On New Approximations of the Generalized Logarithmic Functional Equation in 2-Banach Spaces

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Abstract: In this paper, we investigate the stabillty for the generalized logarithmic functional equation. By using the fixed point method of Brzdęk and Ciepliński ['On a fixed point theorem in 2-Banach spaces and some of its applications', *Acta Math. Scientia, 38(2)*, 377-390], the generalized Hyers-Ulam stability results for the generalized logarithmic functional equation are proved. Our results generalize, extend and complement some earlier classical results concerning the stability of the logarithmic functional equations. In addition, the stability of logarithmic functional equation in 2-Banach spaces can be derived from our main results.

Key words: Fixed point theorem, generalized logarithmic functional equation, Hyers-Ulam stability, 2-Banach spaces.

1. Introduction

In the sequel, $\mathbb{N}_0, \mathbb{N}, \mathbb{Z}$ and $\mathbb{R}_+$ denote the set of nonnegative integers, the set of positive integers, the set of integers, and the set of nonnegative real numbers, respectively. Also, denotes Y^X the set of all functions from a set $X \neq \emptyset$ to a set $Y \neq \emptyset$.

First of all, let us bring back to the history in the stability theory for functional equations. The first idea of the stability for the functional equation was motivated by Ulam's question [1] in 1940. Its partial answer was published by Hyers [2] and then Aoki [3] extended Hyers' result in 1950. Afterwards, Rassias [4] gave the following generalized result in 1978.

Theorem 1.1. ([4]) Let E_1 and E_1 be two Banach spaces and $f: E_1 \to E_2$ be a mapping. If f satisfies the following inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\left(\|x\|^p + \|y\|^p\right)$$

for some $\theta \in [0,\infty)$, for some $0 \leq p < 1$ and for all $x, y \in E_1$, then there exists a unique additive mapping $A: E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$$

for each $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the mapping A is linear.

In 1990, during the 27th International Symposium on Functional Equations, Rassias [5] asked that "Can we prove Theorem 1.1 for case $p < 0$ and $p \geq 1$?". Later, Rassias observed that Theorem 1.1 holds for case $p < 0$ whenever we suppose that $\|0\|^p = \infty$. Later, Gajda [6] gave a solution to Rassias [5] in the case of $p > 1$ and gave an example to show that Theorem 1.1 fails when $p = 1$. Afterwards, Găvruţa [7] gave the generalization of Theorem 1.1 in term of the control function.

In the mid 1960s, the concept of a 2-normed space and the theory of 2-normed spaces were first introduced by Gähler [8].

Definition 1.2 ([8]). Let X be a real vector space with $\dim(X) \geq 2$. A function $\|\cdot,\cdot\|: X \times X \to \mathbb{R}_+$ is called a 2-norm on X if it satisfies the following conditions for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$:

1) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
2) $\|x, y\| = \|y, x\|$;
3) $\|\alpha x, y\| = |\alpha| \|x, y\|$;
4) $\|x + z, y\| = \|x, y\| + \|z, y\|$.

The ordered pair $(X, \|\cdot,\cdot\|)$ is also called a 2-normed space.

Example 1.3. Let $X = \mathbb{R}^2$ and $\|\cdot,\cdot\|: X \times X \to \mathbb{R}_+$ be defined by $\|x, y\| = |x_1 y_2 - x_2 y_1|$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then $\|\cdot,\cdot\|$ is a 2-norm on $\mathbb{R}^2$.

Definition 1.4 A sequence $\{x_n\}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is called a Cauchy sequence if there are linearly dependent $y, z \in X$ such that

$$\lim_{n,m\to\infty} \|x_n - x_m, y\| = 0 = \lim_{n,m\to\infty} \|x_n - x_m, z\|,$$

whereas $\{x_n\}$ is said to be convergent if there exists a point $x \in X$ (called a limit of this sequence and denoted by $\lim_{n\to\infty} x_n$) with

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in X$.

Definition 1.5 A 2-Banach space is a 2-normed space in which every Cauchy sequence is convergent.

Lemma 1.6 ([9]). Suppose that X is a 2-normed space. If $x, y_1, y_2 \in X$ y_1, y_2 are linearly independent and $\|x, y_1\| = 0 = \|x, y_2\|$, then $x = 0$.

Lemma 1.7 ([9]). Suppose that X is a 2-normed space. If $\{x_n\}$ is a convergent sequence in X, then $\lim_{n\to\infty} \|x_n, y\| = \left\| \lim_{n\to\infty} x_n, y \right\|$ for all $x, y \in X$.

In the few decades, many mathematicians investigated several stability results of various functional equations by using the fixed point method in many spaces (see [9]-[13] and references therein).

In 2018, Brzdęk and Ciepliński [9] proved the following fixed point theorem in 2-Banach spaces and gave its applications to the stability of the Cauchy functional equation in 2-Banach spaces.

Theorem 1.8 ([9]). Let U be a nonempty set, $(Y, \|\cdot,\cdot\|)$ be a 2-Banach space, Y_0 be a subset of Y containing two linearly independent vectors, $k \in \mathbb{N}$, $f_i : U \to U$, $g_i : Y_0 \to Y_0$, and $L_i : U \times Y_0 \to \mathbb{R}_+$ are given mappings for $i = 1, 2, \ldots, k$. Suppose that $\mathrm{T}: Y^U \to Y^U$ is an operator satisfying the inequality

$$\left\| (\mathrm{T}\xi)(x) - (\mathrm{T}\mu)(x), y \right\| \leq \sum_{i=1}^{k} L_i(x, y) \left\| \xi\left(f_i(x)\right) - \mu\left(f_i(x)\right), g_i(y_2) \right\| \tag{1}$$

for all $\xi, \mu \in Y^U, x \in U$ and $y \in Y_0$. Assume that there are functions $\varepsilon: U \times Y_0 \to \mathbb{R}_+$ and $\varphi: U \to Y$ fulfil the following conditions for each $x \in U$ and $y \in Y_0$:

$$\left\| (\mathrm{T}\varphi)(x) - \varphi(x), y \right\| \leq \varepsilon(x, y) \tag{2}$$

And

$$\varepsilon^*(x, y) \leq \sum_{n=0}^{\infty} \left(\Lambda^n \varepsilon\right)(x, y) < \infty, \tag{3}$$

where $\Lambda: \mathbb{R}_+^{U \times Y_0} \to \mathbb{R}_+^{U \times Y_0}$ is defined by

$$(\Lambda\delta)(x, y) := \sum_{n=0}^{\infty} L_i(x, y) \delta(f_i(x), g_i(y)) \tag{4}$$

for all $\delta \in \mathbb{R}_+^{U \times Y_0}$ and $y \in Y_0$. Then there exists a unique fixed point Ψ of T defined by

$$\Psi(x) := \lim_{n \to \infty} \left(\mathrm{T}^n \varphi\right)(x)$$

for all $x \in U$ such that

$$\left\| \varphi(x) - \Psi(x), y \right\| \leq \varepsilon^*(x, y)$$

for all $x \in U$ and $y \in Y_0$.

Inspired by the above facts, we are interested in studying the stability of the generalized logarithmic functional equations of the form

$$f\left(x^a y^b\right) = Af(x) + Bf(y) \tag{5}$$

where f is a mapping from a field of real or complex numbers to a 2-Banach space X over a field E, $a, b \in \mathbb{Z} - \{0\}$, $A, B \in \mathrm{E} - \{0\}$. The stability result of the functional equation (5) are proved by using Theorem 1.6.

2. Main Results

In this section, we investigate the stability of generalized logarithmic functional equations by using the Brzdęk and Ciepliński's fixed point result in 2-normed space (Theorem 1.8).

Theorem 2.1. Let K, E be two fields of real or complex numbers, $\left(X, \|\cdot,\cdot\|\right)$ be a 2-Banach space over field E and X_0 be a subset of X containing two linearly independent vectors and let

$a,b\in\mathbb{Z}-\{0\}$, $A,B\in\mathrm{E}-\{0\}$ and $h:\mathrm{K}\to\mathbb{R}_+$ be a function such that

$$M_0:=\left\{n\in\mathbb{N}:\left|\frac{1}{A}\right|s(a+bn)+\left|\frac{B}{A}\right|s(n)<1\right\}\neq\varnothing \tag{6}$$

where $s(n):=\{t\in\mathbb{R}_+:h(x^n,z)\le th(x,z)\quad\forall x\in\mathrm{K}\}$. Suppose that $f:\mathrm{K}\to X$ satisfies the following inequality

$$\left\|f\left(x^a y^b\right)-Af(x)-Bf(y),z\right\|\le Ch(x,z)+Dh(y,z) \tag{7}$$

for all $x,y\in\mathrm{K}$ and $z\in X_0$, where $C,D\in\mathbb{R}_+$. Then there exists a unique function $G:\mathrm{K}\to X$ such that it satisfies the generalized logarithmic functional equation (5) for all $x,y\in\mathrm{K}$ and

$$\left\|f(x)-G(x),z\right\|\le s_0h(x,z) \tag{8}$$

for all $x\in\mathrm{K}$ and $z\in X_0$, where

$$s_0:=\inf\left\{\frac{C+Ds(n)}{1-\left|\frac{1}{A}\right|s(a+bn)-\left|\frac{B}{A}\right|s(n)}:n\in M_0\right\}.$$

Proof. For each $m\in\mathbb{N}$ and $x\in\mathrm{K}$, replacing y by x^m in (7), we obtain

$$\left\|f\left(x^{a+bm}\right)-Af(x)-Bf\left(x^m\right),z\right\|\le\left(C+Ds(m)\right)h(x). \tag{9}$$

Define operators $\mathrm{T}_m:X^{\mathrm{K}}\to X^{\mathrm{K}}$ and $\Lambda_m:\mathbb{R}_+^{\mathrm{K}}\to\mathbb{R}_+^{\mathrm{K}}$ by

$$(\mathrm{T}_m\xi)(x):=\frac{1}{A}\xi\left(x^{a+bm}\right)+\frac{B}{A}\xi\left(x^m\right),\quad m\in\mathbb{N},x\in\mathrm{K},\xi\in X^{\mathrm{K}},$$
$$(\Lambda_m\delta)(x,z):=\left|\frac{1}{A}\right|\delta\left(x^{a+bm},z\right)+\left|\frac{B}{A}\right|\delta\left(x^m,z\right),\quad m\in\mathbb{N},\ x\in\mathrm{K},\ \delta\in\mathbb{R}_+^{\mathrm{K}},\ z\in Y_0. \tag{10}$$

Then it is easily seen that, for each $m\in\mathbb{N}$, Λ_m has the form described in (4) with $U=\mathrm{K}$, $k=2$ $L_1(x,z)=\left|\frac{1}{A}\right|$, $L_2(x,z)=\left|\frac{B}{A}\right|$, $g_1(z)=z=g_2(z)$, $f_1(x)=x^{a+bm}$ and $f_2(x)=x^m$. From (9) for all $m\in\mathbb{N},x\in\mathrm{K},\xi,\mu\in X^{\mathrm{K}}$ and $z\in Y_0$, we have

$$\left\|(\mathrm{T}_mf)(x)-f(x),z\right\|\le\left(C+Ds(m)\right)h(x):=\varepsilon_m(x,z) \tag{11}$$

and

$$\begin{aligned}\left\|(\mathrm{T}_m\xi)(x)-(\mathrm{T}_m\mu)(x),z\right\| &= \left\|\frac{1}{A}\left(\xi\left(x^{a+bm}\right)+\frac{B}{A}\xi\left(x^m\right)\right)-\frac{1}{A}\left(\mu\left(x^{a+bm}\right)+\frac{B}{A}\mu\left(x^m\right)\right),z\right\| \\ &= \left\|\frac{1}{A}(\xi-\mu)\left(x^{a+bm}\right)+\frac{B}{A}(\xi-\mu)\left(x^m\right),z\right\| \\ &= \left|\frac{1}{A}\right|\left\|\xi\left(x^{a+bm}\right)-\mu\left(x^{a+bm}\right),z\right\|+\left|\frac{B}{A}\right|\left\|\xi\left(x^m\right)-\mu\left(x^m\right),z\right\|.\end{aligned} \tag{12}$$

So (1) holds with $\mathrm{T}=\mathrm{T}_m$ for each $m\in\mathbb{N}$. By the definition of $s(n)$, we get

$$h\left(x^n,z\right)\le s(n)h(x,z) \tag{13}$$

for all $n\in\mathbb{N}, x\in\mathrm{K}$ and $z\in Y_0$. Here, we will prove that for each $n\in\mathbb{N}_0$,

$$\Lambda_m^n\varepsilon_m(x,z)\le\left(C+Ds(m)\right)\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^n h(x,z) \tag{14}$$

for all $m\in M_0, x\in\mathrm{K}$ and $z\in Y_0$. From (11), we obtain that the inequality (14) holds for $n=0$. Next, we will assume that (14) holds for $n=k$, where $k\in\mathbb{N}_0$. Then we obtain

$$\begin{aligned}\Lambda_m^{k+1}\varepsilon_m(x,z)&=\Lambda_m\left(\Lambda_m^k\varepsilon_m(x,z)\right)=\left|\frac{1}{A}\right|\Lambda_m^k\varepsilon_m\left(x^{a+bm},z\right)+\left|\frac{B}{A}\right|\Lambda_m^k\varepsilon_m\left(x^m,z\right)\\ &\le\left(C+Ds(m)\right)\left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^k h\left(x^{a+bm},z\right)+\left(C+Ds(m)\right)\left|\frac{B}{A}\right|\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^k h\left(x^m,z\right)\\ &\le\left(C+Ds(m)\right)\left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^k s(a+bm)h(x,z)+\left(C+Ds(m)\right)\left|\frac{B}{A}\right|\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^k s(m)h(x,z)\\ &\le\left(C+Ds(m)\right)\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^{k+1} h(x,z)\end{aligned}$$

$m\in M_0, x\in\mathrm{K}$ and $z\in Y_0$. Then

$$\varepsilon^*(x,z):=\sum_{n=0}^{\infty}\Lambda_m^n\varepsilon_m(x,z)\le\left(C+Ds(m)\right)h(x,z)\sum_{n=0}^{\infty}\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^n\le\frac{\left(C+Ds(m)\right)h(x,z)}{1-\left|\frac{1}{A}\right|s(a+bm)-\left|\frac{B}{A}\right|s(m)}$$

for all $m\in M_0, x\in\mathrm{K}$ and $z\in Y_0$. Now, we apply Theorem 1.6 with $\varepsilon=\varepsilon_m$ and $\varphi=f$. According to it, the limit $G_m(x):=\lim_{n\to\infty}\left(\mathrm{T}_m^n f\right)(x)$ exists for each $x\in\mathrm{K}$ such that

$$\left\|f(x)-G_m(x),z\right\|\le\frac{(C+Ds(m))h(x,z)}{1-\left|\frac{1}{A}\right|s(a+bm)-\left|\frac{B}{A}\right|s(m)} \tag{15}$$

for all $m\in M_0, x\in\mathrm{K}$ and $y\in Y_0$. Next, we will claim that

$$\left\|\mathrm{T}_m^n f\left(x^a y^b\right)-A\mathrm{T}_m^n f(x)-B\mathrm{T}_m^n f(y),z\right\|\le\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^n\left(Ch(x,z)+Dh(x,z)\right) \tag{16}$$

for all $n\in\mathbb{N}_0, m\in M_0, x,y\in K$ and $z\in Y_0$. If $n=0$, then (16) comes from (7). By taking $r\in\mathbb{N}_0$ and assuming (16) holds for $n=r$ and for each $x,y\in K$, $z\in Y_0$, we have

$$\begin{aligned}
&\left\|T_m^{r+1}f\left(x^ay^b\right)-AT_m^{r+1}f(x)-BT_m^{r+1}f(y),z\right\| \\
&=\left\|\frac{1}{A}T_m^r f\left(\left(x^ay^b\right)^{a+bm}\right)-\frac{B}{A}T_m^r f\left(\left(x^ay^b\right)^m\right)-A\left(\frac{1}{A}T_m^r f\left(x^{a+bm}\right)-\frac{B}{A}T_m^r f\left(x^m\right)\right)-B\left(\frac{1}{A}T_m^r f\left(y^{a+bm}\right)-\frac{B}{A}T_m^r f\left(y^m\right)\right),z\right\| \\
&=\left\|\frac{1}{A}T_m^r f\left(\left(x^ay^b\right)^{a+bm}\right)-A\frac{1}{A}T_m^r f\left(x^{a+bm}\right)-B\frac{1}{A}T_m^r f\left(y^{a+bm}\right),z\right\|+\left\|\frac{B}{A}T_m^r f\left(\left(x^ay^b\right)^m\right)-A\frac{B}{A}T_m^r f\left(x^m\right)-B\frac{B}{A}T_m^r f\left(y^m\right),z\right\| \\
&\le\left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^r\left(Ch\left(x^{a+bm},z\right)+Dh\left(y^{a+bm},z\right)\right)+\left|\frac{B}{A}\right|\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^r\left(Ch(x^m,z)+Dh(x^m,z)\right) \\
&\le\left[\left|\frac{1}{A}\right|s(a+bm)+\left|\frac{B}{A}\right|s(m)\right]^{r+1}\left(Ch(x,z)+Dh(x,z)\right).
\end{aligned}$$

By the mathematical induction, we have shown that (16) holds for every $n\in\mathbb{N}_0, m\in M_0, x,y\in K$ and $z\in Y_0$. Letting $z\in Y_0$, $n\to\infty$ in (16) and using Lemmas 1.6 and 1.7, we obtain the equality

$$G_m\left(x^ay^b\right)=AG_m(x)+BG_m(y) \tag{17}$$

for all $m\in M_0, x,y\in K$ and $z\in Y_0$ and $G_m: K\to X$ defined in (17), is a solution of the equation

$$G(x)=\frac{1}{A}G\left(x^ay^b\right)-\frac{B}{A}G(y). \tag{18}$$

Next, we will prove that each generalized logarithmic function $G: K\to X$ satisfying the inequality

$$\left\|f(x)-G(x),z\right\|\le Lh(x,z) \tag{19}$$

for all $x\in K$ and $z\in Y_0$ with $L>0$, is equal to G_m for each $m\in M_0$. Fix $m_0\in M_0$ and $G: K\to X$ satisfying (19). Then we observe that

$$\begin{aligned}
\left\|G(x)-G_{m_0}(x),z\right\|&\le\left\|G(x)-f(x),z\right\|+\left\|f(x)-G_{m_0}(x),z\right\| \\
&\le Lh(x,z)+\left(C+Ds(m_0)\right)h(x,z)\sum_{n=0}^{\infty}\left[\left|\frac{1}{A}\right|s(a+bm_0)+\left|\frac{B}{A}\right|s(m_0)\right]^n \\
&\le h(x,z)\left(L\left(1-\left|\frac{1}{A}\right|s(a+bm_0)-\left|\frac{B}{A}\right|s(m_0)\right)+\left(C+Ds(m_0)\right)\right)\sum_{n=0}^{\infty}\left[\left|\frac{1}{A}\right|s(a+bm_0)+\left|\frac{B}{A}\right|s(m_0)\right]^n \\
&\le h(x,z)L_0\sum_{n=0}^{\infty}\left[\left|\frac{1}{A}\right|s(a+bm_0)+\left|\frac{B}{A}\right|s(m_0)\right]^n,
\end{aligned} \tag{20}$$

where $L_0=L\left(1-\left|\frac{1}{A}\right|s(a+bm_0)-\left|\frac{B}{A}\right|s(m_0)\right)+\left(C+Ds(m_0)\right)$ (the case $h(x,z)\equiv 0$ is trivial, so we exclude it here). Note that G and G_{m_0} are solutions to equation (18) for all $m_0\in M_0$. Here we will show that for each $j\in\mathbb{N}_0$, we have

$$\left\|G(x)-G_{m_0}(x),z\right\| \le h(x,z)L_0 \sum_{n=j}^{\infty}\left[\left|\frac{1}{A}\right|s(a+bm_0)+\left|\frac{B}{A}\right|s(m_0)\right]^n \tag{21}$$

for all $x\in K$ and $z\in Y_0$. The case $j=0$ is inequality (20). So we fix $l\in\mathbb{N}_0$ and suppose that (21) hold for $x\in K$. For $m_0\in M_0, x\in K$ and $z\in Y_0$, we have

$$\begin{aligned}\left\|G(x)-G_{m_0}(x),z\right\| &\le \left\|\frac{1}{A}G\left(x^{a+bm_0}\right)-\frac{B}{A}G\left(x^{m_0}\right)-\frac{1}{A}G_{m_0}\left(x^{a+bm_0}\right)+\frac{B}{A}G_{m_0}\left(x^{m_0}\right),z\right\| \\ &\le \left|\frac{1}{A}\right|\left\|G\left(x^{a+bm_0}\right)-G_{m_0}\left(x^{a+bm_0}\right),z\right\|+\left|\frac{B}{A}\right|\left\|G\left(x^{m_0}\right)-G_{m_0}\left(x^{m_0}\right),z\right\| \\ &\le \left|\frac{1}{A}\right|Lh\left(x^{a+bm_0},z\right)+\left|\frac{B}{A}\right|(C+Ds(m_0))h\left(x^{m_0},z\right)\sum_{n=l}^{\infty}\left[\left|\frac{1}{A}\right|s(a+bm_0)+\left|\frac{B}{A}\right|s(m_0)\right]^n \\ &\le L_0\left(\left|\frac{1}{A}\right|h\left(x^{a+bm_0},z\right)+\left|\frac{B}{A}\right|h\left(x^{m_0},z\right)\right)\sum_{n=l}^{\infty}\left[\left|\frac{1}{A}\right|s(a+bm_0)+\left|\frac{B}{A}\right|s(m_0)\right]^n \\ &\le h(x,z)L_0\sum_{n=l+1}^{\infty}\left[\left|\frac{1}{A}\right|s(a+bm_0)+\left|\frac{B}{A}\right|s(m_0)\right]^n.\end{aligned}$$

Therefore, (21) holds for all $n\in\mathbb{N}_0$. Letting $j\to\infty$ in (21) and using Lemmas 1.6 and 1.7, we get

$$G=G_{m_0}. \tag{22}$$

This implies $G_m=G_{m_0}$ for each $m\in M_0$. So (15) implies

$$\left\|f(x)-G_{m_0}(x),z\right\| \le \frac{(C+Ds(m_0))h(x,z)}{1-\left|\frac{1}{A}\right|s(a+bm_0)-\left|\frac{B}{A}\right|s(m_0)}$$

for all $m_0\in M_0, x\in K$ and $z\in Y_0$. This implies (8) with $G=G_{m_0}$. Also, (22) yields the uniqueness of G.

Remark 2.2 If $a=1=b$ in Theorem 1.1, then the stability result of logarithmic functional in 2-Banach spaces. By using the same technique in Corollary 6.2 of Brzdęk and Ciepliński [9] , we get the stability result for inhomogeneous generalized logarithmic functional equations in 2-Banach spaces.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

LA and WS conceived of the presented idea. LA developed the theory and proved it. WS verified the analytical methods. All authors discussed the results and contributed to the final manuscript.

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