Determining the Structures of the Differential Equations Admitting Multi-parameter Special Symmetries

Mehmet Pakdemirli
Department of Mechanical Engineering, Manisa Celal Bayar University, Yunusemre, Manisa, Turkey.

Email: pakdemirli@gmail.com
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Abstract: Special Lie group of transformations are considered. Multi-parameter transformations of translational, scaling, spiral and combined types are employed. Both ordinary and partial differential equations are considered. The general forms of the differential equations admitting such special transformations are given by theorems. The theorems are proven and sample differential equations are discussed as direct applications of the theorems. In this paper, we conduct the first systematic classification of multi-parameter special symmetries for ODEs and PDEs, and derive the general forms of differential equations admitting multi-parameter translational, scaling, spiral, and combined transformations, which fill the gap that the existing single-parameter symmetry classification cannot cover complex multi-parameter physical scenarios.

Keywords: differential equations, special lie group of transformations

1. Introduction

Lie group theory is a unified and systematic way of constructing solutions for differential equations [1–8]. The method can be virtually employed to any kind of differential equation, whether the equation is linear or nonlinear, ordinary or partial, constant coefficient or variable coefficient, homogenous or non-homogenous etc. Many of the different ad hoc techniques developed for solving differential equations has strong links to the theory and can be identified as special transformations within the theory. The general Lie group theory however requires extensive computations and the theory is much involved. On the contrary, special group transformations admitted by the differential equations are simple and easy to apply which does not require extensive algebra. Translational symmetries, scaling symmetries, spiral group symmetries are among the most common symmetries that are inherited by the differential equations. It is an easy task to check whether such symmetries are admitted by the given equation or not. The direct method is to apply the special transformation to the given specific equation and then construct the solutions based on the symmetries. The indirect or reverse method is to determine the most general form of a differential equation that is admitted by a given special transformation.

For the direct method, as far as applications of special Lie groups of transformations to physical problems are considered, see [9–21].

As far as the indirect method is considered, tables are provided for the general structure of the ordinary differential equations admitting some special transformations. Those tables are given separately for first order and second order ordinary differential equations [6]. The transformations are single-parameter Lie group of transformations. In this work, the special transformations are multi-parameter and the general structures of the equations are determined for such multi-parameter transformations for the first time. In addition to ODEs, PDEs are also considered as well. As an example, the one parameter uniform scaling is classified in the mentioned tables whereas here, two-parameter non-uniform scaling transformations for ODEs and three-parameter scaling transformations for PDEs are considered.

In physical problems, multi-parameter transformations occur frequently. For example, for boundary layer equations of Newtonian fluids, two-finite parameter scaling Lie group of transformations and one-parameter translational symmetries in the horizontal direction and an infinite parameter translational symmetry in the vertical coordinate appears [22]. For the famous Korteweg-de-Vries equation describing shallow water waves, two parameter translational symmetries exist leading to the travelling wave solutions [8]. The stationary transonic gas flow equation admits two-parameter scaling transformations and three parameter translational symmetries [4]. A multi-parameter analysis will then be more informative to describe the general form of the differential equation admitting such symmetries.

While single parameter classifications are reported for ordinary differential equations of first and second orders, classifications for partial differential equations are not given due to their more general complex forms. In this study, for a scalar PDE of second order, classification result has been given for scaling symmetries for the first time.

2. Special Symmetries

Multi-parameter special groups of transformations, namely the translational, scaling, spiral and a combination of all are considered. For a general k'th order ordinary differential equation of the form

$$F\left(x, y, y', y'', \ldots, y^{(k)}\right) = 0, \tag{1}$$

the 2-parameter translations is

$$x^* = x + \epsilon a\,, \; y^* = y + \epsilon b\,, \tag{2}$$

where ϵ is the continuous Lie group parameter and a and b are some constant parameters. Due to the two parameters a and b, the translational symmetry is two-parameter Lie group of transformations. If one of the parameters is zero, then one can speak of a single parameter transformation. For a k'th order partial differential equation of two independent (x,y) and one dependent variable (u=u(x,y))

$$F\left(x, y, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\right) = 0, \tag{3}$$

where $u_{(i)}$, $i = 1,2, \ldots k$ stands for all i'th order derivatives, the 3-parameter translations is

$$x^* = x + \epsilon a, \quad y^* = y + \epsilon b, \; u^* = u + \epsilon c\,, \tag{4}$$

for the group parameter ϵ and for some constants a, b and c. This transformation corresponds to the rigid translation of the coordinate system. Geometrically, it does not change the shape and gradient of the solution of the equation, but only changes the position of the solution in the coordinate system.

Scaling symmetries express geometrical reductions/extensions in size. Two similar triangles are the consequence of uniform scaling transformations. For non-uniform scalings, the size reductions/extensions are not the same in different coordinate directions. This transformation corresponds to the uniform/non-uniform stretching of the coordinate system, which geometrically describes the self-similarity of the solution and is the core basis of similarity solution. The two-parameter non-uniform scaling transformations for ODE given in Eq. (1) can be written as

$$x^* = \lambda^a x, y^* = \lambda^b y, \tag{5}$$

with $\lambda = 1 + \epsilon$. Alternatively the scaling symmetries can be expressed as

$$x^* = e^{\epsilon a} x, y^* = e^{\epsilon b} y, \tag{6}$$

where ϵ is the continuous group transformation parameter. For the PDE given in Eq. (3), the 3-parameter scalings may be

$$x^* = e^{\epsilon a} x, y^* = e^{\epsilon b} y, u^* = e^{\epsilon c} u. \tag{7}$$

This transformation is a combination of translation and scaling. In polar coordinates, it corresponds to the coupling of angular rotation and radial stretching, and can geometrically describe the self-similar solution structure of a spiral shape. Sample spiral transformations for the ODE given in Eq. (1) may be

$$x^* = x + \epsilon a, y^* = e^{\epsilon b} y, \tag{8}$$

or alternatively,

$$x^* = e^{\epsilon a} x, y^* = y + \epsilon b\,, \tag{9}$$

One sample spiral group for the PDE given in Eq. (3) may be

$$x^* = x + \epsilon a, y^* = e^{\epsilon b} y, u^* = e^{\epsilon c} u. \tag{10}$$

Many other spiral group alternatives can be written by imposing translations to some of the coordinates and scalings to the remaining ones.

Finally, all previous special transformations can be generalized in a 6-parameter Lie group of transformations for the PDE given in Eq. (3) which may be called combined transformations [9].

$$x^* = e^{\epsilon a} x + \epsilon b, y^* = e^{\epsilon c} y + \epsilon d, u^* = e^{\epsilon e} u + \epsilon f. \tag{11}$$

If $a = c = e = 0$, the transformation is purely translational, if $b = d = f = 0$, the transformation is purely scaling. There are many combinations of different spiral symmetries, three of such are $a = d = f = 0$, $b = c = f = 0$, and $b = d = e = 0$. This transformation is the fully coupled form of translation and scaling, which can cover the widest range of linear coordinate transformation scenarios and is suitable for symmetry analysis of complex multi-physics field equations.

Note that since the variables are two in scalar ODEs (one dependent and one independent variables), the special transformations may be two in maximum. However, if there is a coupled system of ODE with two dependent variables, then the multi-parameter transformation can be maximum three and so on. In general, for a PDE with m independent variables and n dependent variables, each variable can independently correspond to one transformation parameter, so the maximum dimension of the multi-parameter transformation is m+n.

3. Theorems for Determining the General Structure of the Differential Equations

Theorems that determine the most general forms of differential equations for the given specific symmetry will be posed and proven. The theorems may guide the researchers about the special symmetry admitted by the equation without actually performing the calculations.

Theorem 1. The nonlinear ordinary differential equation

$$F\left(y', y'', \dots, y^{(k)}\right) = 0, \tag{12}$$

with $y^{(i)}=\frac{d^i y}{dx^i}$, $y^{(0)}=y$, $i=0,1,2,\ldots k$, admits the two-parameter translational symmetries

$$x^*=x+\epsilon a,\, y^*=y+\epsilon b \tag{13}$$

Proof

Since the derivatives are transformed as

$$y^{*(i)}=y^{(i)},\;\; i=1,2,\ldots,k, \tag{14}$$

the equation takes the form

$$F\left(y^{*\prime},y^{*\prime\prime},\ldots,y^{*(k)}\right)=0, \tag{15}$$

which indicates that the equation is invariant (unchanged) under the given transformation. In fact, if a variable is missing in the equations, then translational symmetry of the missing variable is admitted by the equation. For example, for the differential equation $F\left(y,y',y'',\ldots,y^{(k)}\right)=0$, the translational symmetry is one-parameter, i.e., $x^*=x+\epsilon a$, $y^*=y$. On the contrary, it may happen that y coordinate is missing in the differential equation $F\left(x,y',y'',\ldots,y^{(k)}\right)=0$, which results in the admittance of the group $x^*=x$, $y^*=y+\epsilon a$.

Theorem 2. If the nonlinear ordinary differential equation

$$F\left(x,y,y',y'',\ldots,y^{(k)}\right)=0, \tag{16}$$

accepts translational symmetries

$$x^*=x+\epsilon a,\, y^*=y+\epsilon b, \tag{17}$$

then the equation can be cast into the following form

$$F\left(y-\frac{b}{a}x,y',y'',\ldots,y^{(k)}\right)=0. \tag{18}$$

For the special case of $b=a$, the form is

$$F\left(y-x,y',y'',\ldots,y^{(k)}\right)=0 \tag{19}$$

In fact, a general way of proving the above theorem exists and it will be similar to the one given in the next theorem and therefore skipped. Eq. (18) is valid only for two parameter translational symmetries to avoid any singularities.

Theorem 3. If the nonlinear ordinary differential equation

$$F\left(x,y,y',y'',\ldots,y^{(k)}\right)=0, \tag{20}$$

accepts the scaling transformations

$$x^*=e^{\epsilon a}x,\, y^*=e^{\epsilon b}y, \tag{21}$$

then the equation can be cast into the following form

$$F\left(x^{-b/a}y,x^{1-b/a}y',x^{2-b/a}y'',\ldots,x^{k-b/a}y^{(k)}\right)=0 \tag{22}$$

Proof

Substitute the scaling transformations to the original Eq. (20),

$$F\left(e^{-\epsilon a}x^*,e^{-\epsilon b}y^*,e^{\epsilon(a-b)}y^{*\prime},\ldots,e^{\epsilon(ka-b)}y^{*(k)}\right)=0. \tag{23}$$

It is clear that for an arbitrary F function, the invariance requires $a=b=0$, hence the most general form does not accept scaling symmetries. However, for some special forms of F, the scaling symmetry may be admitted by the equation. To find this special form, differentiate F with respect to the group parameter ϵ and evaluate the derivatives at $\epsilon = 0$

$$\left[F_x \frac{dx}{d\epsilon} + F_y \frac{dy}{d\epsilon} + F_{y'} \frac{dy'}{d\epsilon} + \cdots + F_{y^{(k)}} \frac{dy^{(k)}}{d\epsilon}\right]_{\epsilon=0} = 0, \tag{24}$$

or since $\left.\frac{dx}{d\epsilon}\right|_{\epsilon=0} = -ax, \left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = -by, \left.\frac{dy'}{d\epsilon}\right|_{\epsilon=0} = -(b-a)y', \left.\frac{dy^{(k)}}{d\epsilon}\right|_{\epsilon=0} = -(b-ka)y^{(k)}$, the equation takes the form

$$axF_x + byF_y + (b-a)y'F_{y'} + \cdots + (b-ka)y^{(k)}F_{y^{(k)}} = 0. \tag{25}$$

The above equation is a first order PDE which can be solved by the method of characteristics

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{dy'}{(b-a)y'} = \cdots = \frac{dy^{(k)}}{(b-ka)y^{(k)}} = \frac{dF}{0}. \tag{26}$$

Taking the first two equations, $\frac{dx}{ax} = \frac{dy}{by}$ and integrating, the first invariant is $x^{-b/a}y$. Taking the first and third equation $\frac{dx}{ax} = \frac{dy'}{(b-a)y'}$ and integrating, the second invariant is $x^{1-b/a}y'$ and so on. Substituting the invariants back into the original equation

$$F\left(x^{-b/a}y, x^{1-b/a}y', x^{2-b/a}y'', \ldots, x^{k-b/a}y^{(k)}\right) = 0. \tag{27}$$

For uniform scaling symmetry, $b = a$ and the above form reduces to

$$F\left(x^{-1}y, y', xy'', \ldots, x^{k-1}y^{(k)}\right) = 0. \tag{28}$$

In Eq. (22) note that $a \neq 0$ to avoid any singularity. The following theorem is about the combined symmetries.

Theorem 4. If the nonlinear ordinary differential equation

$$F\left(x, y, y', y'', \ldots, y^{(k)}\right) = 0, \tag{29}$$

accepts a combination of scaling, spiral and translational symmetries

$$x^* = e^{\epsilon a}x + \epsilon b\,, \quad y^* = e^{\epsilon c}y + \epsilon d, \tag{30}$$

then the equation can be cast into the following form

$$F\left((cy+d)(ax+b)^{-c/a}, (ax+b)^{1-c/a}y', \ldots, (ax+b)^{k-c/a}y^{(k)}\right) = 0 \tag{31}$$

Proof

The steps of proof are as follows: i) The transformations are substituted into the original equation (See Eq. (32)). ii) The differential equation is differentiated with respect to the group parameter and the derivatives are evaluated at $\epsilon = 0$ (See Eqs. (33) and (34)). iii) The resulting linear first order PDE in terms of the variables (independent, dependent and derivatives) is solved by the method of characteristics (See Eqs. (35)–(37)). iv) Substituting the invariants back to the original equation completes the proof (See Eq. (38)).

As an initial step, substitute the transformations (30) into the original Eq. (29)

$$F\left(e^{-\epsilon a}(x^* - \epsilon b), e^{-\epsilon c}(y^* - \epsilon d), e^{\epsilon(a-c)}y^{*\prime}, \ldots, e^{\epsilon(ka-c)}y^{*(k)}\right) = 0. \tag{32}$$

It is clear that for an arbitrary F function, the invariance requires $a = b = c = d = 0$, hence the most general form does not accept combined symmetries. However, for some special forms of F, the combined symmetries may be admitted by the equation. To find this special form, differentiate F with respect to the group parameter ϵ and evaluate the derivatives at $\epsilon = 0$

$$\left[F_x \frac{dx}{d\epsilon} + F_y \frac{dy}{d\epsilon} + F_{y'} \frac{dy'}{d\epsilon} + \cdots + F_{y^{(k)}} \frac{dy^{(k)}}{d\epsilon}\right]_{\epsilon=0} = 0, \tag{33}$$

or since $\left.\frac{dx}{d\epsilon}\right|_{\epsilon=0} = -(ax+b)$, $\left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = -(cy+d)$, $\left.\frac{dy'}{d\epsilon}\right|_{\epsilon=0} = -(c-a)y'$, $\left.\frac{dy^{(k)}}{d\epsilon}\right|_{\epsilon=0} = -(c-ka)y^{(k)}$, the equation takes the form

$$(ax+b)F_x + (cy+d)F_y + (c-a)y'F_{y'} + \cdots + (c-ka)y^{(k)}F_{y^{(k)}} = 0. \tag{34}$$

The above equation can be solved by the method of characteristics

$$\frac{dx}{ax+b} = \frac{dy}{cy+d} = \frac{dy'}{(c-a)y'} = \cdots = \frac{dy^{(k)}}{(c-ka)y^{(k)}} = \frac{dF}{0}. \tag{35}$$

The first two equations $\frac{dx}{ax+b} = \frac{dy}{cy+d}$ lead to the invariant

$$v_0 = (cy+d)(ax+b)^{-c/a} \tag{36}$$

In general, the k'th order term and the first term $\frac{dx}{ax+b} = \frac{dy^{(k)}}{(c-ka)y^{(k)}}$ can be solved leading to the k'th invariant.

$$v_k = (ax+b)^{k-c/a}y^{(k)} \tag{37}$$

Finally, solving $\frac{dx}{ax+b} = \frac{dF}{0}$, $F = v_{k+1} = 0$ or the constant v_{k+1} can be expressed in terms of the other constants (invariants) $v_{k+1} = F(v_0, v_1, \dots . v_k)$ yielding

$$F\left((cy+d)(ax+b)^{-c/a}, (ax+b)^{1-c/a}y', \dots, (ax+b)^{k-c/a}y^{(k)}\right) = 0. \tag{38}$$

Note that $a \neq 0$ and $c \neq 0$ to prevent any singularity. Actually $a = c = 0$ corresponds to the translational symmetries the general form of which was already given in Theorem 2.

The following theorem determines the form of the equation for PDEs in case of scaling symmetry.

Theorem 5. If the second order nonlinear partial differential equation with two independent variables

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}\right) = 0, \tag{39}$$

admits three parameter non-uniform scaling symmetry

$$x^* = e^{\epsilon a}x, y^* = e^{\epsilon b}y, u^* = e^{\epsilon c}u, \tag{40}$$

then the equation can be cast into the following form

$$F\left(\frac{y}{x^{b/a}}, \frac{u}{x^{c/a}}, \frac{u_x}{x^{(c-a)/a}}, \frac{u_y}{x^{(c-b)/a}}, \frac{u_{xx}}{x^{(c-2a)/a}}, \frac{u_{yy}}{x^{(c-2b)/a}}, \frac{u_{xy}}{x^{(c-a-b)/a}}\right) = 0 \tag{41}$$

Proof follows similar steps with the previous theorems. If the transformation is uniform, then $a = b = c$ and Eq. (39) takes the simplified form

$$F\left(\frac{y}{x}, \frac{u}{x}, u_x, u_y, xu_{xx}, xu_{yy}, xu_{xy}\right) = 0. \tag{42}$$

If the transformation is non-uniform, for example, if $a = b = 1$ and $c = 2$, then from Eq. (41), the general form is

$$F\left(\frac{y}{x}, \frac{u}{x^2}, \frac{u_x}{x}, \frac{u_y}{x}, u_{xx}, u_{yy}, u_{xy}\right) = 0. \tag{43}$$

Theorem 5 can be extended in an analogous manner to cover n-independent variable and m-dependent variable PDEs. Such extensions are straightforward generalizations of the present analysis and hence are not given for brevity. Note that $a \neq 0$ in Eq. (41) to avoid singularities.

4. Applications of the Theorems

In this section, the applications of the theorems are given.

Application 1. From Theorem 1

$$y'y''' + y'' = 0, \tag{44}$$

admits translational symmetries in both x and y directions since both variables x and y are missing from the equation.

Application 2. From the special case of Theorem 1,

$$y'' + e^{y}\cos y' = 0, \tag{45}$$

admits translational symmetry in the x direction only since x is the missing variable in the equation.

Application 3. From Theorem 2, the equation $y'' + (y - 3x)y' + y = 3x$ accepts the special translational symmetry for $b = 3a$ since the equation is a function of $y - 3x$, y' and y'' which makes the equation a special form of Eq. (18).

Application 4. From Theorem 3, the ordinary differential equation

$$y''' - \frac{\tan y'^2}{x^2} = 0, \tag{46}$$

admits uniform scaling, i.e., $x^* = e^{\epsilon}x$,$y^* = e^{\epsilon}y$, since the equation can be written as

$$x^2y''' - \tan y'^2 = 0, \tag{47}$$

which is a function of x^2y''' and y', a special form of Eq. (28).

Application 5. To determine whether the equation

$$yy'' - x^2y'^3 = 0, \tag{48}$$

admits scaling symmetry and to determine the form of the symmetry, the general form given in Theorem 3 will be adapted. Multiply the equation by x^m

$$x^m yy'' - x^{m+2}y'^3 = 0, \tag{49}$$

since the equation may be divided or multiplied by an arbitrary power of x^m to resemble the canonical form. From Eq. (22), since the variables y and y'' accompany the powers $x^{-b/a}$, $x^{2-b/a}$, and y' accompany the power $x^{1-b/a}$, the powers should be equated between Eqs. (49) and (22)

$$-\frac{b}{a} + 2 - \frac{b}{a} = m\,, \quad 3\left(1 - \frac{b}{a}\right) = m + 2. \tag{50}$$

Solving the equations, $m = 4$ and $\frac{b}{a} = -1$ which lead to the scaling transformation $x^* = e^{\epsilon}x, y^* = e^{-\epsilon}y$ where parameter a may be augmented into the group parameter ϵ without loss of generality.

Application 6. From Theorem 4, the equation

$$(2x + 1)^2y'' + (2x + 1)y' - 4y = 3, \tag{51}$$

accepts a special form of the combined symmetries. To see this, take all terms to the left and divide by $(2x + 1)^2$

$$y'' + \frac{y'}{2x+1} - \frac{4y+3}{(2x+1)^2} = 0. \tag{52}$$

Comparing Eq. (52) with the general form Eq. (31), one may conclude that $a = 2, b = 1, c = 4, d = 3$. Hence the specific transformation admitted by the equation is

$$x^* = e^{2\epsilon}x + \epsilon, y^* = e^{4\epsilon}y + 3\epsilon, \tag{53}$$

which can be verified by direct substitution into the original equation.

Application 7. From Theorem 5, the nonlinear partial differential equation

$$u_y = uu_{xx}, \tag{54}$$

admits a special scaling symmetry. To determine the specific form of this symmetry, from Eq. (41).

$$\frac{u_y}{x^{(c-b)/a}} = \frac{u}{x^{c/a}} \frac{u_{xx}}{x^{(c-2a)/a}}, \tag{55}$$

which requires

$$\frac{c-b}{a} = \frac{c}{a} + \frac{c-2a}{a} \quad \text{or} \quad c = 2a - b, \tag{56}$$

with parameters a and b remaining arbitrary. With this choice of parameters, the denominators cancel and Eq. (55) looks exactly as Eq. (54). Hence the two-parameter scaling transformation is

$$x^* = e^{\epsilon a}x, y^* = e^{\epsilon b}y, u^* = e^{\epsilon(2a-b)}u. \tag{57}$$

This result can be verified by direct substitution of the general form of the scaling transformation

$$x^* = e^{\epsilon a}x, y^* = e^{\epsilon b}y, u^* = e^{\epsilon c}u, \tag{58}$$

into Eq. (54). Substituting $u = e^{-\epsilon c}u^*$, $u_y = e^{\epsilon(b-c)}u^*_{y^*}$, $u_{xx} = e^{\epsilon(2a-c)}u^*_{x^*x^*}$ the equation takes the form

$$e^{\epsilon(b-c)}u^*_{y^*} = e^{\epsilon(2a-2c)}u^*u^*_{x^*x^*}. \tag{59}$$

Multiplying the equation by $e^{\epsilon(c-b)}$, the transformed equation is

$$u^*_{y^*} = e^{\epsilon(2a-b-c)}u^*u^*_{x^*x^*}. \tag{60}$$

Hence, for Eq. (60) to look similar to Eq. (54), the invariance condition is

$$c = 2a - b \tag{61}$$

which verifies the transformation Eq. (57).

5. Concluding Remarks

In this work, the general forms of ODEs which admit translational, scaling, spiral and a combination of all of them are determined. For second order PDEs, the general form of the equations for admitting scaling symmetry is also given. The classifications in the literature are for one-parameter Lie group of transformations whereas here multi-parameter group of transformations are considered. Constructions of the group invariant solutions or reductions in the order of equations are straightforward and can be traced in any of the references given below.

The classification presented here with respect to multi-parameter groups has some limitations. The theorems in this paper assume $a \neq 0$ and $c \neq 0$ to avoid singularities. When $a = 0$ or $c = 0$, the transformation degenerates into pure translational symmetry, and the corresponding general form of the equation needs to be derived separately. Meanwhile, the theorems in this paper only cover scalar ODEs and binary second-order PDEs, and are not applicable to coupled systems of differential equations and higher-order PDEs for the time being.

In the future, the theorems can be extended to coupled systems of differential equations and higher-order multi-variable PDEs. Meanwhile, the equation structure of discrete multi-parameter symmetries and the symmetry classification under singular parameters can be further analyzed.

Conflict of Interest

The author declares no conflict of interest.

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