FC-Projective Comodules and FC-Projective Dimensions

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Manuscript submitted February 28, 2023; revised April 29, 2023; accepted August 14, 2023; published January 30, 2024.
doi: 10.17706/ijapm.2024.14.1.12-19

Abstract: The purpose of this paper is to investigate FC-projective comodules and FC-projective dimensions. We introduce the concepts of copure exact sequences and FC-projective comodules, and give some equivalent characterizations of FC-projective modules. We also define FC-projective dimensions, which is similar to projective dimensions. We characterize semihereditary and coregular coalgebras related to FC-projective comodules.

Keywords: Coalgebra, comodule, FC-projective comodule, FC-projective dimension, semihereditary coalgebra, coregular coalgebra

1. Introduction

Note that absolutely pure modules are precisely FP-injective modules. Maddox [1] and Stenström [2] introduced absolutely pure modules (i.e., FP-injective modules). Enochs [3] and Stenström [2] gave that a ring is right Noetherian (resp. coherent) if and only if the class of injective (resp. FP-injective) right modules is closed under direct limits. Mao and Ding [4] wondered if every right R-module over any ring has an FP-injective precover. They investigated the FI-injective and FI-flat modules with respect to FP-injective modules [5]. Li *et al.* [6] introduced the strong FP-injective modules and characterized coherent rings. If R be a commutative ring with unitary. Recall that an exact sequence of right R-modules $0 \to K \to M \to N \to 0$ is called pure exact [7], if for every left R-module A, we have the exactness of $0 \to K \otimes A \to M \otimes A \to N \otimes A \to 0$. We say that $f(K) \subseteq M$ is a pure submodule in this case. An R-module M is called absolutely pure [1] if M is a pure submodule in every R-module which contains M as a submodule. According to Ref. [8], an R-module M is absolutely pure if and only if $\mathrm{Ext}_C^1(F, M) = 0$ for all finitely presented R-modules F.

The coalgebra theory as a dual of algebra has been more and more prosperous. Many scholars devoted themselves to investigating this aspect and came up with some open problems [9, 10]. For instance, Professor Simson [11] in Copernicus University, pointed nine questions out, about the structure of coalgebras and the research for representations.

Motivated by the above research mentioned, our main aim of this present paper is to study the FC-projective comodules and FC-projective dimension. Then we use FC-projective comodules to characterize special coalgebras. Throughout this paper, C denotes the semiperfect coalgebra over filed k.

2. FC-Projective Comodules

2.1. Definition

A short exact sequence of C-comodules

$$0 \to K \to M \to N \to 0 \tag{1}$$

is said to be copure if every finitely copresented comodule is injective with respect to this sequence.

2.2. Definition

A subcomodule K of M is said to be copure in M if the canonical short exact sequence $0 \to K \to M \to M/K \to 0$ is copure.

Remark: We can obtain that the exact sequence (1) is copure if and only if K is the copure subcomodule of M from Definitions 2.1 and 2.2.

2.3. Definition

Let C be a coalgebra, and M a right C-comodule. We call M a FC-projective comodule, if $\mathrm{Ext}_C^1(M, F) = 0$ for any finitely copresented right C-comodule F.

Remark: It is clear that the class of FC-projective comodules is closed under direct sums and direct summands.

2.4. Proposition

Assume that C is a right semiperfect coalgebra and M is a right C-comodule. Then the following are equivalent.

(1) M is FC-projective;

(2) Every exact sequence $0 \to A \to B \to M \to 0$ is copure;

(3) There is a copure exact sequence $0 \to K \to P \to M \to 0$ with P being a projective right C-comodule;

(4) There exists a copure exact sequence of right C-comodules

$$0 \to M_1 \to M_2 \to M \to 0$$

with M_2 being FC-projective;

(5) Let F be a finitely cogenerated free right C-comodule, and N be a finitely cogenerated quotient comodule of F, then for any right C-comodule homomorphism $h: M \to L$, there is a right C-comodule homomorphism $g: M \to F$ such that the following diagram is commutative.

$$\begin{tikzcd} & M \arrow[dl, dotted, "g"'] \arrow[d, "h"] & \\ F \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

Proof: (1) $\Rightarrow$ (2) Take any finitely copresented right C-comodule F, it is easy to get by the assumption that $\mathrm{Ext}_C^1(M, F) = 0$. Applying the functor $\mathrm{Com}_C(-, F)$ to the short exact sequence

$$0 \to A \xrightarrow{f} B \to M \to 0,$$

Then we can obtain the following induced exact sequence

$$0 \to \mathrm{Com}_C(M, F) \to \mathrm{Com}_C(B, F) \xrightarrow{\mathrm{Com}_C(f,F)} \mathrm{Com}_C(A, F) \to \mathrm{Ext}_C^1(M, F) = 0.$$

Then, $\mathrm{Com}_C(f, F)$ is an epimorphism. It follows by the definition that the exact sequence

$$0 \to A \to B \to M \to 0$$

is copure.

(2) ⇒ (3) It is clear.

(3) ⇒ (4) Since C is a semiperfect coalgebra, there is a projective right C-comodule P such that M is the epimorphic image of P, i.e. $f: P \to M$ is an epimorphism with P projective. Let $K = \mathrm{Ker} f$, then we have the following exact sequence

$$0 \to K \to P \to M \to 0.$$

Since projective right C-comodule P is FC-projective, the conclusion holds.

(4) ⇒ (1) Let P be a finitely copresented right C-comodule and the exact sequence

$$0 \to M_1 \to M_2 \to M \to 0$$

be copure with M_2 FC-projective. Applying the functor $\mathrm{Com}_C(-, P)$ to the above sequence, then we have the long exact sequence

$$\cdots \to \mathrm{Com}_C(M_2, P) \to \mathrm{Com}_C(M_1, P) \to \mathrm{Ext}_C^1(M, P) \to \mathrm{Ext}_C^1(M_2, P) \to \cdots.$$

It yields by M_2 FC-projective that $\mathrm{Ext}_C^1(M_2, P) = 0$. Since the exact sequence

$$0 \to M_1 \to M_2 \to M \to 0$$

is copure, we can learn that $\mathrm{Com}_C(M_2, P) \to \mathrm{Com}_C(M_1, P)$ is epimorphic. Then, $\mathrm{Ext}_C^1(M, P) = 0$, i.e. M is FC-projective.

(1) ⇔ (5) Applying the functor $\mathrm{Com}_C(M, -)$ to the short exact sequence

$$0 \to K \to F \xrightarrow{p} L \to 0$$

There is a long exact sequence

$$0 \to \mathrm{Com}_C(M, K) \to \mathrm{Com}_C(M, F) \to \mathrm{Com}_C(M, L) \to \mathrm{Ext}_C^1(M, K) \to \cdots$$

Since F is a finitely cogenerated free comodule and L is a finitely cogenerated comodule, we have that K is finitely copresented. It follows by the assumption that $\mathrm{Ext}_C^1(M, K) = 0$. Hence, $\mathrm{Com}_C(M, p)$ is epimorphic. So, $h: M \to L$ can be extended to $g: M \to F$.

On the other hand, let P be any finitely copresented right C-comodule, then there is an exact sequence

$$0 \to P \to F \to L \to 0$$

with F finitely cogenerated free and L finitely cogenerated. Applying the functor $\mathrm{Com}_C(M, -)$ to the above exact sequence, then we can have the following exact sequence

$$\cdots \to \mathrm{Com}_C(M, F) \xrightarrow{\mathrm{Com}_C(M,p)} \mathrm{Com}_C(M, L) \to \mathrm{Ext}_C^1(M, P) \to \mathrm{Ext}_C^1(M, F) = 0.$$

It follows by the assumption that $\mathrm{Com}_C(M, p)$ is epimorphic. Therefore, $\mathrm{Ext}_C^1(M, P) = 0$. So, M is FC-projective.

2.5. Lemma

Let C be a right cocoherent coalgebra, if M is a FC-projective right C-comodule, then $\mathrm{Ext}_C^n(M, F) = 0$ for $n > 0$ and any finitely copresented right C-comodule F.

Proof: Let F be a finitely copresented right C-comodule, since C is right cocoherent, there is an exact sequence of right C-comodules

$$0 \to F \to I_0 \to I_1 \to \cdots \to I_{n-2} \to K_{n-2} \to 0$$

with $K_{n-2} = \mathrm{Im}(I_{n-2} \to I_{n-1})$ being finitely copresented and I_i being finitely cogenerated injective for $0 \le i \le n-2$. Thus, we have that $\mathrm{Ext}_C^n(M, F) \cong \mathrm{Ext}_C^1(M, K_{n-2}) = 0$

2.6. Lemma

Let C be a right cocoherent coalgebra. If $0 \to K \to M \to N \to 0$ is an exact sequence of right C-comodules with M being FC-projective, then $\mathrm{Ext}_C^i(K,F) \cong \mathrm{Ext}_C^{i+1}(N,F)$ for $i > 0$ and any finitely copresented right C-comodule F.

Proof: Consider the exact sequence of right C-comodules

$$0 \to K \to M \to N \to 0$$

with M being FC-projective. Applying the functor $\mathrm{Com}_C(M,-)$ to the above exact sequence for each finitely copresented right C-comodule F, we have the following exact sequence

$$\mathrm{Ext}_C^i(M,F) \to \mathrm{Ext}_C^i(K,F) \to \mathrm{Ext}_C^{i+1}(N,F) \to \mathrm{Ext}_C^{i+1}(M,F)$$

Hence, $\mathrm{Ext}_C^i(M,F) = 0$ and $\mathrm{Ext}_C^{i+1}(M,F) = 0$ by Lemma 2.5. Thus, $\mathrm{Ext}_C^i(K,F) \cong \mathrm{Ext}_C^{i+1}(N,F)$.

2.7. Proposition

Let C be a right cocoherent coalgebra and $0 \to K \to M \to N \to 0$ an exact sequence of right C-comodules.

(1) If K and N are FC-projective, so is M.

(2) If M and N are FC-projective, so is K.

(3) If K and M are FC-projective, then N is FC-projective if and only if $\mathrm{Ext}_C^1(N,F) = 0$ for all finitely copresented right C-comodules F.

Proof: (1) Applying the functor $\mathrm{Com}_C(-,F)$ to the exact sequence for any finitely copresented right C-comodule F, we can obtain the following exact sequence

$$\cdots \to \mathrm{Ext}_C^1(N,F) \to \mathrm{Ext}_C^1(M,F) \to \mathrm{Ext}_C^1(K,F) \to \cdots$$

Since K and N are FC-projective, we have that $\mathrm{Ext}_C^1(K,F) = \mathrm{Ext}_C^1(N,F) = 0$ for any finitely copresented right C-comodule F. It follows that $\mathrm{Ext}_C^1(M,F) = 0$, i.e. M is FC-projective.

(2) Similarly to the proof of (1), we can obtain the result by Lemmas 2.5 and 2.6.

(3) Applying the functor $\mathrm{Com}_C(-,F)$ to the exact sequence for any finitely copresented right C-comodule F, we can obtain the following exact sequence

$$0 \to \mathrm{Com}_C(N,F) \to \mathrm{Com}_C(M,F) \to \mathrm{Com}_C(K,F) \to \mathrm{Ext}_C^1(N,F) \to \mathrm{Ext}_C^1(M,F) \to \mathrm{Ext}_C^1(K,F) \to \cdots$$

By the definition of FC-projective, we can easily get the conclusion.

According to Ref. [12], a class $\mathcal{X}$ of right C-comodules is called projectively resolving, if $\mathcal{P} \subseteq \mathcal{X}$, here $\mathcal{P}$ stands for the class of projective C-comodules, and for every short exact sequence

$$0 \to K \to M \to N \to 0$$

with $N \in \mathcal{X}$, then $K \in \mathcal{X}$ if and only if $M \in \mathcal{X}$. Dually, $\mathcal{X}$ is called injectively resolving, if $\mathcal{I} \subseteq \mathcal{X}$, where $\mathcal{I}$ stands for the class of injective C-comodules, and for every short exact sequence

$$0 \to K \to M \to N \to 0$$

with $K \in \mathcal{X}$, then $M \in \mathcal{X}$ if and only if $N \in \mathcal{X}$.

2.8. Theorem

Let C be a right cocoherent coalgebra, then the class of FC-projective right C-comodules is projectively resolving.

Proof: Note that every projective right C-comodule is FC-projective. Consider the exact sequence of right C-comodules

$$0 \to K \to M \to N \to 0,$$

where K is FC-projective. On the other hand, if N is also FC-projective, we have the following exact sequence

$$0 = \mathrm{Ext}_C^1(N,F) \to \mathrm{Ext}_C^1(M,F) \to \mathrm{Ext}_C^1(K,F) = 0$$

for each finitely copresented right C-comodule F. Thus, $\mathrm{Ext}_C^1(M,F) = 0$. It follows that M is FC-projective.

On the other hand, if M is also FC-projective, we have the following exact sequence

$$\mathrm{Ext}_C^1(M,F) \to \mathrm{Ext}_C^1(K,F) \to \mathrm{Ext}_C^2(N,F)$$

for each finitely copresented right C-comodule F. Since M is FC-projective, we have that $\mathrm{Ext}_C^1(M,F) = 0$. It follows by Lemma 2.5 that $\mathrm{Ext}_C^2(N,F) = 0$. Thus, $\mathrm{Ext}_C^1(K,F) = 0$, i.e. K is FC-projective.

2.9. Definition

Let C be a right cocoherent coalgebra and M a right C-comodule, the FC-projective dimension of M, denoted by $\mathrm{FC}\text{-}pd(M)$, is defined as follows,

$$\mathrm{FC}\text{-}pd(M) = \inf\{\, n \mid \text{there is an exact sequence of right } C\text{-comodules } 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0, \text{ where } P_i \text{ are } FC\text{-projective}\}.$$

If no such exact sequence exists, we define $\mathrm{FC}\text{-}pd(M) = \infty$.

2.10. Proposition

Assume that C is a right cocoherent coalgebra and n is a nonnegative integer. Then the following conditions are equivalent.

(1) $\mathrm{FC}\text{-}pd(M) \le n$;

(2) $\mathrm{Ext}_C^{n+1}(M,F) = 0$ for any finitely copresented right C-comodule F;

(3) $\mathrm{Ext}_C^{n+j}(M,F) = 0$ for any finitely copresented right C-comodule F and any j;

(4) If $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is an exact sequence of right C-comodules with P_i being FC-projective $0 \le i \le n-1$, then P_n is FC-projective;

(5) There is an exact sequence of right C-comodules

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

such that P_i $(0 \le i \le n)$ is FC-projective.

Proof: (1) ⇒ (2) Since $\mathrm{FC}\text{-}pd(M) \le n$, there is an exact sequence of right C-comodules

$$0 \to P_m \to P_{m-1} \to \cdots \to P_0 \to M \to 0$$

where P_i $(0 \le i \le m)$ is FC-projective and $0 \le m \le n$. Thus $\mathrm{Ext}_C^{n+1}(M,F) \cong \mathrm{Ext}_C^{n-m+1}(P_m,F) = 0$ for every finitely copresented right C-comodules F by Lemmas 2.6 and 2.7.

(2) ⇒ (3) The case $j = 1$ is clear. We may assume $j \ge 2$. Since C is right cocoherent and F is finitely copresented, there is an exact sequence of right C-modules

$$0 \to F \to I_0 \to I_1 \to \cdots I_{j-2} \to K_{j-2} \to 0,$$

where $K_{j-2} = Im(I_{j-2} \to I_{j-1})$ is finitely copresented and each I_i $(0 \le i \le n-1)$ is finitely cogenerated injective. Thus, $\mathrm{Ext}_C^{n+j}(M,F) \cong \mathrm{Ext}_C^{n+1}(M,K_{j-2}) = 0$.

(3) $\Rightarrow$ (4) Suppose that $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is an exact sequence of right C-comodules with each P_i FC-projective, then $\mathrm{Ext}_C^1(P_n, F) \cong \mathrm{Ext}_C^{n+1}(M, F) = 0$ for every finitely copresented right C-comodule F. Thus P_n is FC-projective.

(4) $\Rightarrow$ (5) Let $0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be a projective resolution of M, where each P_i $(0 \leq i \leq n-1)$ is projective. Thus, K is FC-projective by (4).

(5) $\Rightarrow$ (1) It is easy to be obtained by the definition.

3. Some Special Cases

3.1. Definition

Let C be a coalgebra, we call C a right semihereditary coalgebra if every finitely cogenerated quotient comodule of finitely cogenerated injective right C-comodule is injective.

3.2. Lemma

Let C be a right semiperfect coalgebra, then the following are equivalent.

(1) C is a right semihereditary coalgebra;

(2) The subcomodules of FC-projective comodules are FC-projective;

(3) The subcomodules of projective comodules are FC-projective.

Proof: (1) $\Rightarrow$ (2) Let M be a FC-projective comodule and K a subcomodule of M. For any finitely cogenerated injective right C-comodule I, and I_2 is a finitely cogenerated quotient comodule of I, we can obtain that I_2 is injective. Then we have the following commutative diagram

$$\begin{tikzcd} I_2 & I \arrow[l, two heads, "p"'] \\ K \arrow[u, "f"] \arrow[r, tail, "i"'] & M \arrow[u, dotted, "h"'] \arrow[ul, dotted, "g"] \end{tikzcd}$$

Indeed, since I_2 is injective, we have that for any $f: K \to I_2$, there is $g: M \to I_2$ such that $f = gi$. It is easy to obtain the following exact sequence

$$0 \to I_1 \to I \xrightarrow{p} I_2 \to 0 \tag{2}$$

with $I_1 = \mathrm{Ker}(I \to I_2)$. Then we can get that I_1 is finitely copresented. Applying the functor $\mathrm{Com}_C(M, -)$ to this exact sequence, since M is FC-projective, we have the induced exact sequence

$$0 \to \mathrm{Com}_C(M, I_1) \to \mathrm{Com}_C(M, I) \xrightarrow{\mathrm{Com}_C(M,p)} \mathrm{Com}_C(M, I_2) \to \mathrm{Ext}_C^1(M, I_1) = 0.$$

So, the $\mathrm{Com}_C(M, p)$ is an epimorphism. This means that for $g: M \to I_2$, there is $h: M \to I$ such that $g = ph$. It follows that for any $f: K \to I_2$, there is $hi: K \to I$ such that $f = phi$. Therefore, the conclusion holds by Proposition 2.4 (5).

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Let I be a finitely cogenerated injective right C-comodule and I_1 a finitely cogenerated quotient comodule of I. Let $I_2 = \mathrm{Ker}(I \to I_1)$, it is easy to see that I_2 is finitely copresented. Take any projective comodule P, and P_1 is a subcomdule of P, then by the assumption that P_1 is FC-projective. We can easily obtain the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I_2 \arrow[r] & I \arrow[r, two heads, "p"] & I_1 \arrow[r] & 0 \\ & & P \arrow[u, dotted, "h"] & P_1 \arrow[l, tail, "i"] \arrow[u, "f"'] \arrow[ul, dotted, "g"] & \end{tikzcd}$$

In fact, since P_1 is FC-projective, it yields that for any $f: P \to I$, there is $g: P_1 \to I$ such that $f = pg$.

Because I is injective, we have that there is $h: P \to I$ such that $g = hi$. Therefore, $f = phi$. So, I_1 is injective.

3.3. Definition

A coalgebra C is said to be right coregular if every finitely copresented right C-comodules is injective.

3.4. Proposition

The following statements are equivalent.

(1) C is a right coregular coalgebra;

(2) Every right C-comodule is FC-projective;

(3) Every finitely cogenerated right C-comodule is FC-projective.

Proof: $(1) \Rightarrow (2) \Rightarrow (3)$ It is obvious.

$(3) \Rightarrow (1)$ Let M be a finitely copresented right C-comodule, $E(M)$ the injective envelope of M. It is easy to see that $E(M)/M$ is finitely cogenerated. By the assumption, we have that $E(M)/M$ is FC-projective. It follows by Proposition 2.5 that the following exact sequence

$$0 \to M \to E(M) \to E(M)/M \to 0 \tag{3}$$

is copure. Applying the functor $\mathrm{Com}_C(-, M)$ to the above sequence, since $E(M)/M$ is FC-projective, we have the following induced exact sequence

$$0 \to \mathrm{Com}_C(E(M)/M, M) \to \mathrm{Com}_C(E(M), M) \to \mathrm{Com}_C(M, M) \to \mathrm{Ext}_C^1(E(M)/M, M) = 0$$

Then we can learn that 1_M factors through $E(M)$. Therefore, the sequence (3) is split. Hence, M is injective. The proof is completed.

4. Conclusion

The concepts of FC-projective comodules and FC-projective dimensions are given. We show the equivalent characterizations of FC-projective modules. Moreover, we investigate FC-projective dimensions. Finally, we use FC-projective comodules to give new characterizations of semihereditary and coregular coalgebras.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Qianqian Yuan and Hailou Yao. The first draft of the manuscript was written by Qianqian Yuan. All authors commented on previous versions of the manuscript, read and approved the final manuscript.

Funding

This work was supported by National Natural Science Foundation of China (Grant No.12071120).

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