

Fractional-Order Chaotic Lorenz-84 Atmosphere System Stabilization Using a Feedback State Space Control

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Manuscript submitted April 18, 2025; accepted May 19, 2025; published June 23, 2025.

doi: 10.17706/ijapm.2025.15.1.69-77

Abstract: This paper explores the stabilization of the three-dimensional Fractional-Order (FO) chaotic Lorenz-84 atmosphere model through control strategies. Employing the Grünwald-Letnikov approximation for fractional integration, the chaotic system is simulated to understand its intricate dynamics. The main focus lies in utilizing state-space feedback control to adapt input signals based on system states, aiming to steer chaotic dynamics towards stability. Additionally, control gains optimization is conducted through Particle Swarm Optimization (PSO) to guarantee robust stabilization of the chaotic atmosphere model.

Keywords: Lorenz-84 Atmosphere model, chaotic system, fractional-order system, fractional integrator, Particle Swarm Optimization (PSO)

1. Introduction

For decades, chaotic systems—those that exhibit unpredictable but deterministic behavior—have captivated scientists [1]. Now, a new layer of complexity is emerging: partial chaotic systems. These systems are described by more complex mathematics that go beyond the realm of integer-order derivatives, and exhibit richer dynamics, full of potential applications across various disciplines [2]. However, harnessing this potential depends on our ability to control these complex systems.

Controlling partial chaotic systems represents an exciting frontier with broad implications across engineering, biology, and secure communications [3]. Methodologies include traditional control techniques, partial-order controllers, adaptive strategies, optimization, and chaos synchronization [4]. Challenges include limited understanding, robust controller design, scalability, computational complexity, and experimental validation [2]. Addressing these issues requires multidisciplinary collaboration to unleash their full potential.

Among the countless complex dynamic behaviors observed in the natural sciences, atmospheric models stand out as an exemplary case study. In 1984, Edward Lorenz made modifications to his system of equations, giving birth to the Lorenz-84 model [5]. By taking advantage of hyper-graphs, we detect chaos situations characterized by incorrect values, leading to deeper exploration [6, 7].

In recent years, research into fractional calculus and its applications has experienced a boom in both the

quantity of work and the variety of application subjects and methods used [8–10].

In the field of chaotic systems, the introduction of fractional order operators has made it possible to model several physical phenomena and, above all, to add a degree of complexity to data encryption systems, particularly for secure communication [11, 12]. Several fractional order control and synchronization techniques have been proposed in the specialized literature such as adaptive control [13], sliding mode control [14], PID control [15], etc.

The proposed methodology is based on the Grünwald-Letnikov approximation of partial integral to simulate and understand the complex behavior of the system [16] with the primary goal of reducing the chaotic tendencies inherent in the model and guiding it towards stability using advanced control methodologies. Our approach is based on the use of state space feedback control, a powerful technique for manipulating system dynamics by modifying input signals based on system states [17]. Furthermore, we harness the capabilities of Particle Swarm Optimization (PSO)—a nature-inspired transformational algorithm—to determine the optimal control gains, ensuring effective stability of the chaotic Lorenz-84 atmospheric model [18].

This paper is organized as follows: Section 2 presents some basic definitions of fractional calculus. Section 3 is dedicated to the description of the fractional order Lorenz-84 system. Then, Section 4 presents the proposed control strategy. Simulation results are given in Section 5 whereas Section 6 concludes this paper.

2. Fractional-Order Operators

With a 300-year history, fractional calculus extends differentiation/integration for non-integer orders, finding recent application in control systems through definitions such as Caputo and Grünwald-Letnikov derivatives [6, 19].

2.1. Grünwald-Letnikov (G-L) Definition

For $q > 0$, the definition of the G-L fractional order derivative is.

$$D_{GL}^q f(t) = \lim_{h \rightarrow 0} h^{-q} \sum_{j=0}^h (-1)^j \binom{q}{j} f(kh - jh) \tag{1}$$

where h is a sample period, and coefficients:

$$W_j^q = \binom{q}{j} = \frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)} \tag{2}$$

With $W_0^q = \binom{q}{0} = 1$ are the following binomial coefficients:

$$(1 - z)^q = \sum_{j=0}^{\infty} (-1)^j \binom{q}{j} z^j = \sum_{j=0}^{\infty} W_j^{(q)} z^j \tag{3}$$

The fractional G-L integration is formulated as:

$$I_{GL}^\gamma f(t) = \lim_{h \rightarrow 0} h^\gamma \sum_{j=0}^h (-1)^j \binom{-\gamma}{j} f(kh - jh) \tag{4}$$

With $W_0^{-\gamma} = \binom{-\gamma}{0} = 1$ are the following binomial coefficients:

$$(1 - z)^{-\gamma} = \sum_{j=0}^{\infty} (-1)^j \binom{-\gamma}{j} z^j = \sum_{j=0}^{\infty} W_j^{(-\gamma)} z^j \tag{5}$$

2.2. Grunwald-Letnikov (G-L) Approximation

Due to sampling in industrial control, numerical approximations are essential for capturing the fractional dynamics.

The Grünwald-Letnikov (G-L) approximation for the fractional derivative integral of causal function $f(t)$ at time $t = kh$ is given by:

$$D^q f(kh) \approx h^{-q} \sum_{j=0}^k w_j^q f(kh - jh) \tag{6}$$

$$I_{GL}^\gamma f(kh) \approx h^\gamma \sum_{j=0}^k w_j^{-\gamma} f(kh - jh) \tag{7}$$

The coefficients w_j^q and $w_j^{-\gamma}$ represent binomial coefficients from the Eqs. (8) and (10), respectively. These coefficients can be computed using the following two recursive formulas for $j = 1, 2, \dots, k$.

$$w_0^q = w_0^{-\gamma} = 1$$

$$w_j^q = \left(1 - \frac{1+q}{j}\right) w_{j-1}^q, w_j^{-\gamma} = \left(1 - \frac{1-\gamma}{j}\right) w_{j-1}^{-\gamma}$$

2.3. Fractional-Order Chaotic Lorenz-84 Model

The equations below define the fractional Lorenz-84 system [9]:

$$D^q x_1(t) = -ax_1 - x_2^2 - x_3^2 + aF$$

$$D^q x_2(t) = -x_2 - x_1x_2 - bx_1x_3 + G \tag{8}$$

$$D^q x_3(t) = -x_3 + bx_1x_2 + x_1x_3$$

x_1 , x_2 , and x_3 represent global westerly current, cosine phase strength, and sine phase strength, respectively. Parameters a and b represent advection of wave strength by easterly and westerly currents, respectively. F and G are positive thermal forcing terms that typically depend on temperature constants.

By a simple analysis, we obtain an unstable equilibrium point. $E = (7.99626863, 0.00652641402, 0.029837024)$.

To select the appropriate fractional order, we analyze the bifurcation diagram that shows the evolution of x_2 with respect to q in Fig. 1.

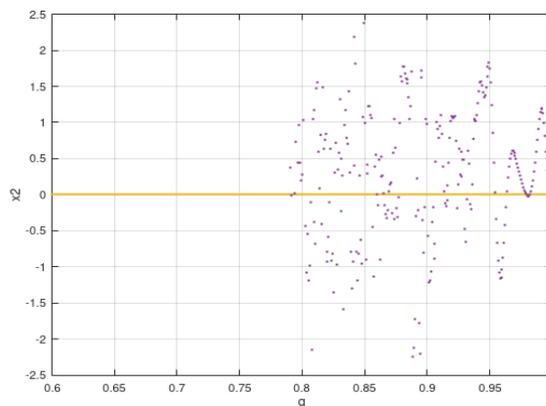


Fig. 1. Bifurcation diagram $x_2 = f(q)$.

When $(F, G, a, b) = (8, 1, 0.25, 4)$ [20], the diagram allows us to identify the interval over which chaos exists. In our investigation, we selected a specific fractional system ($q = 0.83$) and analyzed the resulting attractor, which is shown in Figs. 2 and 3. The initial conditions used were $x(0) = 1, y(0) = 1$, and $z(0) = 1$.

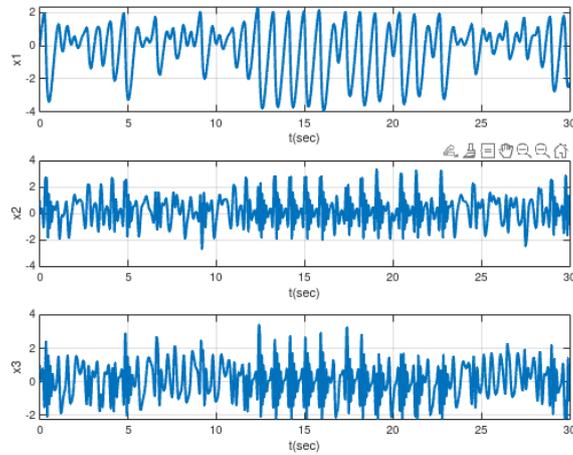


Fig. 2. Phase plane of fractional chaotic Lorenz-84.

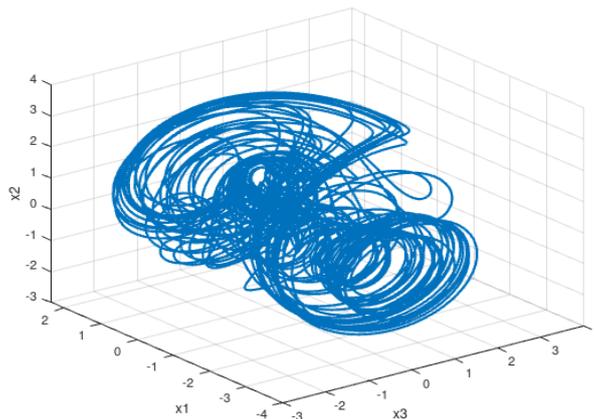


Fig. 3. State variables of fractional Lorenz-84 system.

3. Control Strategy

To control chaos in the Lorenz-84 model, we use a combination of State Feedback Control and Particle Swarm Optimization (PSO) based on ITAE as the key function.

3.1. State Space Feedback Control

Guiding a dynamic system to an equilibrium point using state-space equations and feedback control, where the control input is

$$U = kx \tag{9}$$

k is the feedback gain matrix.

3.2. Particle Swarm Optimization (PSO)

An algorithm inspired by animal social behavior optimizes control gains for a state-space feedback controller, stabilizing the FO-chaotic Lorenz-84 model. This is achieved through iterative optimization of particle velocities and positions, guided by individual and neighbor experiences as follows [6]:

Velocity update:

$$v(t + 1)_{ij} = w \cdot v(t)_{ij} + c_1 r_1 pbest(t)_{ij} - x(t)_{ij} + c_2 r_2 (gbest_j - x(t)_{ij}) \quad (10)$$

Position update:

$$x(t + 1)_{ij} = x(t)_{ij} + v(t + 1)_{ij} \quad (11)$$

where:

$v(t)_{ij}$ is the velocity of particle i in dimension j at time t .

$x(t)_{ij}$ is the position of particle i in dimension j at time t .

w is the inertia weight, controlling the influence of the previous velocity.

c_1 and c_2 are acceleration coefficients representing the cognitive and social components, respectively.

r_1 and r_2 are random numbers sampled from a uniform distribution in the range $[0, 1]$.

p_{bestij} is the best position achieved by particle i in dimension j so far.

g_{bestj} is the best position achieved by any particle in dimension j so far.

3.3. Integral Time Absolute Error (ITAE)

The system's performance is evaluated by looking at the total error over time, calculated using the formula [21]:

$$ITAE = \int_0^{\infty} |e(t)| t dt \quad (12)$$

where $e(t)$ is the error at a given time.

4. Simulation Results

The following model describes the controlled fractional chaotic Lorenz-84 system incorporating the previously defined controller:

$$\begin{aligned} \frac{dx_1^q}{dt^q} &= -ax_1 - x_2^2 - x_3^2 + aF - k_1x_1 \\ \frac{dx_2^q}{dt^q} &= -x_2 - x_1x_2 - bx_1x_3 + G - k_2x_2 \\ \frac{dx_3^q}{dt^q} &= -x_3 + bx_1x_2 + x_1x_3 - k_3x_3 \end{aligned} \quad (13)$$

In this study, we look for the stabilization of the fractional chaotic Lorenz-84 system on its unstable equilibrium point.

In order to find the best values for controller parameters k_1 , k_2 , and k_3 , we employed a PSO algorithm. The optimized values of k are shown in Fig. 4 and the corresponding reduction in error is evident in Fig. 5.

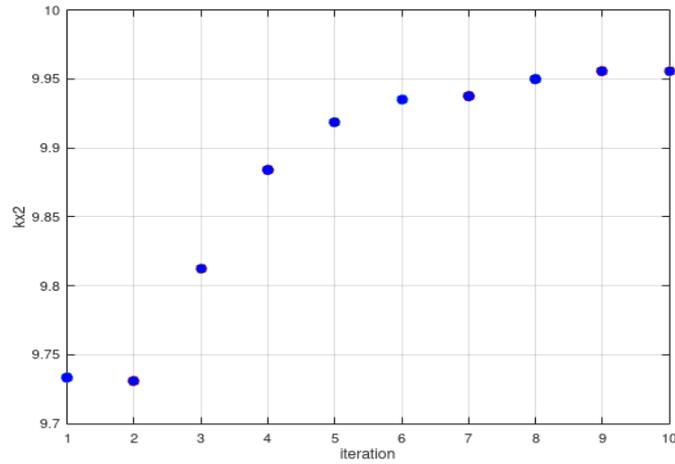


Fig. 4. Optimized controller parameters using PSO algorithm.

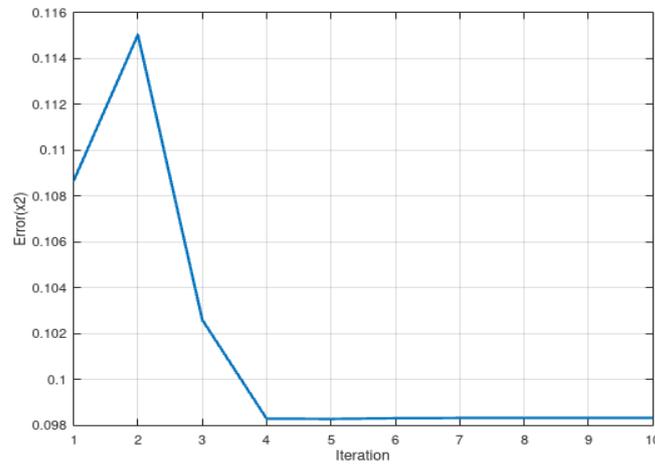


Fig. 5. Reduction in error corresponding to optimized controller parameters.

We utilize feedback with gains $k_1= 0.9$, $k_2 = 0.01$, and $k_3= 0.05$ to stabilize the Lorenz-84 fractional chaotic system at a desired equilibrium point. The results are presented in Figs. 6 and 7.

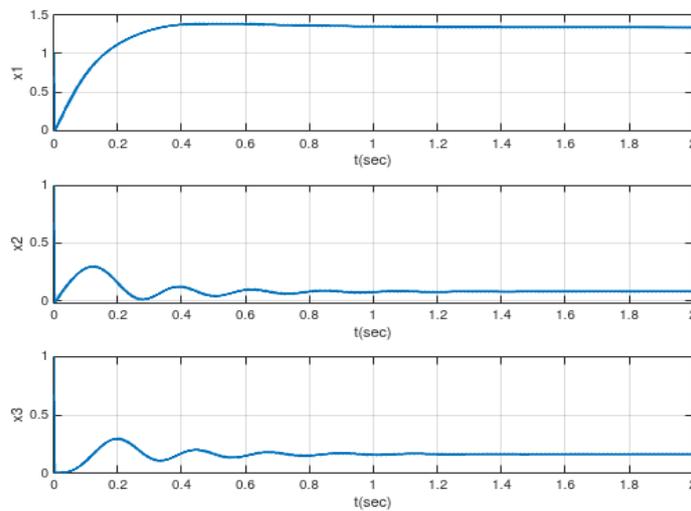


Fig. 6. State variables of fractional Lorenz-84 System with control.

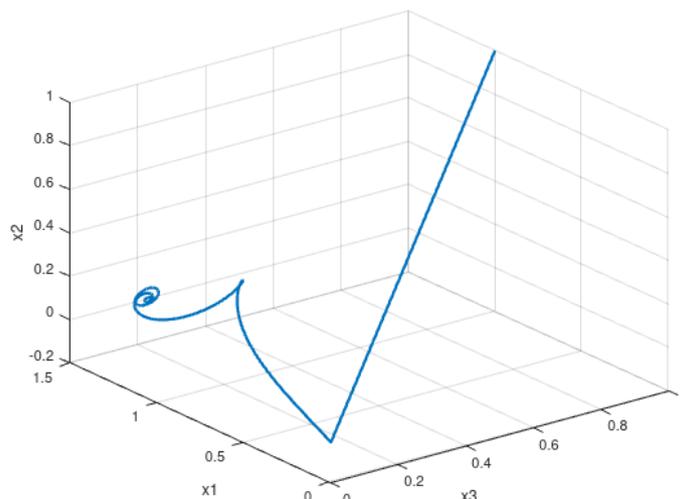


Fig. 7. Phase plane of fractional chaotic Lorenz-84 with control.

Simulation results show that the proposed controller is able to stabilize the fractional order chaotic system within a reasonable time.

5. Conclusion

This study explores stabilizing the three-dimensional fractional-order chaotic Lorenz-84 atmosphere model using control strategies. We employ the Grünwald-Letnikov approximation for fractional integration to accurately capture system dynamics. State-space feedback control adjusts input signals based on system states, guiding chaotic dynamics towards stability. Additionally, Particle Swarm Optimization (PSO) optimizes control gains to ensure robust stabilization. Results illustrate the effectiveness of state-space feedback control and PSO in stabilizing chaotic behavior of the Fractional-order Chaotic Lorenz-84 Atmosphere System.

Future work will address synchronization and control techniques for fractional-order chaotic systems with uncertainties and disturbances.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

N-E.M. conducted the research; S.B. analyzed the data; S.L. supervised the research and corrected the paper; all authors had approved the final version.

Funding

This research was funded by Algerian MESRS PRFU project N° A10N01ES160220230001.

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