Proposal for Developing Learning Materials Focusing on Modified Problem-Posing Using a Probability Game to Promote Understanding of the Problem Structure in Basic Engineering Mathematics

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Abstract: This study proposes a new method of teaching and practicing problem-posing based on a probability game that enhances students' problem structure comprehension. We employed problems with probability elements as subjects that interest numerous students and developed a problem modification/transformation method that allows them to create new problems independently. We suggest that this methodology can connect to basic engineering mathematics skills.

Keywords: Understanding of the problem structure, problem-posing, probability, basic mathematics

1. Introduction

This study aimed to develop a problem-posing learning material based on probability games that develop students' ability to understand the structure of problems, and to propose a practical method for its implementation.

In the globally developing information society, the skills required of students are becoming increasingly diverse. In addition to traditional learning, it is becoming critical to develop the content of each curriculum subject area and to cultivate learners' ability to formulate and solve problems independently and boldly. Enhancing basic relevant skills in this regard is essential, as well as utilizing them to solve advanced problems and create new ones.

In engineering mathematics, students study models that can represent phenomena in the world. In engineering mathematics, we think about models that can describe the world's phenomena and study the mathematical and computational methods for realizing them. In education, Science, Technology, Engineering, and Mathematics (STEM) subjects [1], which incorporate technology and engineering education in addition to traditional science and mathematics education are important. In addition, because STEM education was born out of the need to strengthen industrial competitiveness in American society, the subjects were biased toward science and engineering. Therefore, Science, Technology, Engineering, Art, and Mathematics (STEAM) education has been proposed to address not only industrial and economic activities but also the complex and diverse problems of the real world to link a broader range of subjects [2]. In the future, problem-solving skills based on understanding the essential structure of problems while utilizing industrial and mathematical methods will be increasingly important. Engineering mathematics includes fundamental fields that apply to a wide range of engineering, such as calculus, linear algebra, ordinary/partial differential equations, probability and statistics, graph theory, and so on. In addition, there are various subjects such as algorithms, graph theory, mathematical programming, combinatorial optimization, game theory, operations research, natural language processing, programming, simulation, data mining, and basic information theory in each specialized area. Therefore, it is essential to provide students of all ages with the primary content of these fields systematically and methodically to enrich engineering mathematics, which is oriented toward problem-solving.

The main focus of engineering mathematics is to solve the problems at hand, and it is crucial to represent the world's phenomena as models. Therefore, when creating a model, it is essential to understand the target problem's structure and create a model based on that understanding. In addition, with the recent increase in computing power, problems can be solved using MATLAB and Artificial Intelligence (AI). To improve such skills, it is necessary to understand the structure of the problem to be solved and to perform appropriate modeling.

This study focuses on probability and statistics as the subject matter. This is because studying probability and statistics is essential in data science. In other words, probability and statistics should be used as subject matter to enhance basic modeling skills while making students understand the structure of the problem to enrich engineering mathematics.

In this study, we propose a method of practice focusing on problem-posing learning. We also present an overview of the activities and provide insights into the enrichment of basic engineering mathematics.

2. Design of Materials and Activities

2.1. Problem-Posing and Modified Problem-Posing

Polya [3] states that understanding the structure of a mathematical problem is the first step in solving it. Anderson *et al*. [4] said that the five phases are "Define Phase, where they identified the problem to be solved," "an Encode Phase, where they encoded the needed information, a Compute Phase where they performed the necessary arithmetic calculations," "a Transform Phase where they performed any mathematical transformations," and "a Respond Phase where they entered an answer." This process is expected to lead to the development of computational thinking [5] and of activities designed to support the understanding of the problems' structure as well as the extraction and encoding of necessary parameters. Performing calculations is critical for developing the skills needed in mathematics, computer science and so on. Furthermore, by ensuring an understanding of the structure, it is expected that students will have a better perspective on problems and will be better able to identify and solve new developmental problems.

This study focuses on "problem-posing" to realize the above contents. Problem-posing is an activity in which learners create problems, which improves their problem-solving skills and measures their understanding of the learning content. Its effectiveness has been widely recognized [6]. Besides, it has been pointed out that problem-posing is one of the essential aspects of mathematics [7], effective in promoting an understanding of the problem structure [8] and fostering creativity [6]. Additionally, there is a connection between learning to formulate problems and creativity [9]. Polya maintained that changing/modifying original problems and creating new problems are significant for increased understanding and resolution capacity [3]. Research on problem-posing learning has been conducted from various perspectives, and its usefulness has been pointed out [10].

Here, some recent research on problem-posing is discussed. Rochaminah *et al*. [11], in a qualitative study to understand students' creative thinking mathematics in problem posing, reported that when comparing the problem posing of participants with high, medium, and low mathematical ability, each participant used different concepts. Cai and Leikin [12] suggest that focusing on the relationship between mathematics problem posing and emotions is essential for future research on mathematics problem-posing. Saeed *et al*. [13], for example, reported on a practice aimed at improving the problem-posing skills of prospective mathematics teachers and found that it was effective. This study was conducted from multiple perspectives, including structured and semi-structured situations, and is considered to provide important insights for future classroom practice with students. Studies have also been conducted in science education that incorporate a problem-posing approach, and the method is considered applicable beyond mathematics [14].

Thus, problem-posing is a promising method for improving learners' understanding of the structure of problems and fostering creativity. It also has many applications and is expected to be used in cross-curricular classes. Therefore, it is necessary to develop relevant materials that can foster developmental learning while increasing students' interest.

However, Mestre [15] reported that the range of associations between situations and the resolution methodologies is limited, especially for novice students, and that bias in problem-posing may occur. Considering this, to promote understanding of the problem structure, it is necessary to use a subject that is easy for beginners to work on and to develop activities that strengthen these skills. In Addition, it is essential to make it possible for non-beginners and experts to generate new problems as well. And it is still being determined whether there is a strong focus on understanding the problem structure. However, there are few such studies, and further investigation is required.

Fukui *et al.* [16, 17] proposes "modified problem-posing" to solve these problems and points out its usefulness. He also discusses how to use it in programming education [16] and what flow of problem-based learning activity [17] should be implemented in mathematics education. However, he did not propose any concrete, functional problems, and it is necessary to select a suitable subject matter for actual implementation in school education. Therefore, the purpose of this study was to develop teaching materials that students would be interested in and to connect them to the creation of classes that many students can work on in the future.

2.2. Design of Materials

In this section, we explain how to develop problem-posing learning materials that promote understanding of the problem structure and meet the following conditions:

(1) Utilization of well-structured problems.

(2) The subject matter should be of interest to students and not require a lot of prior knowledge.

(3) The material should encourage understanding of the problem structure, including through appropriate activities.

For Condition (1), a well-structured problem is clear, easy to model, and has a clear starting point and goal. Almost all mathematical problems in school education fall into this category, and the Tower of Hanoi is a well-known example in this sense due to its game format. In contrast, ill-structured issues are challenging to use in general education because most of them are unsolvable by teachers, and the teaching methods still need to be clarified. Well-structured problems are promising subjects because the first goal is to understand a structured problem, and the second goal is to modify or improve the problem to create a new problem.

For Condition (2), in this study, we will develop probability statistics subject matter, and it is important to develop practices using subject matter that is easy to understand even for beginning students. For example, probability problems with a game element are used as a subject that can raise students' interest and only require little prior knowledge. The usefulness of game material in education has been pointed out for a long time. Finke *et al*. [18] highlighted that constraints are necessary for creativity to be exercised. Using games stimulates students to create various problems within the constraints of a rule-based activity and to solve these independently.

For Condition (3), to promote understanding of the problem structure and balance this with creating and solving new problems, a phase of solving probabilities with game elements and transforming and solving new problems based on the actual problem is set up.

2.3. Development of Materials

Russian roulette is a classic example of a probability problem; however, due to ethical educational considerations, we used the problem of removing a card from a jar/box as a subject that satisfied the conditions in the previous section. This was inspired by Yamamoto and Kumakura [19], who developed probability teaching material for taking a lottery ticket from a bag that fostered developmental thinking and showed its usefulness in practice.

In addition, to make it easier for high school students actually to work on it in the classroom, there is a practice of preparing actual objects called cards and clarifying their mathematical structure, which has been shown to be useful [20].

Problem 1: The cards are shuffled into a pile of one red card and five white cards, and the two players take turns to remove a card from the top of the deck and check its. If white appears, it is the next player's turn; if red appears, the player loses. The probability of each player losing is then calculated. Based on the result, do you choose to go first or second?

We defined the number of trials without distinguishing between players. Thus, we counted the number of trials without distinguishing between players. Specifically, the first player plays for the first, third, and fifth times overall, and the second player plays for the second, fourth, and sixth times overall.

Let p_1 be the probability that the first player A loses on the first attempt, p_2 the probability that the second player B loses on the second attempt, ..., and p6 the probability that the second player B loses on the sixth attempt.

$$p_1 = \frac{1}{6}$$

$$p_2 = \frac{5}{6} \times \frac{1}{5} = \frac{1}{6}$$

$$p_3 = \frac{5}{6} \times \frac{4}{5} \times \frac{1}{4} = \frac{1}{6}$$

$$p_4 = \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{6}$$

$$p_5 = \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$p_6 = \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{6}$$

The probability of first player A losing p_A is as follows:

$$p_A = p_1 + p_3 + p_5 = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

The probability that the second player B loses, p_B is as follows:

$$p_B = p_2 + p_4 + p_6 = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

In this problem, the probabilities of losing for the first and second player are identical.

Next, we set up the following problem.

Problem 2: The cards are shuffled into a pile of two red cards and four white cards, and the two players take turns to remove a card from the top of the deck and check its color. If white appears, it is the next player's turn; if red appears, the player loses. The probability of each player losing is then calculated. Based on the result, do you choose to go first or second?

Define p_1, ..., p_5 as in Problem 1. In Problem 2, the second player cannot make the last play of the latter hand because there are two red cards.

$$p_1 = \frac{1}{3}(= \frac{5}{15})$$

$$p_2 = \frac{4}{6} \times \frac{2}{5} = \frac{4}{15}$$

$$p_3 = \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} = \frac{1}{5}(= \frac{3}{15})$$

$$p_4 = \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} = \frac{2}{15}$$

$$p_5 = \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2} = \frac{1}{15}$$

Probability of first player A losing p_A is as follows:

$$p_A = p_1 + p_3 + p_5 = \frac{5}{15} + \frac{3}{15} + \frac{1}{15} = \frac{3}{5}$$

The probability that the second player B loses, p_B is as follows:

$$p_B = p_2 + p_4 = \frac{4}{15} + \frac{2}{15} = \frac{2}{5}$$

In this case, the first player is at a disadvantage, and the second player has the advantage.

Problem 3: A game is played under the same losing conditions as Problems 1 and 2, but with any number of red and white cards (at least one red card and one white card each). Determine the probability of losing for the first and second players. Based on the results, would you choose to play first or second?

Let there be x number of red cards and y number of white cards, and let p_n be the probability of drawing red and losing on the nth play (n is finite, $x \geqq 1, y \geqq 1$).

In general, $p_1 \geqq p_2 \geqq p_3 \geqq, \ldots, \geqq p_{n-1} \geqq p_n$. Focusing on p_1 and p_2,

$$p_1 = \frac{x}{x+y}$$

$$p_2 = \frac{x}{x+y} \times \frac{x}{x+y-1} = \frac{xy}{(x+y)(x+y-1)}$$

Now, looking at the relationship between p_1 and p_2, we see that

$$p_1 - p_2 = \frac{x(x-1)}{(x+y)(x+y-1)} \geq 0$$

$$p_1 \geq p_2$$

Performing the same calculations below, $p_1 \geqq p_2 \geqq p_3 \geqq, \ldots, \geqq p_{n-1} \geqq p_n$.

Thus, when $x = 1$ and y is an odd number, the probabilities of losing for the first and second players are the same, but in other cases, the first player is disadvantaged, and the second player is advantaged.

Therefore, the game is a tie when there is one red card and an odd number of white cards, and we know that the number of red cards must be one for the probability of losing to be equal when the cards are stacked in one pile in a two-player game. This fact may come as a surprise to students. Furthermore, calculating the losing probability allows them to understand that they can choose the first or the second move, and this is expected to make them realize the usefulness of probability calculation.

The following activity is then set up to allow students to create new problems based on this one and solve them. Students are asked to create a new problem for themselves, check it, verify it, modify it into a problem they think is solvable, and then solve it.

Problem 4: Create your own problems by modifying and improving Problem 1 and determine the losing probability of each player in each situation. If you can, classify the cases in which the modified problem is solvable and the instances in which it is not, and try to make the necessary modifications to make it solvable.

Problem 1 calculates the probability that the first or second player loses in this game. It is expected that Problem 2 will help students understand the structure of Problem 1. Based on this essential skill, Problems 3 and 4 aim to allow them to understand the existing problems even better, which enables them to conceive new challenges.

2.4. Example of Modifying a Problem with Modified Problem-Posing

When we formulate a question by modifying or improving Problem 1, we can divide the possible modifications or improvements into three main parts: "initial conditions," "gameplay," and the "end condition." Specifically, the initial conditions are the number of cards in each deck, the number of participants, and how the cards are arranged (in a single deck). The gameplay is taking a card from the deck." The end condition is the number of red cards in each deck. The condition is "you lose if you draw a red card." The above contents can be captured immediately after reading a given problem and can be modified and improved in various ways by adding problem statements and combining variations.

For example, the initial condition could be "increase the number of card types," and the end condition could be "you lose if you draw a red card." The gameplay can be "draw two cards." Another example would be "take two cards" as a gameplay condition and "return one card to the pile after taking two white cards" as an additional gameplay rule.

The following are examples of modifications and enhancements, such as adding some rules to the original problem:

(1) Example 1: Changing the number of cards

Two players take turns drawing a card from the top of the deck and turning it over. If a white card is extracted, it is the next player's turn; if a red card is drawn, the player loses.

(2) Example 2: Changing the number of players

Three players take turns drawing a card from the top of the deck and turning it over. If a white card is extracted, it is the next player's turn; if a red card is drawn, the player loses.

(3) Example 3: Changing the card order

The cards, consisting of one red card and five white cards, are shuffled, and divided into two piles. The two players then select a deck of cards and turn over a card from the top of the deck. If a white card is extracted, it is the next player's turn; if a red card is drawn, the player loses.

(4) Example 4: Changing the method of drawing cards

The two players take turns taking one or two cards from the top of the deck and turning them over. If a white card is drawn, it is the next player's turn. If a white card is extracted, it is the next player's turn; if a red card is drawn, the player loses.

There can be other variations and improvements. If the end condition is "the game ends when the player draws two red cards at the same time," it is necessary to adjust the number of cards, consider whether a tie is allowed, and add or adjust other rules, such as putting the cards back in a pile. Thus, the modifications may also require adjustments to other parts of the game. However, adjusting the problem to render it solvable and having the students explain the adjustment is assumed to be effective in promoting a deeper understanding of the problem structure and developing the ability to create new problems based on one's own thinking.

There can also be changes that keep the structure of the problem the same. For example, if the problem is to find the probability of winning by drawing the white card when the one who takes the red card loses, the probability of winning is the same. Therefore, we are asked to think about how deformations affect the game. Additionally, it is possible to alter more than one part simultaneously, so students require flexibility to generate lots of modified problems.

2.5. Examples of Instruction

Examples of instructions are provided in continuation.

(1) Example of instruction for changing the number and types of cards

For changing the number of cards, when the game is played in two, the students who have already made this change in Problems 1 and 2 are instructed to make another alteration. Additionally, the original problem has two types of cards (red and white), and another one can be added (black). In this case, however, it is necessary to have the players decide how the game can be lost.

(2) Example of instruction for changing the number of participants

It is relatively easy to change the number of participants to three, four, or s. However, if the number of participants is not changed, the calculation can be performed in the same way as in the original problem. Therefore, students who are only altering the number of participants should be instructed to make a different variation. If the calculation is complicated, it may be effective to adjust the number of cards and the number of players to adjust the difficulty level of the problem.

(3) Example of instruction for changing the arrangement of cards

This could include dividing the cards into multiple piles or spreading all cards on the table so that players can take any card they wish. If this modification is made, it may also be necessary to vary how the cards are drawn. Therefore, it would be effective to instruct students to consider how the card extraction method can be modified to solve the problem.

(4) Example of instruction for changing the card extraction and return method

Examples of changes in card draws include introducing a play in which the player can take either one or two cards at a time and not take a card only once. This change is expected to depend on the way the cards are arranged. Therefore, it would be effective to instruct students to consider the possible forms of this variation, which can make the problem solvable by changing the arrangement of the cards as well.

(5) Example of instruction for changing the game end condition

One example of changing the end condition would be when a player loses if two white cards are taken or, allowing to put back a red card just once, a player would lose the next time they drew one. This change may require adjustments to how the cards are extracted and put back while maintaining the problem as solvable. Therefore, it would be effective to instruct the players to consider this.

(6) Examples of instruction for other changes

There could be a variant where each time the cards are taken, they are shuffled each time. Tell the students that the original problem will be used as a reference if the original problem is not used as a reference. Or ask them to explain the problem they created.

Such potential modifications show that this is an open-ended problem, and students can modify it with considerable freedom.

Fukui *et al*. [21] reported that most high school students only changed the initial conditions of the problem, such as the number and/or types of cards or the number of players. Therefore, we also incorporated instructions to encourage students to vary other game parts. Thus, designing a problem that progressively develops from narrow to wide and open-ended is possible. Finally, it is also possible to develop activities that make students realize the importance of asking questions by having them select one or more interventions from the six ones previously suggested and formulating a new problem independently.

3. Conclusions

This study aimed to develop problem-posing learning materials based on a probability game that develops students' ability to understand the problems' structure and proposes a practical method for its application to enhance basic engineering mathematics. The developed teaching materials can be used in actual educational settings and are expected to have wide appeal for students. It is also essential to quantitatively evaluate the changes in computational thinking obtained this way and to connect this to relevant research. In addition, the proposed subject matter can be used in engineering mathematics and mathematics education in general school education. Furthermore, it is assumed that this subject matter can be further improved and applied to developing subject matter in physics and other fields.

However, we have yet to be able to examine the learning effects of the subject matter developed in this study. The validity of the subject matter as a subject matter for basic industrial mathematics has yet to be examined either. In the future, it will be necessary to investigate the effectiveness of the subject matter by practicing it with high school and university students in a practical manner. Therefore, in the future, it is necessary to conduct practical lessons using this material, to evaluate it quantitatively, and to develop subjects for other subjects. These are issues to be addressed in the future.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

MF and MK Conceptualization; MF methodology; PC and YS validation; MF and MK formal analysis; MF, MK, and YS investigation; MF resources; MF writing original draft preparation; PC writing, review, and editing; MF project administration. All authors had approved the final version.

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