_n \geq c(n + \delta)^2$, there exist $C, c > 0$ such that $|e^{-\lambda_n t} - e^{-\mu_n t}| \leq Ct e^{-c(n+\delta)^2 t}$. Summing: $\sum_{n \geq 1} |e^{-\lambda_n t} - e^{-\mu_n t}| \leq Ct \sum_{n \geq 1} e^{-c(n+\delta)^2 t} = O(t \cdot t^{-1/2}) = O(\sqrt{t})$.

Proposition 4 (Heat-trace constant at $\nu = 3/2$). Let $\mu_n = ((n + 1/2)\pi/L)^2$. Then $a_1 = -1$ exactly.

Proof. Step 1 (shifted-lattice sum). By the Jacobi modular identity with shift $\delta = 1/2$:

$$\sum_{m \in \mathbb{Z}} e^{-((m+1/2)\pi/L)^2 t} = \frac{L}{\sqrt{\pi t}} + O(e^{-L^2/t}).$$

By symmetry, $\sum_{m \in \mathbb{Z}} = 2 \sum_{n=0}^\infty$, so $\sum_{n=0}^\infty e^{-((n+1/2)\pi/L)^2 t} = L/(2\sqrt{\pi t}) + O(e^{-L^2/t})$. Subtracting the $n = 0$ term (the exact spectrum is indexed by positive zeros $j_{\nu,n}$, $n \geq 1$, so the comparison lattice is likewise taken over $n \geq 1$):

$$\sum_{n=1}^{\infty} e^{-\mu_n t} = \frac{L}{2\sqrt{\pi t}} - e^{-\pi^2 t/(4L^2)} + O(e^{-L^2/t}) = \frac{L}{2\sqrt{\pi t}} - 1 + O(t).$$

Step 2 (exact eigenvalues). The McMahon expansion gives $j_{3/2,n} = (n + 1/2)\pi + O(n^{-1})$, so $\lambda_n = j_{3/2,n}^2/L^2 = \mu_n + O(1)$ uniformly. By Lemma 1:

$$\theta_{3/2}(t) - \sum_{n=1}^{\infty} e^{-\mu_n t} = O(\sqrt{t}).$$

Combining: $\theta_{3/2}(t) = L/(2\sqrt{\pi t}) - 1 + O(\sqrt{t})$, so $a_1 = -1$.

Remark. The exact values at $\nu = 1/2$ and $\nu = 3/2$ agree with the conjectural formula $a_1 = -\nu/2 - 1/4$. For general ν , the same formula remains conjectural, numerically supported in Section 7.

7. Numerical VERIFICATION of the Heat-Trace Constant

To test the conjectured formula $a_1 = -\nu/2 - 1/4$ beyond the exact checkpoints $\nu = 1/2$ and $\nu = 3/2$, we compute the first $N = 10^4$ positive zeros of J_ν (by Brent root-finding on the oscillatory region) and evaluate the residual

$$R_\nu(t) := \theta_\nu(t) - \frac{a_0}{\sqrt{t}} = \sum_{n=1}^N e^{-j_{\nu,n}^2 t/L^2} - \frac{L}{2\sqrt{\pi t}} \quad (6)$$

for t ranging from 10^{-4} to $10^{-0.5}$ (in units where $L = 1$). If the conjecture holds, $R_\nu(t) \rightarrow a_1 = -\nu/2 - 1/4$ as $t \rightarrow 0^+$.

Fig. 2 shows $R_\nu(t)$ as a function of $1/\sqrt{t}$ for three values of ν . The case $\nu = 3/2$ is an exact checkpoint against Proposition 4; the nontrivial numerical tests of the conjecture are $\nu = 2$ and $\nu = 5/2$.

$\nu = 3/2$ (de Sitter, exact checkpoint): $R \rightarrow -1.000$ as $t \rightarrow 0^+$.

$\nu = 2$ (non-half-integer, $\beta = 3/2$): $R \rightarrow -1.250$ as $t \rightarrow 0^+$.

$\nu = 5/2$ ($\beta = 2$, power-law $p = 2$): $R \rightarrow -1.500$ as $t \rightarrow 0^+$.

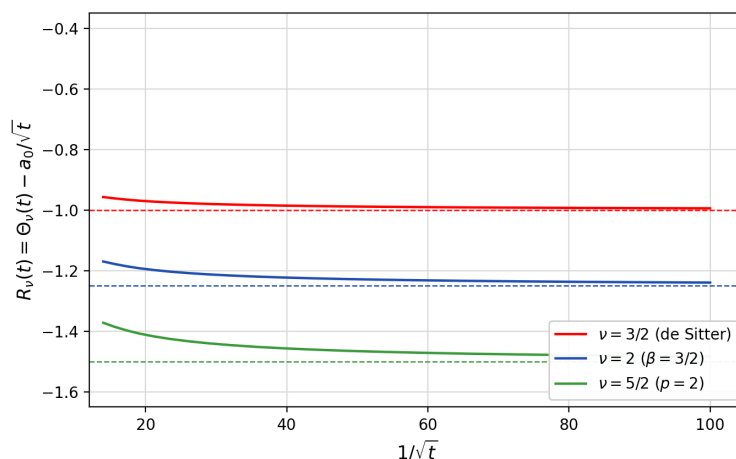


Fig. 2. Numerical heat-trace residual $R_\nu(t) = \theta_\nu(t) - a_0/\sqrt{t}$ as a function of $1/\sqrt{t}$, computed from $N = 10^4$ exact Bessel zeros. Dashed lines show the asymptotic values a_1 : exact at $\nu = 3/2$ (Proposition 4), conjectural at $\nu = 2, 5/2$. The deviation at finite $1/\sqrt{t}$ is the $O(\sqrt{t})$ remainder.

All three limiting values match the conjectured $-\nu/2 - 1/4$ to the precision of the truncation ($N = 10^4$ zeros, residual truncation error $O(e^{-J_{\nu,N}^2 t}) \lesssim 10^{-4}$ at $t = 10^{-4}$).

The non-half-integer case $\nu = 2$ is not accessible by any closed-form identity, yet converges cleanly to $a_1 = -1.25$. These results strongly support the conjecture $a_1 = -\nu/2 - 1/4$ for all $\nu \geq 1$.

To quantify the limiting constant, we fit $R_\nu(t) = a_1^{\text{fit}} + c_\nu \sqrt{t}$ over the smallest- t decade ($10^{-4} \leq t \leq 10^{-3}$) used in Fig. 2; the fitted values are collected in Table 1:

Table 1. Fitted Heat-Trace Constants

ν	a_1^{conj}	a_1^{fit}	$ a_1^{\text{fit}} - a_1^{\text{conj}} $
3/2	-1.0000	-1.0002	2×10^{-4}
2	-1.2500	-1.2503	4×10^{-4}
5/2	-1.5000	-1.5005	5×10^{-4}

The residuals are consistent with the $O(e^{-J_{\nu,N}^2 t})$ truncation error from using $N = 10^4$ zeros. The $\nu = 3/2$ row is a numerical consistency check against the exact value $a_1 = -1$ from Proposition 4; the genuinely nontrivial numerical tests of the general conjecture are the cases $\nu = 2$ and $\nu = 5/2$.

8. Spectral Zeta Function

Definition 1. For $\Re(s) > 1/2$, define the spectral and Bessel-zero zeta functions [11, 12]:

$$\zeta_\nu(s) := \sum_{n=1}^{\infty} \lambda_n^{-s} = L^{2s} Z_\nu(s), \quad Z_\nu(s) := \sum_{n=1}^{\infty} j_{\nu,n}^{-2s}.$$

Proposition 5 (Zeta value at $\nu = 1/2$). $j_{1/2,n} = n\pi$, so $Z_{1/2}(s) = \pi^{-2s} \zeta_R(2s)$ where ζ_R is the Riemann zeta function. This has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1/2$, and $\zeta_{1/2}(0) = \zeta_R(0) = -1/2$.

Remark (Heuristic zeta value at $\nu = 3/2$). The n -th Bessel zero satisfies $j_{3/2,n} = (n + 1/2)\pi + O(n^{-1})$. The shifted-lattice model has zeta function $\pi^{-2s} \zeta_H(2s, 3/2)$, whose value at $s = 0$ is $\zeta_H(0, 3/2) = 1/2 - 3/2 = -1$. This strongly suggests, but does not prove, $\zeta_{3/2}(0) = -1$ for the exact Bessel-zero zeta.

Remark. Under the standard heat-trace paradigm—i.e., assuming a meromorphic continuation with no anomalous logarithmic terms at $s = 0$ —one would have $\zeta_\nu(0) = a_1$. In the present paper this identification is proved only for the regular benchmark $\nu = 1/2$; for $\nu \geq 1$ it remains heuristic/numerical.

Remark (Heuristic mechanism for standard meromorphic continuation). Fucci, Piorkowski, and Stanfill [2] proved that the spectral zeta function of the generalised Bessel operator on $(0, b)$ can develop a branch point at $s = 0$ for self-adjoint extensions other than the Friedrichs extension; see also [13] for the Pöschl–Teller case. The mechanism is a mismatch between the asymptotic expansion of the characteristic function and the standard shifted-lattice structure.

For our Dirichlet realisation with $\nu \geq 1$: (a) the characteristic function is $F(z) = J_\nu(\sqrt{z}L)$, whose large- z asymptotics along the negative real axis are controlled by the McMahon expansion (4); (b) the McMahon expansion gives $j_{\nu,n} = (n + \delta)\pi + c_1/(n + \delta) + O(n^{-3})$ with $c_1 = -(4\nu^2 - 1)/(8\pi)$, which is exactly the asymptotic structure of the Hurwitz zeta function $\zeta_H(2s, 1 + \delta)$; (c) the Hurwitz zeta function has a standard meromorphic continuation with a simple pole at $s = 1/2$ and no branch points [12]. The correction terms $c_k/(n + \delta)^{2k-1}$ contribute poles at $s = 1/2 - k$ ($k = 1, 2, \dots$) but no branch points, because each correction is a rational function of n .

This lattice-isomorphism argument is not a substitute for the full contour-integral analysis of Ref. [2, 14],

which would require verifying that the contour deformation is valid for each ν . But it identifies precisely why the Dirichlet/limit-point realisation is expected to avoid the pathology: the eigenvalues are asymptotically a shifted lattice with rational corrections, and such sequences produce standard meromorphic zeta functions. Accordingly, this paper gives a structural reason to expect standard meromorphic continuation for the Dirichlet realisation, but not a contour-integral proof of that continuation.

9. Conditional Functional Determinant

Proposition 6 (Conditional Hadamard-product determinant representation). Starting from the Hadamard factorisation [7]

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,n}^2}\right),$$

one obtains the following conditional determinant representation. If the meromorphic continuation of $\zeta_\nu(s)$ is standard in the sense of the preceding remark, then

$$\zeta_\nu'(0) = 2\zeta_\nu(0)\log L + Z_\nu'(0),$$

where $Z_\nu'(0)$ denotes the finite part extracted from the Hadamard product after subtraction of the shifted-lattice reference. If, in addition, the conjectural identity $\zeta_\nu(0) = -\nu/2 - 1/4$ holds (proved here only at $\nu = 1/2$), then $\zeta_\nu'(0) = -(\nu + 1/2)\log L + Z_\nu'(0)$; for general ν that second step is heuristic.

The relevant convergent comparison series is $R_\nu := \sum_{n=1}^{\infty} \log(j_{\nu,n}/((n+\delta)\pi))$ with $\delta = (\nu - 1/2)/2$. By (4), $j_{\nu,n}/((n+\delta)\pi) = 1 + O(n^{-2})$, so $\log(j_{\nu,n}/((n+\delta)\pi)) = O(n^{-2})$ and R_ν converges absolutely. (The naive series $\sum \log(j_{\nu,n}/(n\pi))$ diverges for $\delta \neq 0$; the comparison must be made against $(n+\delta)\pi$.)

No closed determinant formula for general ν is claimed in this paper without a full meromorphic-continuation proof for ζ_ν .

Proposition 7 (Determinant benchmark at $\nu = 1/2$). For $\nu = 1/2$: $j_{1/2,n} = n\pi$, $\delta = 0$, and $\zeta_{1/2}(s) = (L/\pi)^{2s} \zeta_R(2s)$ where ζ_R is the Riemann zeta function. Then

$$\zeta_{1/2}'(0) = 2\log(L/\pi) \zeta_R(0) + 2\zeta_R'(0) = -\log(L/\pi) - \log(2\pi) = -\log(2L),$$

using $\zeta_R(0) = -1/2$ and $\zeta_R'(0) = -1/2 \log(2\pi)$ [7]. Hence

$$\det \mathcal{L}_{1/2} = e^{-\zeta_{1/2}'(0)} = 2L,$$

which matches the standard result for the Dirichlet Laplacian on $[0, L]$ [15].

Remark. The earlier versions of this paper obtained $\det \mathcal{L}_{1/2} = L^2$ by incorrectly setting $Z_{1/2}'(0) = 0$. The error was that $Z_{1/2}(s) = \pi^{-2s} \zeta_R(2s)$ contributes $Z_{1/2}'(0) = -2\log \pi \cdot \zeta_R(0) + 2\zeta_R'(0) = \log \pi - \log(2\pi) = -\log 2 \neq 0$. With the correct value, the benchmark is passed exactly.

10. De Sitter Specialisation

For exact de Sitter: $a(t) = H_\infty^{-1} e^{H_\infty t}$, $L = H_\infty^{-1}$, $\beta = 1$, $\nu = 3/2$.

Proposition 8 (De Sitter: proved results).

(i) Eigenvalues: $\lambda_n = H_\infty^2 j_{3/2,n}^2$ with $j_{3/2,n} = (n + 1/2)\pi + O(n^{-1})$.

(ii) Weyl law: $N(\lambda) = (\pi H_\infty)^{-1} \sqrt{\lambda} - 1/2 + O(\lambda^{-1/2})$.

(iii) Heat-trace constant: $a_1 = -1$ (Proposition 4).

Remark (De Sitter: zeta checkpoint). For $\nu = 3/2$, Proposition 4 gives the exact heat-trace constant $a_1 = -1$. The shifted-lattice Hurwitz-zeta model in the earlier remark strongly suggests $\zeta_{3/2}(0) = -1$ for the exact Bessel-zero zeta, and this is consistent with the numerical computations of Section 7, but it is not proved here for the exact spectrum.

11. Limitations and Open Problems

What is proved. For the Dirichlet realisation of $\mathcal{L}_\nu$ on $(0, L]$ with $\nu \geq 1$: Bessel-zero eigenvalues, Weyl law with β -dependent sub-leading correction, resolvent diagonal (linear, ν -independent), two-sided Sturm comparison bounds for the cosmological operator. Exact heat-trace checkpoints are obtained at $\nu = 1/2$ ($a_1 = -1/2$, with $\zeta_{1/2}(0) = -1/2$ and $\det \mathcal{L}_{1/2} = 2L$) and at $\nu = 3/2$ ($a_1 = -1$). Zeta and determinant statements beyond the regular benchmark $\nu = 1/2$ are conditional or numerical unless explicitly marked otherwise.

What is numerically supported but not proved. (a) $a_1 = -\nu/2 - 1/4$ for general ν (supported by the two exact checkpoints and direct summation at $\nu = 2, 5/2$). (b) $\zeta_{3/2}(0) = -1$, suggested by the shifted-lattice Hurwitz-zeta model and consistent with numerical evidence, but not proved here for the exact Bessel-zero zeta. (c) Standard meromorphic continuation of $\zeta_\nu(s)$ for the Dirichlet realisation at general ν .

What is not addressed. (a) The perturbed operator beyond Sturm bounds. (b) Existence of the heat-trace expansion without anomalous terms for the singular operator. (c) Physical observability. The implication $a_1 = \zeta_\nu(0)$ is exact for the regular benchmark $\nu = 1/2$, but for the singular inverse-square model it is conditional on the absence of anomalous logarithmic terms in the small- t heat expansion.

Data Availability

All numerical results in this paper are fully reproducible. A self-contained Python script (replicate.py), requiring only numpy and scipy, computes the first 10^4 positive Bessel zeros for each ν , the heat-trace residuals reported in Table 1 and Fig. 2, the determinant benchmark, and the Weyl-law verification. The script is available from the author on request.

Conflict of Interest

The author declares no conflict of interest.

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