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Clifford Algebra of Projective Space $PG(3,2)$—Fano Tetrahedron Following a Conventional Quaternion Basis

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Abstract: In finite geometry the symbolic word $PG(3,2)$ denotes the smallest three-dimensional projective space. It can be assembled by four Fano planes $PG(2,2)$ that form a pyramid, the so called Fano tetrahedron. In this sense the $PG(3,2)$ is a geometric extension of the Fano plane $PG(2,2)$. In the Clifford algebra community, in order to extend the equations of motion in quantum physics to geometric algebras of quaternions and octonions, the Fano plane mnemonic has occasionally been used but not the Fano tetrahedron. This is still some undeveloped topic in geometric algebra although it deserves our full attention, as it is related in a surprising way to the group structures of Pauli algebra and some peculiar Hamiltonian group. The special relatedness of the three-dimensional projective space with the Clifford algebra of Minkowski space has suggested a physical question. Namely, is it possible that the hyperbolic transformation laws of Special Relativity (SR) follow from the arrangement of the observation of physical motion in projective Euclidean space (SRP connection)? I am not yet ready to accept this assumption. I still agree with the AI on this point[1]. But I decided to provide some Clifford algebraic tools that allow us to go deeper into this question and enable us to give precise answers. The SRP question was asked repeatedly on various oral occasions by Rolf Dahm. I assume that Dahm's considerations on the significance of projective geometry for high-energy physics may become relevant. Structures presented there can provide a new way of representing observations and degrees of freedom of particle states provided the appropriate Clifford map is used. At present all considerations are not really microscopic, but there applies to them a good old saying of Louis de Broglie: "The Theory of Relativity is the culmination of the older macroscopic Physics, while the Quantum Theory, on the other hand, has its origins in the investigation of the universe of corpuscules and atoms," and a few pages further down "there is an apparent Determinism in macroscopic phenomena, which in no way conflicts with a certain indeterminateness in phenomena on the microscopic scale".

Keywords: Clifford algebra, projective space, Fano tetrahedron, Fano plane, Minkowski algebra, Pauli algebra

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[1] Asking "Can the Lorentz transformation be traced back to projective geometry alone?" (Google search) the AI answered: The Lorentz transformation cannot be derived from projective geometry alone; it requires additional physical or geometric principles. However, there are deep mathematical connections, and the two are closely related through concepts like hyperbolic geometry and Möbius transformations, to which the Lorentz group is isomorphic.

1. Introduction

1.1. Motive

The Fano plane mnemonic is widely adopted in octonion and Clifford algebra studies [1, 2]. The article is aimed at mathematical Physicists who have seen the entry $PG(3,2)$ on the Fano-tetrahedron in Wikipedia but want to know more. There it is said that "In finite geometry, $PG(3,2)$ is the smallest three-dimensional projective space. It can be thought of as an extension of the Fano plane, $PG(2,2)$" [3]. It can indeed be assembled by four Fano planes $PG(2,2)$ that form a pyramid, the so called Fano tetrahedron. The Fano plane is sometimes used in octonion algebra, but not so the Fano tetrahedron. This is still some undeveloped topic in geometric algebra. But it deserves attention, as it is related in a surprising way to the group structures of Pauli algebra and some peculiar Hamiltonian group. Ultimately the Clifford map allows us to represent space-time spinors in projective 3-space.

1.2. Wayback Supplementaries

During several gatherings and visits by Rolf Dahm to my home in Vienna, the idea for a project "On Unifying $Cl_{3,1}$ and Geometry" emerged [4]. Fundamental to this should be my work on Majorana isospinor representations in the Clifford Algebra (CA) of Minkowski space [5, 6], a list of matrix double quaternions, by Dahm then called Quaternions [7], and a list of real- and imaginary-valued matrix units that may be dated back to Hermann Weyl. The supplementary data QQik_docxFin I had to trace back to Dahm's dissertation and way back further to Bjorken and Drell (1964/65) [8].

Using lists of two-fold quaternions QQik and such Weyl-like matrix units with a view on Dahm's "Complex Extensions of Quaternions" and "Identification of SU(4) Generators" in his Diss [9]. I calculated the first CA assignment and added a new matrix representation to the file QQik_docxFin. The matrices now had the familiar shape of graded multivectors in the complexified algebra $Cl_{3,1}$.

1.3. Trigonal Rotations

The Fano tetrahedron could be wonderfully represented using the Clifford algebra of Minkowski space. The algebraic properties of the structure of the Fano tetrahedron were astonishing. I succeeded in transforming the pyramid by some universal trigonal rotations, typical in the space of strongly interacting fermions. This can be done in such a way that the vertical axis leading from the apex through the barycenter into the middle of the base triangle connects the three time-like quaternion generating units e_4, e_{123}, e_{1234} which squared give the negative identity. Doing so, the projective space can be metaphorically called "synchronized" I am using the word synchronized here in an unconventional way, which is mainly due to the fact that the four-volume e_{1234} in relativity theory, whether special or general, is not conserved.

1.4. 3D Graphic Preliminaries

A most important contribution to the first part of this research work is Rolf Dahm's preliminary pictorial draft with QQs that arrived in Vienna on February 26th [10]. This is depicted in Fig. 1 (Section 2.3). With this assignment Dahm strictly preserves the Dirac algebra and defines spin following the convention given by Bjorken and Drell [8]. Referring to his table 12.16 in [9] Dahm writes: "However, the table immediately suggests replacing the initially selected base system $\mathfrak{Q}_{\alpha\beta}$ with a more favorable one." Such replacement is shown in a second part in which a precise definition of a Fermions Fano-line in accord with space-time isospinors is given. Before we do so, in this article we bring out the riches of the old convention that favors Dirac algebra.

2. Preliminary Concepts

2.1. Projective Space $PG(3,2)$

The symbolic word $PG(3{,}2)$ denotes the smallest three-dimensional projective space. It can be assembled by four Fano planes $PG(2{,}2)$ that form a pyramid, the so called Fano tetrahedron. It has 15 points, 35 lines, and 15 planes. Each point is contained in 7 lines and 7 planes. Each line is contained in 3 planes and contains 3 points. Each plane contains 7 points and 7 lines. It has the following properties:

Each plane is isomorphic to the Fano plane. Every pair of distinct planes intersects in a line. A line and a plane not containing the line intersect in exactly one point [3]. Kevin Lamoreau gives a beautiful tetrahedral depiction on Wikipedia https://en.wikipedia.org/w/index.php?title=PG(3,2)&oldid=1289967280/#Tetrahedral_depiction. In the coming sections we investigate representations and automorphisms of this tetrahedral structure in Clifford algebra.

2.2. Minkowski Algebra

Generally, we consider universal real Clifford algebra Cl_n with a Graßmann basis $\mathfrak{B}$, that is, some universal $Cl_n \simeq Cl(V,Q)$ of a non-degenerate quadratic real vector space of dimension n. In particular we select from this 'universe' the Minkowski algebra $Cl_{3,1} \simeq Cl_{2,2}$ as space-time algebra of physical space generated by the Minkowski space $V \simeq \mathbb{R}^{3,1}$ having signature $\{+++-\}$, shortly written as $\varepsilon = (3{,}1)$ (Appendix 1).

2.3. Key Operations

2.3.1. Wedge product

Geometric products have a complex history and they have an equally diverse appearance. In their present form they go back to Graßmann's work before 1844. Graßmann thought of an exterior product as an oriented operation on the most general higher-dimensional geometric forms [11–13]. Elements a, b, etc. are drawn from some n-dimensional vector space. On these acts Graßmann's exterior product such that an orientation is determined by anti-commutativity).

$$a\wedge b = -b\wedge a$$

$$a\wedge a = 0$$

In the visualization space, bivectors $a\wedge b$ can be regarded as oriented flat segments, e.g., such as parallelograms in Euclidean planes.

2.3.2. Clifford product

Clifford's geometric algebra uses the same basis like Graßmann's exterior algebra, but in addition introduces a new kind of product[2]. The Clifford product of two vectors and is obtained by adding the scalar $a \cdot b$ and the bivector $a\wedge b = 0$:

$$ab = a \cdot b + a\wedge b$$

Clifford applies Graßmann's approach [12, 13] by realizing this decomposition of the vector product in higher dimensional exterior algebraic spaces which are direct sums of their subspaces of homogeneous

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[2] "Following a suggestion of Professor Sylvester, I call that kind of multiplication in which the sign of the product is reversed by an interchange of two adjacent factors, polar multiplication; because the product ab has opposite properties at its two ends, so that $ab = -ba$. The ordinary or commutative multiplication I shall call scalar, being that which holds good of scalar numbers. These words answer to Graßmann's outer and inner multiplication; which names, however, do not describe the multiplication itself, but rather those geometrical circumstances to which it applies" [14].

degrees 0, 1, 2, a.s.o. While in Graßmann's approach unit line elements satisfy the multiplication rules

$$e_i \wedge e_j = -e_j \wedge e_i$$

$$e_i \wedge e_i = 0$$

Clifford keeps the first rule, $e_i e_j = -e_j e_i$, that is, $e_i e_j = e_i \wedge e_j$, but replaces the second by $e_i e_j = -1$ in 1878 [12] and $e_j e_i = +1$ in 1882 [14, 15].

From $ab = a \cdot b$ and $ba = a \cdot b - a \wedge b$ we can infer that two vectors are parallel $a \parallel b$ when they commute, and orthogonal $a \perp b$ when they anticommute.

In the Clifford algebra Cl_3 of 3-space, bivectors can be used to carry out reflections, i.e., a reflection of a general vector $r = xe_1 + ye_2 + ze_3$ in thexy-plane brings forth a vector $e_1 e_2 r e_1 e_2 = xe_1 + ye_2 - ze_3$. The trivector $e_{123} \overset{\text{def}}{=} e_1 e_2 e_3$ in such a representation is interpreted as an orientedunit volume. Squaring this trivector yields -1, and due to its property to commute with all elements of the algebra it is compared with the imaginary unit i. It is a pseudoscalar of this algebra. Note however, that the identification $e_1 e_2 e_3 = i$ that is made sometimes by a few authors is avoided. Complexification in the present context has the following meaning: The directed volume e_{123} commutes with the single vectors $e_1 e_2 e_3$. Therefore, every general element of the Clifford algebra having form $x + ye_{123}$ commutes with all the elements of Cl_3. The subalgebra of scalars and 3-vectors $\mathbb{R} \oplus \wedge^3 \mathbb{R}^3$ is the center of Cl_3. We write $e_{123}^2 = -Id$ and say: the center of Cl_3 is isomorphic to the complex field $Cen(Cl_3) = \mathbb{R} \oplus \wedge^3 \mathbb{R}^3 \simeq \mathbb{C}$.

The upcoming rigor is based on representation of matrix units by double quaternions (Appendix 2, Weyl matrices). This has been worked out by Rolf Dahm and was extended by me to the Clifford algebra of Minkowski space using matrices $\mathbb{C} \otimes Mat(4, \mathbb{R})$ in order to investigate $PG(3,2)$ by algebraic tools.

2.4. The Double Quaternion Pyramid

On 9th of March 2025, the suitable transformation then produced the following two images [16].

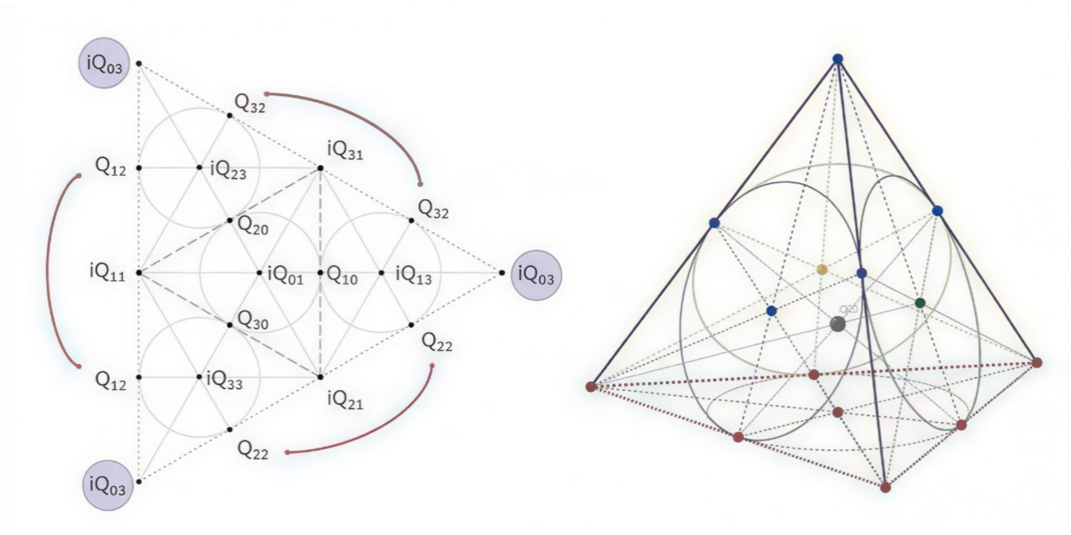

Fig. 1. From left to right "folding" of the fifteen SU^* or Segre operators $Q_{\alpha\beta}$ by arranging them into the Fano space $PG(3,2)$ with Q_{02} as barycenter (big dot in the right middle).

3. Clifford Algebra Representation of the Fano Tetrahedron

The automorphism of Segre operators $Q_{\alpha\beta}$ is presented in Fig. 2.

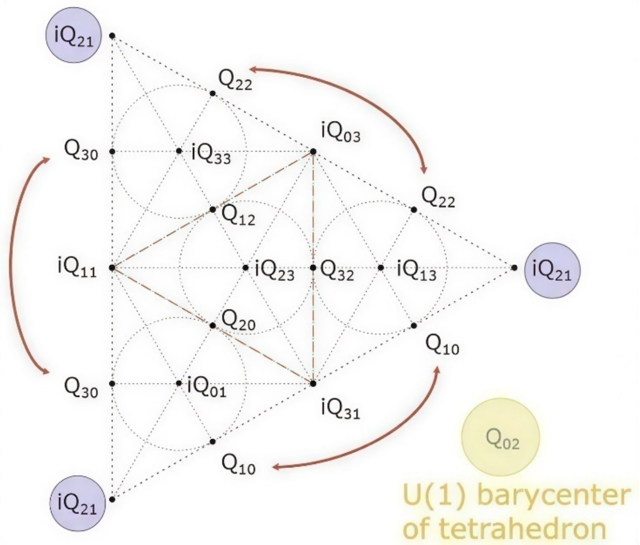

Fig. 2. Suitable automorphism of Segre operators $Q_{\alpha\beta}$.

The unitary space-time element representation in $\mathbb{C}\otimes Cl_{3,1}$ is shown in Fig. 3.

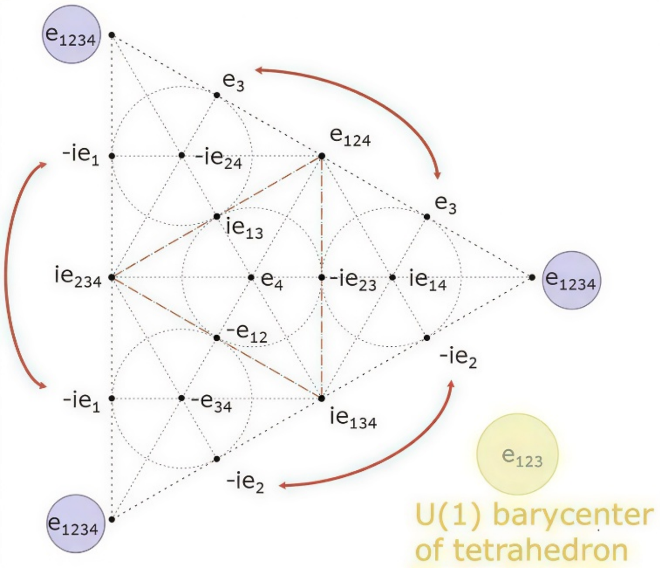

Fig. 3. Representation by unitary space-time elements of Clifford algebra $\mathbb{C}\otimes Cl_{3,1}$.

4. Clifford Product Featuring the Fano Tetrahedron

4.1. Algebraic Properties of the Fano Base

We focus our attention on the base of the Fano tetrahedron. The original tetrahedron is involuted such that the top vortex, represented by the time-like unit line-element e_4 becomes located in the center of the bottom triangle (Fig. 4). This triangle has three vertices, and in the center of each of the three edges are three more points. Together withe_4 the base thus consists of seven points which represent both geometric operations, geometric objects and algebraic Clifford numbers. The triangle is a Fano plane which, as a Clifford algebraic object, looks like that:

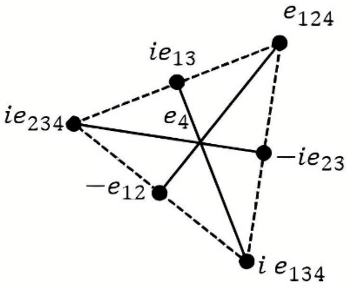

Fig. 4. Fano plane in the base as elements in $\mathbb{C}\otimes Cl_{3,1}$.

We retain the following features of the Clifford product. Consider elements $a, b \in Cl_{3,1}$ and Clifford product $a \wedge b$ $ab = a \cdot b + a \wedge b$.

feature 1 $a \parallel b \leftrightarrow [a, b]$ commutation

feature 2 $a \perp b \leftrightarrow \{a, b\}$= 0 anticommutation

Using a Clifford calculator, we first find out that any neighbors along the perimeter of the triangle are orthogonal to each other.

To put it bluntly, orthogonality at the outer closure contrasts with parallelism on the inside (Figs. 5 and 6).

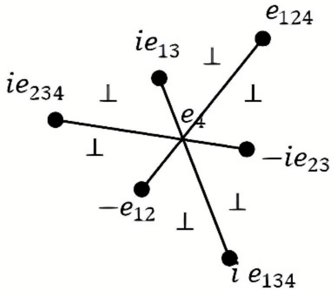

Fig. 5. Orthogonality along the perimeter.

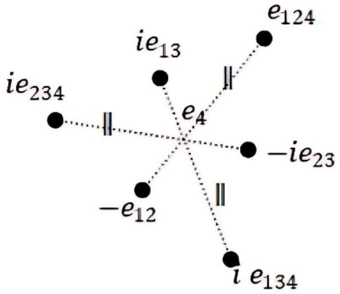

Fig. 6. Parallelism of midpoints along the lines across the center of the triangle.

4.2. Clifford Product and Universal Centrality in $PG(3, 2)$

The representation of the structure of the projective space $PG(3,2)$ using the Fano tetrahedron as in [1] is helpful in some respects, but should not suggest any confusion with the Euclidean tetrahedron. It is only a kind of metaphorical transposition for a better understanding of the geometric elements, which are points, lines and surfaces. It is all the more surprising that there is actually a perfect correlation between the concept of the geometric center, here as the center of the triangular lateral faces, and the algebraic concept of the center of a group or a subalgebra of the Clifford algebra. If we select the seven elements

$$\mathcal{F} \stackrel{\text{def}}{=} \{e_4, -e_{12}, ie_{1,3}, -ie_{2,3}, e_{124}, ie_{134}, ie_{234}\}$$

under Clifford multiplication they generate an algebra with central elements

$$Cen(\mathcal{F}) = \{aId + be_4\}, \qquad a, b \in \mathbb{R}$$

Note, if we add to the generators elements of form $\pm i \cdot Id$, with $= \sqrt{-1}$, all commutation- and anti-commutation relations among elements of $\mathcal{F}$ are preserved. The element e_4 commutes with the six other unit line- and space-time-elements in the base $\mathcal{F}$. So together with the identity Id it forms an algebraic center in parallel to six elements of the closure which are all orthogonal to each other. At the same time the time-line-element $\{e_4\}$ is located in the visual center of the bottom face of the Fano pyramid.

5. Group Structure of the Quaternions Fano Tetrahedron

I would now like to narrow our view in a helpful way, and look at the following downsized Table 1 which is generated by the elements of $\mathcal{F}$.

Table 1. Downsized multiplication table of the set $\mathcal{F}$

	$\boldsymbol{Id}$	$\boldsymbol{e_4}$	$\boldsymbol{-e_{12} = I}$	$\boldsymbol{ie_{13} = J}$	$\boldsymbol{-ie_{23} = K}$	$\boldsymbol{e_{124} = R}$	$\boldsymbol{ie_{134} = S}$	$\boldsymbol{ie_{234} = T}$
Id	Id	e_4	I	J	K	R	S	T
e_4	e_4	$-Id$	$-R$	S	$-T$	I	$-J$	K
I	I	$-R$	$-Id$	$-K$	J	e_4	T	K
J	J	S	K	Id	I	T	e_4	R
K	K	$-T$	$-J$	$-I$	Id	S	R	$-e_4$
R	R	I	e_4	$-T$	$-S$	Id	K	$-J$
S	S	$-J$	$-T$	e_4	$-R$	K	$-Id$	I
T	T	K	S	$-R$	$-e_4$	J	$-I$	$-Id$

5.1. The Clifford Fano Group

The Fano base given by above seven Clifford numbers generates a discrete multivector group by Clifford multiplication. Yet, this group is a bit larger than indicated by the multiplication table. The existence of entries $-Id$ implies that the set of all group elements is at least having size $\mathcal{F} \cup -\mathcal{F} \cup \{Id, -Id\}$. Is it larger than that? Obviously no product can bring on the single imaginary unit, nor spatial line elements e_1, e_2, e_3. The director e_{1234} of the Minkowski algebra is also not an element of the group, hence there is no way either to generate the barycenter e_{123}. Bottom line: the multivector group of the Fano base has sixteen elements. I call this group the Clifford-Fano group

$$Cf \leftrightharpoons \{\pm Id, \pm e_4, \pm e_{12}, \pm ie_{13}, \pm ie_{23}, \pm e_{124}, \pm ie_{134}, \pm ie_{234}\}$$

Let $I \stackrel{\text{def}}{=} -e_{12}, J \stackrel{\text{def}}{=} ie_{13}, K \stackrel{\text{def}}{=} -ie_{23}$ so that we get some new fourfolds satisfying multiplication rules

$$I^2 = -Id, J^2 = Id, K^2 = Id, IJ = -JI = -K, IK = -KI = J, JK = -KJ = I, IJK = -Id$$

In Ref. [17], we gave the algebraic entity the name conectarines, or co-quaternions.

5.2. Identification of Subgroups

So we have identified an important subgroup of the Clifford-Fano group as a group generated by algebraic numbers similar to the fourfold field of quaternions, namely the conectarines. In this set we count five elements of period 2, namely $\{-Id, J, K, -J, -K\}$, that give us 5. $\mathbb{Z}_2$ and two of period 4, $\{I, -I\}$, which belong to $\mathbb{Z}_4$. Together they form the group $G_{(1)} \overset{\text{def}}{=} \{\pm Id, \pm J, \pm K, \pm I\}$.

5.2.1. Dihedral groups D_4

This eight-element group is not the smallest Hamiltonian group (also having 8 elements) but the dihedral group $D_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$, the important thing to observe is that barycenter e_{123} is not the center of this slightly deviant co-quaternion group, as would be the case in the groups G_{16}^{13} or Hamiltonian group Q_8 generated by the Pauli matrices. Instead, the algebraic center is somewhere else, so to speak outsourced towards e_4. D_4 is the space-group of a square. It contains the eight symmetries that create congruence when a square can be rotated not only in one plane but also in 3D. The D_4 thus identified as G_1 is not the only one. But we also have the sets

$$G_2 = \{\pm Id, \pm J, \pm R, \pm T\} \simeq D_4$$

$$G_4 = \{\pm Id, \pm K, \pm R, \pm S\} \simeq D_4$$

So we have three subgroups of type dihedral D_4.

5.2.2. The group G_{16}^{13} generated by Pauli matrices

Considering the generating elements as Table 2 shows:

Table 2. Generating elements of group $G_{16}^{13}$and their order

1	4	4	2	2	2	4	4
Id	e_4	$-e_{12} = I$	$ie_{13} = J$	$-ie_{23} = K$	$e_{124} = R$	$ie_{134} = S$	$ie_{234} = T$

We identify four proper subgroups $\mathbb{Z}_4$ and, incorporating the period two element $-Id$, we get seventimes the $\mathbb{Z}_2$generated by the elements $-Id, J, -J, K, -K, R, -R$. The whole 16 element group has 7times the $\mathbb{Z}_2$ and 4 times the $\mathbb{Z}_4$ and 3 times D_4. This allows us to identify the group as isomorphic with G_{16}^{13}, the group generated by the Pauli matrices. In our habitual understanding this group contains an element e_{123} lying in the center. In our case this is replaced by the time-line-element e_4. Notice, the elements $\pm Id, \pm I, \pm J, \pm K$ have periods 2, 4, 2, 2 and squared give us signature $\{+,-,+,+\}$ while the elements $\pm Id, \pm R, \pm S, \pm T$ have periods 2, 2, 4, 4 and have signature $\{+,+,-,-\}$. The $\{\pm Id, \pm R, \pm S, \pm T\}$sends all its products back to $\{\pm Id, \pm I, \pm J, \pm K\}$. These unit space-time-elements complement the dihedral arrangement of the co-quaternions, and together they bring forth the 'projective center' of the time-line-unit e_4. Further rigor also shows that three faces of $PG(3,2)$ are associated with a peculiar Clifford representation of G_{16}^{13} located in $\mathbb{C} \otimes Cl_{3,1}$ [18]. The multivector group of these faces of the projective space $PG(3,2)$ is isomorphic with the group generated by the Pauli matrices.

5.2.3. The emergence of the Hamiltonian group $G_{16}^{12} \simeq Q_8 \times \mathbb{Z}_2$

One face however is a Hamiltonian group that contains 4 times Q_8 and 6 times $\mathbb{Z}_4$. So, let us consider the mapping from bottom to the right hand face (Table 3).

Table 3. Map of elements from bottom to right hand face of Fano pyramid

1	4	4	2	2	2	4	4
Id	e_4	$-e_{12} = I$	$ie_{13} = J$	$-ie_{23} = K$	$e_{124} = R$	$ie_{134} = S$	$ie_{234} = T$
↓	↓	↓	↓	↓	↓	↓	↓
Id	ie_{14}	ie_2	e_3	$-ie_{23} = K$	$e_{124} = R$	$ie_{134} = S$	e_{1234}

It is a group isomorphism. This mapping is created when we fold the right face of the tetrahedron around the edge $e_{124} \longrightarrow ie_{23} \longrightarrow ie_{134}$ so that it coincides with the base triangle. If we fold the right face into the base, we get isomorphism of the multivector groups of both triangles. Similar statements also apply to a second face (Fig. 1), namely when we fold the left face of the tetrahedron around the edge $e_{124}\longrightarrow$ ie_{13} $\longrightarrow$ ie_{234} so that it coincides with the base triangle.

Table 4. Map of elements from bottom to left hand face of Fano pyramid

1	**4**	**4**	**2**	**2**	**2**	**4**	**4**
Id	e_4	$-e_{12}$	$ie_{13} = J$	$-ie_{23}$	$e_{124} = R$	ie_{134}	$ie_{234} = T$
↓	↓	↓	↓	↓	↓	↓	↓
Id	$-ie_{24}$	$-ie_1$	$ie_{13} = J$	e_3	$e_{124} = R$	e_{1234}	$ie_{234} = T$

Now there is left the third side face whose operators have periods $\{2,4,4,4,4,4,4\}$ (Table 4). A little rigor clarifies, this brings forth a Hamiltonian group $G_{16}^{12} \simeq Q_8 \times \mathbb{Z}_2$ which contains 4 subgroups Q_8 having surprising 4D forms such like $\{\pm Id, \pm I, \pm S, \pm T\} \simeq Q_8$ and 7 subgroups $\mathbb{Z}_2$.

6. Summary

Following a convention by Bjorken and Drell, as Dahm had proposed, it is possible to represent the smallest three-dimensional projective space $PG(3,2)$ in the Clifford algebra $\mathbb{C}\otimes Cl_{3,1}$, hence in the matrix algebra $Mat(4,\mathbb{C})$. Given that Dahm's assignment (Fig. 1) of the double-quaternions to the points of the Fano tetrahedron is obliging, we have three faces that represent three different Fano groups Cf isomorphic with the group G_{16}^{13} generated by the Pauli matrices, and we have one exceptional face having a Hamiltonian multivector group G_{16}^{12}. This distinguishes a special position of the Fano tetrahedron and special automorphisms, where the time-like axis $e_4\longrightarrow$ e_{123} $\longrightarrow$ e_{1234} is perpendicular to the base and the purely spatial unit volume e_{123} forms the barycenter of the tetrahedron.

7. Limits

We know the Clifford algebra of Euclidean 3-space $\mathbb{R}^3$ is the Pauli algebra $Cl_{3,0}$ which can be generated by the Pauli matrices. We also know the Pauli algebra is so much more than Euclidean 3-space; it even incorporates $\mathbb{R}^{3,1}$ the Minkowski 4-space itself. Taking the generating line elements e_1, e_2, e_3 and adding to them the unit 3-volume e_{123}, we get a space with signature $\varepsilon = (3,1)$ isomorphic with the Minkowskispace in Lorentz metric. In such a picture the barycenter takes on the shape of frozen time. But the formal riches of Dirac algebras are even greater. In what Dahm called the $\pi N\Delta$-system, generators of the $SU(4,\mathbb{C})$ and non-compact $SU^*(4,\mathbb{C})$ can be identified. This naturally gives rise to a search for standard models of HEPhy. But what do we gain by lifting the quantum physics, in one way or another, into projective geometry, Plücker coordinates, or barycentric coordinates? The limits of both Relativity and Quantum Theory were already recognized by de Broglie and described in great detail over many chapters as well as in a few short lines such as: "Action is thus a magnitude depending simultaneously on the configuration of the system and on its dynamic state" [19]. "I have stated already that from the fact that the constant $h$has a finite value, there followed the impossibility of knowing simultaneously and exactly the values of two canonically conjugate variables" [19]. "To use Bohr's terminology, the exact localization in Space and Time is one 'idealization', and the concept of a completely defined state of motion is another, so that the two 'complementary idealizations', while almost quite compatible on the macroscopic scale, are not strictly so on the microscopic scale" [19]. It is my experience that tells me Special Relativity is neither replaced nor corrected, nor can be overridden or improved by projective geometry and group theoretical considerations alone.

8. Developments

These considerations on the significance of projective geometry for high-energy physics should promote the discourse. Theoretical Physics can possibly be promoted if we could show that symmetries of space-time are active in the inner of nucleons, or said differently, if it turned out that matter and light are space-time. Together with the paper Space-Time Isospinors in a Space of Matrix Units [4], the figures presented there provide a new way of representing the totality of observations and degrees of freedom of particle states with spin, isospin color and strangeness etc. in projective space. This would demand a slightly different Fano group, namely $\mathcal{F} = \{\pm Id, \pm e_4, \pm e_{12}, \pm ie_{13}, ie_{23}, \pm e_{124}, \pm e_{134}, \pm e_{234}\}$ that provides the necessary idempotents for the space-time group and legitimate subspaces as "real lines" of the Fano tetrahedron. A new progress in physics would require more than just a "farewell to old relations of Lie theory, classical geometry and gauge theory" [20], but rather an explication how and why the symmetries of macroscopic physics propagate into the interior of partons. After all it must be clarified if the existence of the Planck constant and Heisenberg's uncertainty can be a matter of the right selection of algebraic tools alone, as some of us seem to imagine. Although it can be considered correct that "the system space of quantum mechanics is a ray space", "not a vector space" [20] that does not explain the origin of Planck's constant and the associated quantization. It is even possible to represent Weyl's ray spaces in Clifford algebra, and these are vector spaces.

Appendix 1: Abbreviations

- $\mathbb{R}^{1,3}$: the Minkowski space, a 4-dimensional real vector space equipped with a non degenerate, symmetric bilinear form on the tangent space at each point in space-time, the inner product, with metric signature $\{+ - - -\}$.
- $\mathbb{R}^{3,1}$: the Minkowski space in the opposite (Lorentz) metric $\{+ + + -\}$equipped with a quadratic form is used throughout this paper.
- e_1, e_2, e_3, e_4: the basis line elements of $\mathbb{R}^{3,1}$ which squared give $e_1^2 = e_2^2 = e_3^2 = Id$ and $e_4^2 = -Id$.
- e_4: is called the unit time line element
- $Cl_{3,1}$: linear space of multivectors generated by the Minkowski space $\mathbb{R}^{3,1}$, that is, by the relations $e_1^2 = e_2^2 = e_3^2 = Id$ and $e_4^2 = -Id$ and anticommutators $e_i e_k + e_k e_i = 0$ for $i \neq k$.
- $Mat(4, \mathbb{C})$: the Clifford algebra $Cl_{3,1}$is isomorphic as an associative algebra with the algebra of 4×4 matrices $Mat(4, \mathbb{C})$ with real entries.
- Standard basis of $Cl_{3,1}$ is given by the set of graded elements $\{Id, e_1, e_2, e_3, e_4, e_{12}, e_{13}, e_{14}, e_{23}, e_{34}, e_{123}, e_{124}, e_{134}, e_{234}, e_{1234}\}$, each e_{ik} is called a unit bivector, the e_{i4} are called unit space-time areas, every e_{ijk} is a unit trivector, the e_{ij4} are denoted unit space-time volumes. The director of $Cl_{3,1}$ is the unit space-time fourvolume.
- Graßmann's exterior product: satisfying $a \wedge b = -b \wedge a$ and $a \wedge a = 0$.
- Clifford product of two vectors and is obtained by adding the scalar $a \cdot b$ and the bivector $a \wedge b = 0$: $ab = a \cdot b + a \wedge b$.
- Capital letters I, J, K, R, S, T: are selected graded unit multivectors from the basis of $\mathbb{C} \otimes Cl_{3,1}$ that generate discrete multivector-groups forming lines and faces of the Fano pyramid.
- Groups G_α^β: peculiar discrete non-abelian Groups having order $\alpha \leq 16$.

Appendix 2: Matrix Representations

The revolving stage for this relativistic projective magic is essentially Clifford algebra $Cl_{3,1}$ of Minkowski space $\mathbb{R}^{3,1}$ in the opposite Lorentz metric up to the imaginary factor and respectively the largest compact and non-compact Lie groups $(4,\mathbb{C})$, $SL(4,\mathbb{R})$ derived from this together with their representations by quaternion matrices. This sounds complicated, but is straight forward. We build up the $Cl_{3,1}$ by the aid of Id and the four unitary space- and time- line elements in $Mat(4,\mathbb{R})$.

$$Id=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}e_1=\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}e_2=\begin{pmatrix}&&1&\\&&&1\\1&&&\\&1&&\end{pmatrix}e_3=\begin{pmatrix}&&&1\\&&-1&\\&-1&&\\1&&&\end{pmatrix}e_4$$

$$=\begin{pmatrix}&&-1&\\&&&1\\1&&&\\&-1&&\end{pmatrix}e_{12}=\begin{pmatrix}&&1&\\&&&1\\-1&&&\\&-1&&\end{pmatrix}e_{13}=\begin{pmatrix}&&&1\\&&-1&\\&1&&\\-1&&&\end{pmatrix}e_{14}$$

$$=\begin{pmatrix}&&-1&\\&&&1\\-1&&&\\&1&&\end{pmatrix}e_{23}=\begin{pmatrix}&-1&&\\1&&&\\&&&1\\&&-1&\end{pmatrix}e_{24}=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&1\end{pmatrix}e_{34}$$

$$=\begin{pmatrix}&-1&&\\-1&&&\\&&&-1\\&&-1&\end{pmatrix}e_{123}=\begin{pmatrix}&-1&&\\1&&&\\&&&-1\\&&1&\end{pmatrix}e_{124}=\begin{pmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{pmatrix}e_{134}$$

$$=\begin{pmatrix}&-1&&\\-1&&&\\&&&1\\&&1&\end{pmatrix}e_{234}=\begin{pmatrix}&&&-1\\&&-1&\\&-1&&\\-1&&&\end{pmatrix}e_{1234}=\begin{pmatrix}&&&-1\\&&-1&\\&1&&\\1&&&\end{pmatrix}$$

The data file RD-Matrix_defs.rif [5] Dahm sent me contained the following entries:

$$Q_{0,0}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}Q_{0,1}=\begin{bmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{bmatrix}Q_{0,2}=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}Q_{0,3}=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}$$

$$Q_{1,0}=\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\-1&0&0&0\\0&-1&0&0\end{bmatrix}Q_{1,1}=\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix}Q_{1,2}=\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix}Q_{1,3}=\begin{bmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{bmatrix}$$

$$Q_{2,0}=\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}Q_{2,1}=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix}Q_{2,2}=\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}Q_{1,3}=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{bmatrix}$$

$$Q_{3,0}=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}Q_{3,1}=\begin{bmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}Q_{3,2}=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}Q_{3,3}=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$$

I could obtain every matrix unit as a linear combination of four of these quaternion matrices.

Appendix 3: Weyl Matrices

$$E_{00}=(Q_{00}+iQ_{03}+iQ_{30}-iQ_{33})/4$$

$$E_{11}=(Q_{00}-iQ_{03}+iQ_{30}+iQ_{33})/4$$

$$E_{22}=(Q_{00}+iQ_{03}-iQ_{30}+iQ_{33})/4$$

$$E_{33}=(Q_{00}-iQ_{03}-iQ_{30}-iQ_{33})/4$$

$$E_{02} = (iQ_{10} - Q_{13} - Q_{20} - iQ_{23})/4$$

$$E_{13} = (iQ_{10} + Q_{13} - Q_{20} + iQ_{23})/4$$

$$E_{20} = (iQ_{10} - Q_{13} + Q_{20} + iQ_{23})/4$$

$$E_{31} = (iQ_{10} + Q_{13} + Q_{20} - iQ_{23})/4$$

$$E_{03} = (-Q_{11} - iQ_{12} - iQ_{21} + Q_{22})/4$$

$$E_{12} = (-Q_{11} + iQ_{12} - iQ_{21} - Q_{22})/4$$

$$E_{21} = (-Q_{11} - iQ_{12} + iQ_{21} - Q_{22})/4$$

$$E_{30} = (-Q_{11} + iQ_{12} + iQ_{21} + Q_{22})/4$$

$$E_{01} = (iQ_{01} - Q_{02} - Q_{31} - iQ_{32})/4$$

$$E_{10} = (iQ_{01} + Q_{02} - Q_{31} + iQ_{32})/4$$

$$E_{23} = (iQ_{01} - Q_{02} + Q_{31} + iQ_{32})/4$$

$$E_{32} = (iQ_{01} + Q_{02} + Q_{31} - iQ_{32})/4$$

By the aid of these matrix units a matrix representation by “Majorana” matrices $Mat(4, \mathbb{R})$ of the standard basis of $Cl_{3,1}$ could be pinned down. Those are the matrices shown at the beginning of the Appendix 2.

Conflict of Interest

The author declares no conflict of interest.

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