# Ulam-Hyers Stability of the Initial Value Problem for the Pantograph-Type Equation

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**Abstract:** Pantograph-type equations arise in diverse areas of applied mathematics, yet their stability and solvability properties remain difficult to analyze. This work addresses this problem by studying the initial value problem for pantograph-type equations on a fixed interval. The objectives are to establish Ulam–Hyers stability conditions and to derive sufficient conditions for unique solvability. The analysis employs the Banach fixed-point theorem, which provides a rigorous basis for proving existence, uniqueness, and stability. An illustrative example is included to demonstrate the applicability of the theoretical results. The findings indicate that, although exact solutions are often challenging to obtain in practice, the Ulam–Hyers theory offers an effective approach for approximating and estimating solutions of pantograph-type equations. The results contribute to the advancement of stability theory for functional differential equations and highlight the significance of Ulam–Hyers methods in the study of equations with proportional arguments.

**Keywords:** Banach fixed-point theorem, pantograph-type equation, Ulam-Hyers stability

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## 1. Introduction

The Ulam-Hyers stability theory has aroused the interest of researchers in various applied scientific fields because it answers the question asked by Ulam: When does a linear mapping near an approximately linear mapping exist? The theory started from Ulam [1], Hyers [2] and Rassias [3]. There are many publications on this type of stability, e.g., [4–6]. We consider pantograph-type equations, which arise in electrodynamics, quantum mechanics, number theory, astrophysics, and other fields (see, e.g., [7, 8]).

We study a pantograph-type equation

$$D_a^q u(t) = p(t)u(ct) + f\big(u(kt)\big), \qquad t \in [a,b], \tag{1}$$

with the initial value condition

$$u(a) = d, \tag{2}$$

where $D_a^q u(t)$ is the Caputo fractional derivative of an order $q$, $q \in (0,1)$, $p$, $f \in L^\infty([a,b],\mathbb{R})$, $c$ and $k \in \mathbb{R}_+$, $a \le \min\{ca, ka\}$, $\max\{cb, kb\} \le b$, $-\infty < a < b < \infty$, $d \in \mathbb{R}$.

The main aim of the investigation is to establish conditions for Ulam-Hyers stability of the pantograph-type problem (1), (2). The second goal is to find conditions under which a unique solution of the problem (1), (2) exists. Many other results on the solvability of (non)linear pantograph-type problems can be found in, e.g., [9–12].

## 2. Notations and Definitions

### 2.1. Notations

$\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = (0, +\infty)$.

$L^\infty([a,b],\mathbb{R})$ is the Banach space of all Lebesgue integrable functions $u: [a,b] \to \mathbb{R}$ with the norm

$$||u||_\infty = \text{ess sup}_{\text{t}\in[\text{a,b}]}|u(t)|.$$

$C^1([a,b],\mathbb{R})$ is the Banach space of all functions $u: [a,b] \to \mathbb{R}$ that are continuously differentiable with the norm

$$||u|| = \text{sup}_{\text{t}\in[\text{a,b}]}|u(t)| + \text{sup}_{\text{t}\in[\text{a,b}]}|u'(t)|.$$

### 2.2. Definitions

Caputo fractional derivatives are the derivatives that arise in the theory of differential equations of fractional order, e.g., [13–15].

**Definition 1.** For any function $u \in C^1([a,b],\mathbb{R})$ the Caputo fractional derivative $D_a^q u$ exists almost everywhere on the interval $[a,b]$ and

$$D_a^q u(t) := \frac{1}{\Gamma(1-q)} \int_a^t (t-s)^{-q} u'(s) ds\,,\ q \in (0,1),\ t > a,$$

where $\Gamma(q): (0,\infty) \to \mathbb{R}$ is Gamma function and $\Gamma(q) := \int_a^\infty t^{q-1} e^{-t} dt$.

The left-sided fractional integral $I_a^q u$ of order $q$, $q \in (0,1)$, for $u \in L^\infty([a,b],\mathbb{R})$ is defined by

$$I_a^q u(t) := \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} u(s) ds\,,\ q \in (0,1),\ t \in [a,b].$$

**Definition 2.** The pantograph-type problem Eqs. (1), (2) is Ulam-Hyers stable if there exists a constant $\delta > 0$ such that for each $\varepsilon > 0$ and for each function $y$ from $C^1([a,b],\mathbb{R})$ that satisfies [2, 5, 16].

$$\left|D_a^q y(t) - p(t) y(ct) - f\big(y(kt)\big)\right| \le \varepsilon, \qquad t \in [a,b], \tag{3}$$

there exists a solution u $\in C([a,b],\mathbb{R})$ of Eqs. (1), (2) with $d = y(a)$, for which

$$|y(t) - u(t)| \le \delta\varepsilon,\ \ t \in [a,b]. \tag{4}$$

## 3. The Unique Solvability of Eqs. (1), (2)

We will need the following obvious Lemma.

**Lemma 1.** If a function $u \in C^1([a,b],\mathbb{R})$ is a solution of the problem Eqs. (1), (2), then it satisfies the integral equation

$$u(t) := d + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} p(s) u(cs) ds + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(u(ks))\, ds$$

for almost every $t \in [a,b]$ and vice versa.

Assume that there exist constants $M < \infty$, $0 < L < \infty$ such that

$$\sup_{t\in[a,b]} |p(t)| = M,\ M > 0 \tag{5}$$

and function $f$ satisfies the Lipschitz condition:

$$\left|f\big(u(\cdot)\big) - f\big(z(\cdot)\big)\right| \le L|u(\cdot) - z(\cdot)|,\ u,\ z \in C^1([a,b],\mathbb{R}). \tag{6}$$

We define a constant $\kappa$ by

$$\kappa := \frac{(M+L)(b-a)^q(b-a+q)}{\Gamma(q+1)},\quad M \in \mathbb{R}_+,\ L \in \mathbb{R}_+. \tag{7}$$

**Theorem 1.** Suppose that the function $p$ has the property Eq. (5) with some constant $0 < M < \infty$ and the function $f$ satisfies the Lipschitz condition Eq. (6) with some constant $0 < L < \infty$. Moreover, assume that a constant $\kappa$, from Eq. (7), belongs to the interval (0, 1). Then the pantograph-type problem Eqs. (1), (2) has a unique solution.

**Proof of Theorem 1.**

Let us set the operator $\mathcal{P}: C^1([a,b],\mathbb{R}) \to C^1([a,b],\mathbb{R})$:

$$\mathcal{P}u(t) := d + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} p(s)u(cs)ds + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(u(ks))\, ds. \tag{8}$$

Considering Lemma 1, we know that any fixed point of $\mathcal{P}$ is a solution of Eqs. (1), (2).

In view of the conditions Eqs. (5)–(8), we obtain

$$\begin{aligned}||\mathcal{P}u(t) - \mathcal{P}z(t)|| = ||&\frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}p(s)u(cs)ds + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}f(u(ks))\, ds \\ &- \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}p(s)z(cs)ds - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(z(ks))ds|| \\ &\le \left| \frac{M(t-a)^{q-1}(t-a+q)}{\Gamma(q)q} + \frac{L(t-a)^{q-1}(t-a+q)}{\Gamma(q)q} \right| ||u-z|| \le \frac{(M+L)(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)} ||u-z||.\end{aligned}$$

Now we get that $||\mathcal{P}u(t) - \mathcal{P}z(t)|| \le \frac{(M+L)(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)} ||u-z|| \le \kappa||u-z||$, where, $\kappa$ is defined by Eq. (7), and the constant $\kappa \in (0,1)$. So, we can use Banach fixed-point theorem to the operator $\mathcal{P}: C^1([a,b],\mathbb{R}) \to C^1([a,b],\mathbb{R})$. In view of Lemma 1 and Banach fixed-point theorem, we conclude that the pantograph-type problem Eqs. (1), (2) has a unique solution. The theorem is proved.

## 4. Main Result

Assume that there exists at least one function $y$ in the Banach space $C^1([a,b],\mathbb{R})$ that satisfies Eq. (3) and differs from the solution of the problem Eqs. (1), (2).

**Remark.** If the function $y$ satisfies (3), then there exists a function $\xi \in L^\infty([a,b],\mathbb{R})$ with property $\sup_{t\in[a,b]}|\xi(\cdot)| \le \varepsilon$, such that

$$D_a^q y(t) = p(t)y(ct) + f\big(y(kt)\big) + \xi(t), \qquad t \in [a,b]. \tag{9}$$

The function $y$ does not have to satisfy the initial value condition $y(a) = d$, where $d = u(a)$.

The following Lemma is helpful for further investigation.

**Lemma 2.** Assume that $\varepsilon > 0$ and assume that all hypotheses of Theorem 1 are fulfilled, moreover, a function $y \in C^1([a,b],\mathbb{R})$ satisfies the inequality Eq. (3). Then there exists a function $\xi$, with the property $\sup_{t\in[a,b]}|\xi(t)| \le \varepsilon$, $t \in [a,b]$ such that

$$||y(t) - d - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}p(s)u(cs)ds - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f\big(y(ks)\big)ds|| \le \frac{\varepsilon(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}. \tag{10}$$

**Proof of Lemma 2.**

We consider the parameter

$$d = y(a). \tag{11}$$

In view of Theorem 1, Lemma 1 and Remark, we can conclude, that $y$ is a unique solution of the problem Eqs. (9), (11), and

$$\begin{aligned} y(t) &= y(a) + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}p(s)u(cs)ds - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(y(ks))ds - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\,\xi(s)ds \\ &= d + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}p(s)u(cs)ds - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(y(ks))ds - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\,\xi(s)ds. \end{aligned}$$

Using the fact that $\sup_{t\in[a,b]}|\,\xi(t)| \le \varepsilon,\ t \in [a,b]$, we obtain that Eq. (10) is fulfilled.

The next theorem constitutes our primary result.

**Theorem 2.** Assume that all hypotheses of Theorem 1 are fulfilled. Then the pantograph-type problem Eqs. (1), (2) is Ulam-Hyers stable.

**Proof of Theorem 2.**

Assume that the inequality Eq. (3) holds for a function $y$ from the space $C^1([a,b],\mathbb{R})$. Next in view of Theorem 1 the pantograph-type problem Eqs. (1), (2) is uniquely solvable for arbitrary $d$. So, we can take in Eq. (2) the parameter $d = y(a)$.

Taking into account properties Eqs. (5) and (6), Theorem 1 and Lemma 2, we obtain:

$$\begin{aligned} ||y-u|| = ||y(t) - u(a) &\\ &- \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}p(s)u(cs)ds - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f\big(u(ks)\big)ds|| \\ &= ||y(t) - d \\ &- \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}p(s)y(cs)ds - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f\big(y(ks)\big)ds \\ &+ \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}p(s)y(cs)ds + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f\big(y(ks)\big)ds \\ &- \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}p(s)u(cs)ds - \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f\big(u(ks)\big)ds\ || \\ &\le \frac{(b-a)^{q-1}(b-a+q)\varepsilon}{\Gamma(q+1)} + \frac{(M+L)(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}||y-u||. \end{aligned}$$

So, $||y-u|| \le \frac{\frac{(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}}{1-\frac{(M+L)(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}}\varepsilon$. In view of (7) we can conclude that $\frac{\frac{(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}}{1-\frac{(M+L)(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}} > 0$.

Therefore $|y(t) - u(t)| \le ||y-u|| \le \delta\varepsilon$, where $\delta = \frac{\frac{(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}}{1-\frac{(M+L)(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}} > 0$.

We established that the pantograph-type problem Eqs. (1), (2) is Ulam-Hyers stable in $C^1([a,b],\mathbb{R})$.

## 5. Example

Let us consider a pantograph-type initial value problem

$$D_0^{0.5}u(t) = t^2u(0.1t) + 0.05\sin(u(0.2t)),\ \ t \in \left[0, \frac{\pi}{8}\right],\ \ u(0) = d. \tag{12}$$

The problem Eq. (12) is the particular case of the problem Eqs. (1), (2) with $p(t) = t^2$, $c = 0.1$, $f\big(u(kt)\big) = 0.05\sin\big(u(0.2t)\big)$, $k = 0.2$, $a = 0$, $b = \frac{\pi}{8}$, and $d$ is some parameter from $\mathbb{R}$.

Here $M = \frac{\pi^2}{64}$ and $L = 0.05$ (using Mean value theorem: $|\sin\big(u(0.2t)\big) - \sin\big(z(0.2t)\big)| \leq \cos\xi_{\xi\in\left[0,\frac{\pi}{8}\right]} |u(0.2t) - z(0.2t)|$ ), then $\kappa = \frac{(M+L)(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)} = \frac{(\frac{\pi^2}{64}+0.05)(\frac{\pi}{8})^{-0.5}(\frac{\pi}{8}+\frac{1}{2})}{\Gamma(1.5)} \sim 0.33 < 1$. Thus, the assumptions of Theorem 1 are fulfilled, therefore the problem Eq. (12) is uniquely solvable for any $d$.

The assumptions of Theorem 2 also hold, therefore the pantograph-type problem Eq. (12) is Ulam-Hyers stable in the Banach space $C^1\left(\left[0, \frac{\pi}{8}\right], \mathbb{R}\right)$ with

$$\delta = \frac{\frac{(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}}{1-\frac{(M+L)(b-a)^{q-1}(b-a+q)}{\Gamma(q+1)}} = \frac{\frac{1}{\Gamma(1.5)}(\frac{\pi}{8})^{-0.5}(\frac{\pi}{8}+\frac{1}{2})}{1-\frac{(\frac{\pi^2}{64}+0.05)(\frac{\pi}{8})^{-0.5}(\frac{\pi}{8}+\frac{1}{2})}{\Gamma(1.5)}} \sim 2.4 > 0.$$

Now, using Ulam-Hyers stability, we aim to estimate an exact solution $u$. Let $\varepsilon > 0$. The function $y\,(t) = 0.1\varepsilon t$ satisfies (3), indeed, $\left|\varepsilon\frac{0.1\sqrt{t}}{0.5\sqrt{\pi}} - 0.01\varepsilon t^3 - 0.05\sin(\varepsilon(0.02t))\right| \leq \varepsilon\left|\frac{\sqrt{t}}{5\sqrt{\pi}} - 0.01t^3 - 0.001t\right| \leq \varepsilon$ on $\left[0, \frac{\pi}{8}\right]$. Here $d = 0 = y(0)$ and an exact solution $u$ is estimated by $|0.1\varepsilon t - u(t)| \leq 2.4\varepsilon$, $t \in \left[0, \frac{\pi}{8}\right]$.

It is often difficult to compute an exact solution to an applied problem. However, as the example demonstrates, the exact solution to the pantograph-type problem can be easily estimated using Theorem 2 with an appropriate ε.

## 6. Conclusion

We have established Ulam–Hyers stability conditions for the initial value problem of pantograph-type equations on a fixed interval. By employing the Banach fixed-point theorem, sufficient conditions ensuring unique solvability were obtained, and the illustrative example confirmed the validity of the theoretical results. These findings advance the stability theory of functional differential equations with proportional arguments, clarifying the role of Ulam–Hyers stability in providing rigorous estimates of exact solutions within a prescribed tolerance parameter $\varepsilon$. The principal limitation of this study lies in its restriction to bounded intervals, which constrains the generality of the results. Future research may focus on extending the stability and solvability analysis to unbounded domains, developing generalized stability frameworks under weaker assumptions, and further refining the interplay between Ulam–Hyers stability and other stability concepts in the theory of functional differential equations.

## Conflict of Interest

The authors declare no conflict of interest.

## Author Contributions

ND conducted the research; ND analyzed the data; ND and SL wrote the paper; all authors had approved the final version.

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