Exploring the Nonlinear Behavior of Solitary Wave Structure to the Phi-Four Model for Quantum Consequence

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Abstract: In this study, the modified simplest equation method is employed to attain the exact solitary wave solutions to the nonlinear space-time fractional Phi-Four (PF) model incorporating the conformable derivative. The PF model is marked as a specific case of well-known Klein-Fock-Gordon model, abundant quantum consequence can be examined by PF model. The attained wave solutions are in types of trigonometric, rational, hyperbolic as well as their integration functions. This study introduces diverse structure of soliton like: multiple periodic and singular periodic soliton, bell and anti bell shape soliton, kink shape soliton and more. We also scrutinize the impact of fractional parameters on the behavior of wave circulation by generating surface plots of the solutions. The established results are realistic, wide-ranging, proficient and inclusive. This indicates that the employed technique is more efficient to resolve the different nonlinear models.

Keywords: the space-time fractional Phi-four model, the modified simplest equation method, the conformable derivative, soliton

1. Introduction

Nonlinear Fractional Differential Equation (NFDE) is a division of Nonlinear Partial Differential Equations (NPDEs), which are the powerful mathematical instrument, which used to explain different physical and engineering phenomenon. Utilizing this instrument, scholars have developed different mathematical structure of the physical phenomenon come up in the field of plasma physics, optical physics, fluid dynamics, chemical physics, optical fibers, mathematical physics, and more. The pursuit for exact solutions to these models is not just a mathematical attempt, it dispenses deep insights into the core structure of corporeal models, handout specific prophecy and increasing our understanding of composite systems.

In recent age, the scholars have been developed various theorem to analyze the properties of fractional derivatives as for instant, Conformable derivative, truncated M-fractional derivative, Beta-derivative, Caputo-Fabrizio [1–4], etc. Due to the significance exact solution, the scholars have originated and suggested various well-organized methods and techniques [5–29] to resolve the nonlinear models.

This study examines the eminent space-time fractional Phi-Four (PF) model [30], that is:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+u(x,t)-u^3(x,t)=0. \tag{1}$$

where $t > 0$ and $\alpha \in (0,1]$. This model is used in quantum field theory that describes a scalar field with a quartic self-interaction. It is also used to understand concepts like renormalization, spontaneous symmetry breaking, and the triviality problem in higher dimensions. Due to the importance of space-time fractional Phi-Four (PF) model, the scholars have investigated the PF model utilizing a variety of efficient and useful techniques for extracting numerous solitary wave solutions.

Such as, $\exp(-\Phi(\xi)$and modified Kudryashov scheme [31], generalized Kudryashov technique [32], new extended direct algebraic method [33], Weier strass elliptic function technique [34], Mapping method [35], unified scheme [36], MSC technique [37], tanh function method [38], auxiliary equation method [39], more are the top methods to search the exact solutions of the PF model. Our stated method is proficient, powerful and produced broad-spectrum traveling wave solutions to the NPDEs. According to statistical evaluation, it is clear that the fractional PF model is not investigated by the modified simplest equation method.

The primary intention of this research is to establish solitary wave solutions of fractional PF model by employing the modified simplest equation technique incorporating the conformable derivative. Also, we have discussed how wave circulation behaviors are influenced by the fractional parameter α. The physical importance of obtained solutions is delineated by sketching surface plot for absolute and imaginary parts of soliton solutions.

The sequence of this document is: In Section 2, we briefly talk about the conformable derivative. In Section 3, the modified simplest equation technique is explained. In Section 4, we have analytically solved the considered mode. In Section 5, we have explained the dynamical manners and physical significance of the attained solutions by drawing surface plots. In Section 6, we have shown novelty and superiority of our study comparing with earlier study. Finally, the concluding notes are presented.

2. Definition of Conformable Derivative

Primarily, Khalil *et al.* [1] introduce an innovative idea to describe the fractional derivative that is entitled by the conformable derivative.

Definition: Khalil *et al.* [1] originated the conformable derivative of order $\beta > 0$ with respect to$t$ as:

Suppose $w: (0, \infty) \to \mathbb{R}$ is a real function, then the conformable derivative of w is define as.

$$E_t^{\beta} w(t) = \lim_{\varepsilon \to 0} \frac{w(t+\varepsilon t^{1-\beta}) - w(t)}{\varepsilon}, \quad \forall\, t > 0,\ \beta \epsilon (0,1].$$

The conformable derivative was defined as:

Theorem-1: Suppose $\beta \epsilon (0,1]$ and w, σ are β- differentiable at a point t. Then the subsequent properties embrace.

- $E_t^{\beta}(cw + d\sigma) = cE_t^{\beta}(w) + dE_t^{\beta}(\sigma)$
- $E_t^{\beta}(t^q) = qt^{q-\beta},\ \forall\, q \in \mathbb{R}$
- $E_t^{\beta}(\tau) = 0$, Here, $w(t) = \tau$is a constant function.
- $E_t^{\beta}(\sigma w) = \sigma E_t^{\beta}(w) + wE_t^{\beta}(\sigma)$
- $E_t^{\beta}\left(\frac{\sigma}{w}\right) = \frac{wE_t^{\beta}(\sigma) - \sigma E_t^{\beta}(w)}{w^2}$
- $E_t^{\beta}\sigma(t) = t^{1-\beta}\frac{d\sigma}{dt}$, in which $t^{1-\beta}$ specify a fractional conformable functionfor all $c, d \in \mathbb{R}$.

Theorem-2: If $\sigma = \sigma(t)$ is β conformable differentiable function and w is differentiable and well-defined in the range ofσ. Then$E_t^{\beta}(\sigma o w)(t) = t^{1-\beta} w'(t)\sigma'\big(w(t)\big)$.

3. The Modified Simplest Equation Method

Suppose the general nonlinear fractional differential equation is:

$$\mathcal{L}\left(\vartheta, D_t^{\alpha}\vartheta, D_x^{\beta}\vartheta, D_y^{\gamma}\vartheta, D_t^{2\alpha}\vartheta \ldots\right) = 0, \tag{2}$$

Here, $\mathcal{L}$is a polynomial that contains partial derivatives of the wave function $\vartheta(t, x, y)$and it embraces the utmost-order nonlinear and linear terms, and sub-scripts denote partial derivative. The modified simplest equation technique advice the subsequent stairs to achieve the solutions of Eq. (2).

Stair 1: Consider the wave variable

$$\vartheta(x, y, t) = \vartheta(\zeta); \zeta = m\frac{x^{\alpha}}{\alpha} + n\frac{y^{\alpha}}{\alpha} \pm k\frac{t^{\alpha}}{\alpha}, \tag{3}$$

where k is wave velocity. m, n are the real parameters and $\alpha, (0 < \alpha \leq 1)$ is the fractional order derivative.

Stair 2: The transformation Eq. (3) convert Eq. (2) to a Nonlinear Ordinary Differential Equation (NLODE) as follows:

$$N(\vartheta, \vartheta', \vartheta'', \ldots) = 0, \tag{4}$$

where the polynomial N holds $\vartheta(\zeta)$ and differentiation of $\vartheta(\zeta)$, where $\vartheta'(\zeta) = \frac{d\vartheta}{d\zeta}$.

Stair 3: At this point Eq. (4) should be integrated term by term one or more times.

Stair 4: According to the method solution of Eq. (4) structured as:

$$\vartheta(\zeta) = b_0 + \sum_{i=1}^{J} b_i\, g^i(\zeta), \tag{5}$$

Here $b_J \neq 0$ and the value of b_0, b_ihave tosettle afterward. $g(\zeta)$ assure the auxiliary equation as:

$$g'(\zeta) = g^2(\zeta) + \eta. \tag{6}$$

Here η is real constant and$g'(\zeta) = \frac{dg}{d\zeta}$.

The auxiliary Eq. (6) carries the subsequent expressions for diverse values of η:

Family 1: if $\eta < 0$ then we have:

$$g(\zeta) = -\sqrt{-\eta}\,tanh\left(\sqrt{-\eta}\zeta\right)$$

$$g(\zeta) = -\sqrt{-\eta}\,coth\left(\sqrt{-\eta}\zeta\right)$$

$$g(\zeta) = \sqrt{-\eta}\left(-tanh\left(2\sqrt{-\eta}\zeta\right) \pm i\,sech\left(2\sqrt{-\eta}\zeta\right)\right)$$

$$g(\zeta) = \sqrt{-\eta}\left(-coth\left(2\sqrt{-\eta}\zeta\right) \pm csch\left(2\sqrt{-\eta}\zeta\right)\right)$$

$$g(\zeta) = -\frac{\sqrt{-\eta}}{2}\left(tanh\left(\frac{\sqrt{-\eta}}{2}\zeta\right) + coth\left(\frac{\sqrt{-\eta}}{2}\zeta\right)\right)$$

Family 2: if $\eta > 0$ then we have:

$$g(\zeta) = \sqrt{\eta}\,tan\left(\sqrt{\eta}\zeta\right)$$

$$g(\zeta) = -\sqrt{\eta}cot(\sqrt{\eta}\zeta)$$

$$g(\zeta) = \sqrt{\eta}\left(tan(2\sqrt{\eta}\zeta) \pm sec(2\sqrt{\eta}\zeta)\right)$$

$$g(\zeta) = \sqrt{\eta}\left(-cot(2\sqrt{\eta}\zeta) \pm csc(2\sqrt{\eta}\zeta)\right)$$

$$g(\zeta) = \frac{\sqrt{\eta}}{2}\left(tan\left(\frac{\sqrt{\eta}}{2}\zeta\right) - cot\left(\frac{\sqrt{\eta}}{2}\zeta\right)\right)$$

Family 3: if $\eta = 0$ then we have:

$$g(\zeta) = -\frac{1}{\zeta}$$

Stair 5: Utilizing the homogeneous balancing principle into the foremost nonlinear and linear terms occur in Eq. (4), we get the value of J in Eq. (5).

Stair 6: Setting Eqs. (5) and (6) into Eq. (4) generate a polynomial of $g^i(\zeta)$ in which$(i = 1, 2, 3, \dots)$. The terms contain the Powers $g^i(\zeta)$put to zero gives a system of algebraic of equations with parameters b_i,m, n, and k.

Stair 7: By resolving the system of algebraic equations, we obtain the values of unknown parameters. Setting the values of $b_i(i = 0, 1, 2, 3, \dots), m, n, k$ and expressions of $g(\zeta)$ to Eq. (5) gives solutions of Eq. (4).

4. Abundant Wave Solutions to the Space-Time Fractional Phi-Four (PF) Model

This segment organizes the solitary wave solutions to the nonlinear PF model by using the modified simplest equation method.

Suppose the wave variable $u(\zeta) = u(x, t)$, where $\zeta = m\frac{x^\alpha}{\alpha} + k\frac{t^\alpha}{\alpha}$ then the model Eq. (1) is changes to a NODE as:

$$(k^2 - m^2)u'' + u - u^3 = 0. \tag{7}$$

Utilizing the homogeneous balancing principle into the foremost nonlinear and linear terms occur in Eq. (7), produce $J = 1$. Thus, the solution structure of Eq. (7) can be written as:

$$u = b_0 + b_1 g(\zeta). \tag{8}$$

Putting the results of Eqs. (8) and (6) into Eq. (7) and linking the coefficients of the alike powers of $g^i(\zeta)$ and setting zero produce a system of algebraic equations for b_0, b_1, m, k. That is:

$$b_0 + {b_0}^3 = 0.$$

$$2k^2 b_1\eta - 2m^2 b_1\eta + b_1 + 3{b_0}^2 b_1 = 0.$$

$$3{b_0}^2 b_1 = 0.$$

$$2k^2 b_1 - 2m^2 b_1 + {b_1}^3 = 0.$$

Resolving the group of algebraic equations by using the computation software like Maple, the subsequent solutions are recommended:

Addressing the above system of algebraic equations by using the computation program like Maple, we get the following values of unknown parameters.

$$b_0 = 0 \ , \quad b_1 = \pm\frac{1}{\sqrt{\eta}}, \quad k = \pm\frac{\sqrt{(2m^2\eta-1)}}{\sqrt{2\eta}}. \tag{9}$$

Case 1: if $\eta < 0$, then we get the soliton solutions:

$$u_1(\zeta) = \mp itanh(\sqrt{-\eta}\zeta). \tag{10}$$

$$u_2(\zeta) = \mp icoth(\sqrt{-\eta}\zeta). \tag{11}$$

$$u_3(\zeta) = \mp\left(itanh(2\sqrt{-\eta}\zeta) + sech(2\sqrt{-\eta}\zeta)\right). \tag{12}$$

$$u_4(\zeta) = \pm i\left(csch(2\sqrt{-\eta}\zeta) - coth(2\sqrt{-\eta}\zeta)\right). \tag{13}$$

$$u_5(\zeta) = \mp\frac{i}{2}\left(tanh\left(\frac{\sqrt{-\eta}}{2}\zeta\right) + coth\left(\frac{\sqrt{-\eta}}{2}\zeta\right)\right). \tag{14}$$

Case 2: if $\eta > 0$, then we acquire the soliton solutions:

$$u_6(\zeta) = \pm tan(\sqrt{\eta}\zeta). \tag{15}$$

$$u_7(\zeta) = \mp cot(\sqrt{\eta}\zeta). \tag{16}$$

$$u_8(\zeta) = \pm\left(tan(2\sqrt{\eta}\zeta) + sec(2\sqrt{\eta}\zeta)\right). \tag{17}$$

$$u_9(\zeta) = \pm\left(csc(2\sqrt{\eta}\zeta) - cot(2\sqrt{\eta}\zeta)\right). \tag{18}$$

$$u_{10}(\zeta) = \pm\frac{1}{2}\left(tan\left(\frac{\sqrt{\eta}}{2}\zeta\right) - cot\left(\frac{\sqrt{\eta}}{2}\zeta\right)\right). \tag{19}$$

Case 3: if $\eta = 0$, then the soliton solutions is:

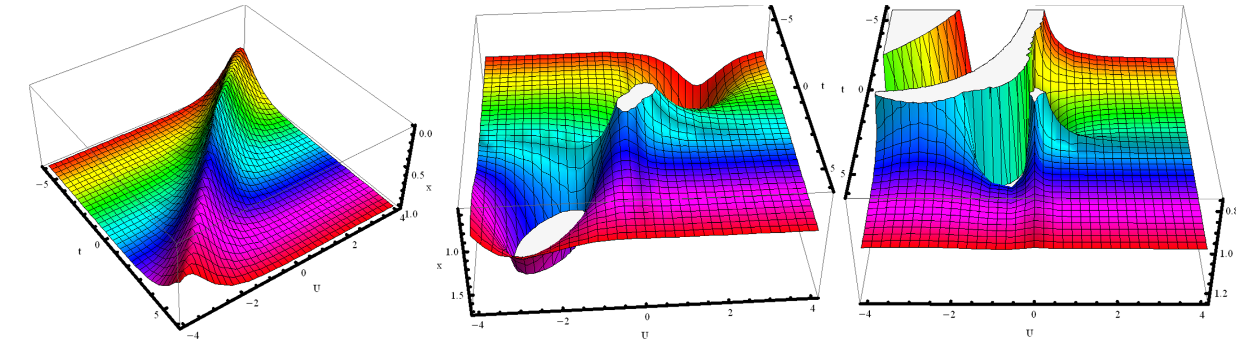
$$u_{11}(\zeta) = \mp\frac{1}{\zeta}. \tag{20}$$

In all soliton solutions, $\zeta = m\frac{x^\alpha}{\alpha} + k\frac{t^\alpha}{\alpha}, k$ is the wave velocity and m is a random constant.

5. Physical Importance of Outcome through the Surface Plots

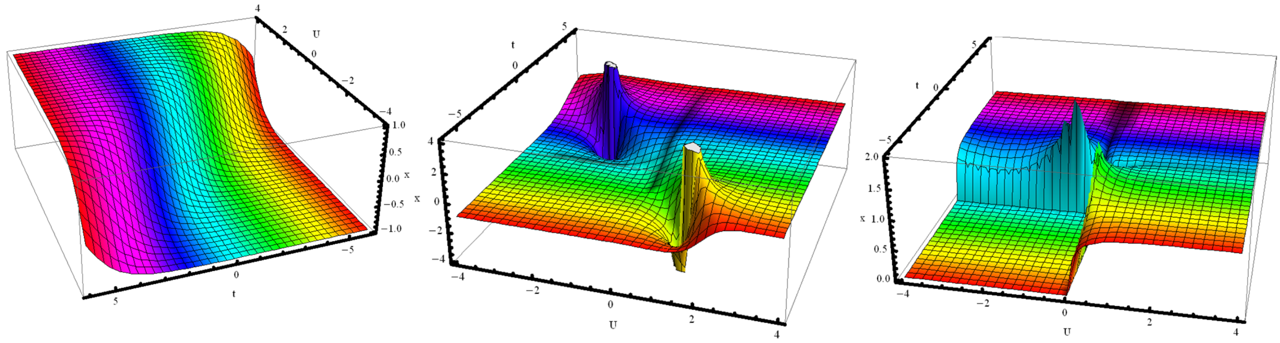
Current segment represents the surface plots of the solutions of the considered model. Key spotlight of the surface plot is to describe the internal mechanism of attained soliton solutions for little changes of α. For simplicity we confer the results Eqs. (10), (12), (16), and (18) and the remaining outcome are skipping now. The impact of fractional parameter on wave circulation behavior is as follows:

The surface plot of solution Eq. (10) repersent bell shape wave, which carries high amplitude and small width. For $t \to \infty$ it characterized as an ongoing soliton (Fig. 1). This graph is sketch for absolute value of u_1 with suitable values of the implied parameters $\eta = -0.10, m = 1.90$ and for $\alpha = 0.96, \alpha = 0.75$, and $\alpha = 0.50$, correspondingly.It is observed that the soliton profile is change with change of α.



(a) (b) (c)

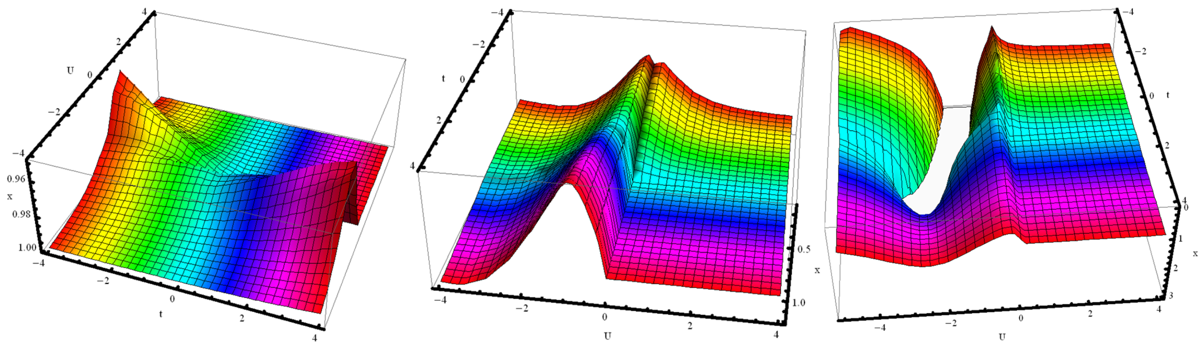
Fig. 1. 3D solitonic layout of the absolute value of Eq. (10) with $\eta = -0.10$, $m = 1.90$. (a) $\alpha = 0.96$; (b) $\alpha = 0.75$; (c) $\alpha = 0.50$.



(a) (b) (c)

Fig. 2. 3D solitonic layout of Eq. (10) for imaginary part of u_1 with $\eta = -0.10, m = 1.90$. (a) $\alpha = 0.96$; (b) $\alpha = 0.75$; (c) $\alpha = 0.50$.

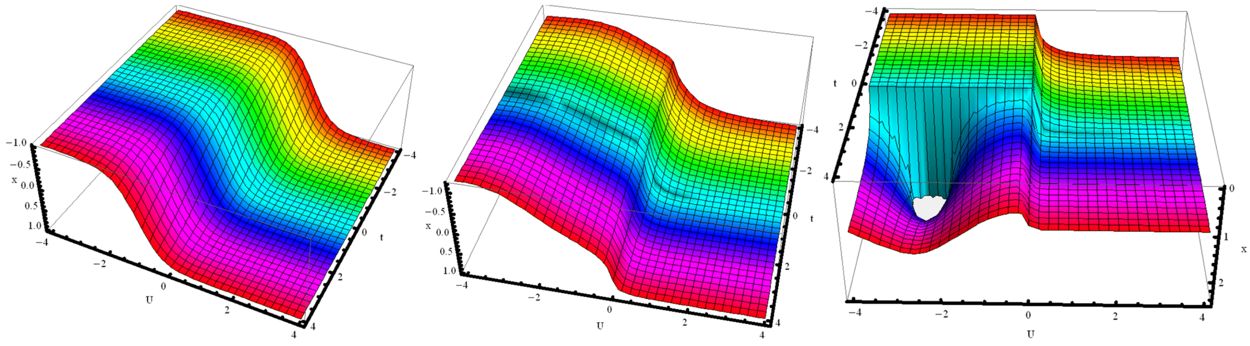
The surface plot of solution Eq. (10) repersent kink shape wave propagation, which rise or fall from one asymptotic position to another and holds steady value for $t \to \infty$ (Fig. 2). This graph is sketch for imaginary part of u_1 with $\eta = -0.10$, $m = 1.90$ and for $\alpha = 0.96, \alpha = 0.75$, and $\alpha = 0.50$, correspondingly. It is observe that the soliton profile is change for the change of α.



(a) (b) (c)

Fig. 3. 3D solitonic layout of the absolute value of Eq. (12) with $\eta = -0.01$, $m = 1.90$. (a) $\alpha = 0.99$; (b) $\alpha = 0.70$; (c) $\alpha = 0.50$.

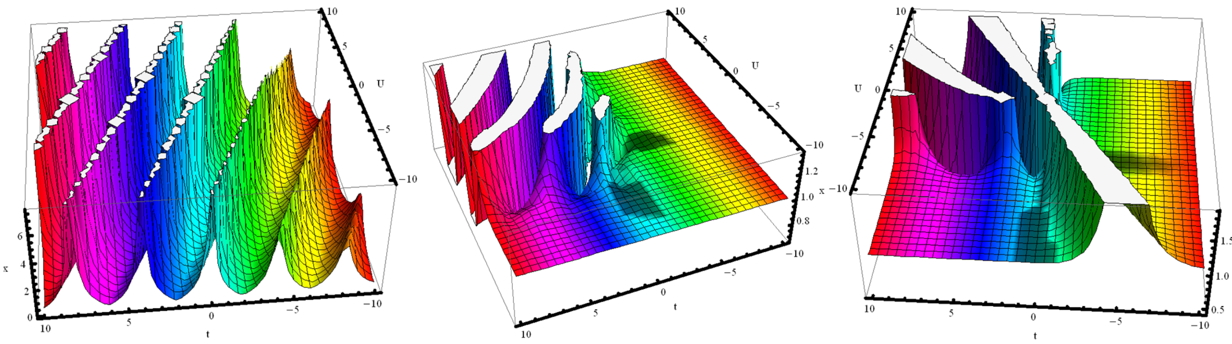
The surface plot of solution Eq. (12) repersent semi bell shape wave propagation. For $t \to \infty$ it characterized as an ongoing soliton (Fig. 3). This graph is sketch for absolute value of u_3 with $\eta = -0.01, m = 1.90$ and for $\alpha = 0.99$, $\alpha = 0.70$, and $\alpha = 0.50$, correspondingly. It is observe that the soliton profile is change for the change of α.



(a) (b) (c)

Fig. 4. 3D solitonic layout of Eq. (12) for imaginary part of u_3 with $\eta = -0.01, m = 1.90$. (a) $\alpha = 0.99$; (b) $\alpha = 0.70$; (c) $\alpha = 0.50$.

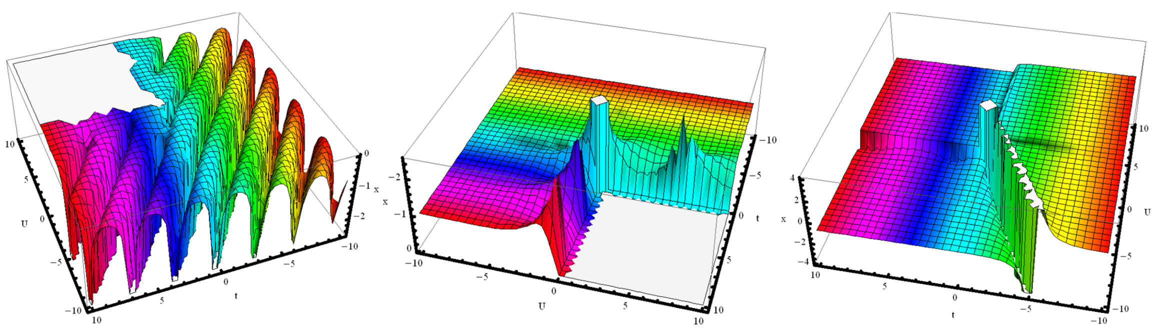
The surface plot of solution Eq. (12) repersent kink shape wave propagation, which rise or fall from one asymptotic position to another and holds steady value for $t \to \infty$ (Fig. 4).This graph is sketch for imaginary part of u_3 with $\eta = -0.01, m = 1.90$ and for $\alpha = 0.99$, $\alpha = 0.70$, and $\alpha = 0.50$, correspondingly. It is observe that the soliton profile is change for the change of α .



(a) (b) (c)

Fig. 5. 3D solitonic layout of the absolute value of Eq. (16) with $\eta = 0.50, m = -1.10$. (a)$\alpha = 0.99$; (b) $\alpha = 0.70$; (c) $\alpha = 0.55$.

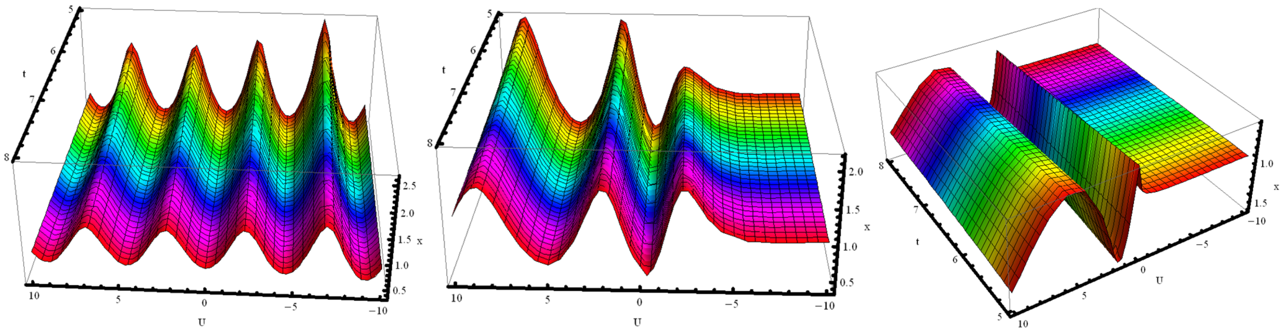
The surface plot of solution Eq. (16) repersent multiple singular periodic shape wave propagation (Fig. 5). This is unsteady soliton and embrace different phase and amplitude. This graph is sketch for absolute value of u_7 with$\eta = 0.50, m = -1.10$ and for $\alpha = 0.99, \alpha = 0.70$, and $\alpha = 0.55$, correspondingly. It is observe that the soliton profile is change for the change ofα .



(a) (b) (c)

Fig. 6. 3D solitonic layout of Eq. (16) for imaginary part of u_7 where $\eta = 0.50, m = -1.10$. (a) $\alpha = 0.99$; (b) $\alpha = 0.70$; (c) $\alpha = 0.55$.

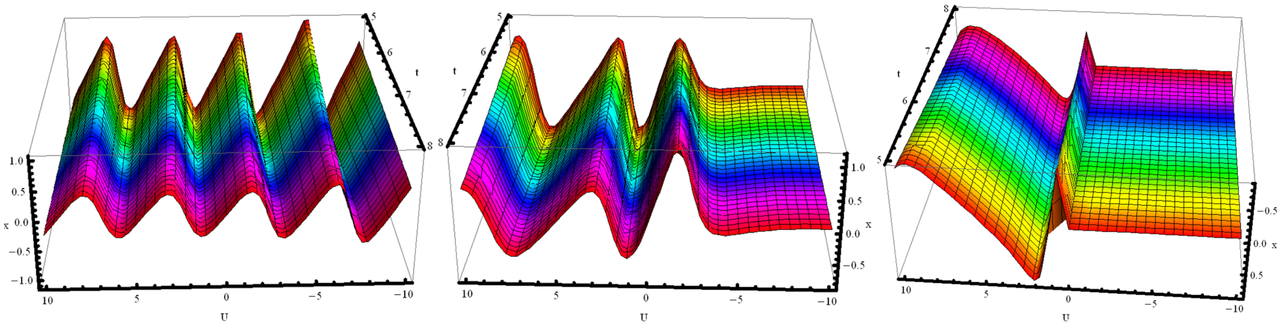
The surface plot of solution Eq. (16) repersent multiple periodic shape wave propagation, which is capable to pass a long way with constant strength and velocity (Fig. 6). These types of waves are stable and hold specific phase and amplitude. This graph is sketch for imaginary part of u_7 with $\eta = 0.50, m = -1.10$ and for $\alpha = 0.99, \alpha = 0.70,$ and $\alpha = 0.55$, correspondingly. It is observe that the soliton profile is change for the change of α .



(a) (b) (c)

Fig. 7. 3D solitonic layout of the absolute value of Eq. (18) with $\eta = 0.01, m = 1.10$. (a) $\alpha = 0.99$; (b) $\alpha = 0.75$; (c) $\alpha = 0.50$.

The surface plot of solution Eq. (18) repersent multiple periodic shape wave propagation. This is capable to pass a long way with constant strength and velocity (Fig. 7). These types of waves are stable and hold specific phase and amplitude. This graph is sketch for absolute value of u_9 with$\eta = 0.01, m = 1.10$ and for $\alpha = 0.99, \alpha = 0.75,$ and $\alpha = 0.50$, correspondingly. It is observe that the soliton profile is change for the change of α.



(a) (b) (c)

Fig. 8. 3D solitonic layout of Eq. (18) for imaginary part of u_9 where $\eta = 0.01, m = 1.10$. (a) $\alpha = 0.99$; (b) $\alpha = 0.75$; (c) $\alpha = 0.50$.

The surface plot of solution Eq. (18) repersent multiple periodic shape wave propagation. This is capable to pass a long way with constant strength and velocity (Fig. 8). These types of waves are stable and hold specific phase and amplitude. This graph is sketch for imaginary part of $u_9$with $\eta = 0.01, m = 1.10$ and for $\alpha = 0.99, \alpha = 0.75,$ and $\alpha = 0.50$, correspondingly. It is observed that the soliton profile is varying for the change ofα .

In all graphs η, m are real parameters. The above graphical demonstration of attained solutions of the considered model discloses the inner procedure of corporeal structure of the nonlinear fission and fusion observed in numerous physical incidents such as: plasma physics, electrodynamics, organic membrane and more. The established solutions symbolize different well-known three dimensional graphs as for instant, bell and semi-bell shape soliton, kink and semi-kink shape soliton, multiple periodic and singular periodic soliton, and others. The above wave profiles are confirmed by putting them into the stated model and found acceptable.

6. The Novelty of Established Results

Roy *et al.* [30] investigate fractional PF model using generalized (G'/G)-expansion technique. The outcome of their study shows notable difference in the methodology, outcome, and contributions with our work. This comparison intends to scrutinize these distinctions to show up the novelty and superiority of our study.

a) Utilized techniques

i) This study utilized a widely used famous technique, which produced about eleven distinct solutions of a nonlinear model.

ii) On the other hand, Roy *et al.* [30] utilizes the comparatively new solution technique that produced about nine distinct solutions of a nonlinear model. This is limiting the span of analytical investigation.

b) Soliton structure

i) Our investigation produced different types of familiar soliton structures such as: singular periodic shape, continuous periodic shape, bell and semi bell shape waves, singular kink and semi kink shape and more. This is a better assortment of consistent structures.

ii) On the other hand, Roy *et al.* [30] shows fewer types of familiar soliton structures such as: singular periodic soliton, this shows a narrower assortment of wave shape.

c) Corporeal insights

i) This study shows deeper insights into solitary wave dynamics employing a well-known efficient method. In this investigation we have shown the impact of fractional parameter on wave structure and also shown the change of 3D shape of the soliton.

ii) On the other hand, Roy *et al.* [30] gives inadequate explanation of results and their corporeal meanings.

d) Contributions

i) This study notably increases the perceptive of nonlinear wave evolution.

ii) On the contrary, Roy *et al.* [30] shows inadequate new knowledge.

In summary, our investigation represents a further comprehensive, rigorous, and novel investigation to the fractional PF model by utilizing a sophisticated analytical method. It reveals new insights, and invents superior assistance to the area of nonlinear wave study compared to Ref. [30].

7. Conclusion

In this study, the nonlinear space-time fractional Phi-Four (PF) model has been successfully resolved by utilizing the modified simplest equation method incorporating the conformable derivative. This study produces numerous soliton solutions as for instant: trigonometric, rational, hyperbolic, and their combined functions. These solutions are practical, resourceful, and easily applicable to plentiful treatments in the areas of natural science, engineering, and more. The graphical representation uncovers many internal characteristics of the considered model. Our attain solutions exhibit different shape of wave circulation as like as: singular periodic shape, continuous periodic shape, bell and semi bell shape waves, singular kink and semi kink shape and more. We observe that the wave propagation related PF model is extensively influenced by the change of the fractional parameter α. The correctness of obtained results are confirmed by placing into the original model with the assist of computational software like Wolfram Mathematica and got acceptable. Therefore, the modified simplest equation method is straightforward, resourceful, efficient and easily applicable, to extract wide-ranging soliton solutions of NLFDEs. In future, due to the efficiency of the modified simplest equation method, various important models in the spheres of sciences, and engineering will be resolved using this technique.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Author Contributions

M. Al-Amin: Conceptualization, Formal analysis, Writing the original draft, Software, Methodology, Investigation, Validation. M. Mizanur Rahman: Writing, review, Supervision. M. Kamrunnaher: Review, Supervision. M. Nurul Islam: Writing, review, Supervision. M. Nazrul Islam: Review, Supervision. All authors had approved the final version.

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