Common Solutions for a Class of Simultaneous Pell Equations

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Abstract: In recent years, the common solution of pell equations is a hot field in indefinite equations. For example, the equations (1) mentioned in the paper. However, due to the diverse forms of such equations, many scholars have done more studies on the smaller values of k and m, and the main conclusions are focused on the estimation of solutions under some special forms of $D_1$ and the specific values of $D$. So there is a lot of room for studying these kinds of equations. In this paper, we studied the common solution of the system of indefinite equations (2) mentioned in this paper by using the elementary method and the recursive property of solution sequence. If $D$ is the case in this paper, the common solution of the equations is given.

Key words: The system of indefinite equations, pell equation, integer solution, common solution, odd prime.

1. Introduction

The Diophantine equation is the oldest branch in number theory, whose content is extremely abundant, and it has close connections with the algebraic number theory, the algebraic geometry, the combinatorics and so on. In the recent 30 years, this field also has developed too much. In such fields as the information encoding theory, the algebraic number theory and the diophantine analysis theory, many types of the results of higher diophantine equation are used, which make it necessary for us to study some basic types of the solutions of higher diophantine equation. We are familiar to study some basic types of the simple diophantine equation and quadratic diophantine equation, while with the solution of higher diophantine equation, there is no general conclusion, so it needs further discussing.

The Diophantine equation not only developed actively itself, but also was apply to else fields of Discretre Mathematics. It plays an important role in people's study and research to solve the actual problems. So many researchers study the Diophantine equation extensively and highly in the domestic and abroad. Along with the development of the Diophantine equation, Algebraic Number Theory obtained the first formation and developments. Currently, Algebraic Number Theory has become a branch of mathematics with abundant contents, is also an important tool of studying of the Diophantine equation.

In recent years, the common solution of pell equations

\[
\begin{align*}
    x^2 - D_1 y^2 &= k \\
    y^2 - D z^2 &= m
\end{align*}
\]

(1)
is a hot field in indefinite equations. The main conclusions are as follows:

1) When \( k=1 \) and \( m=1 \), the research results of the system focus on the scope and estimation of the solution, and the main conclusions are shown in [1], [2].

2) When \( k=1 \) and \( m=4 \),
   a) If \( D=2 \), for the solution of the system, the main conclusion is shown in [3]-[10];
   b) If \( D=6 \), it is shown in [11]-[15];
   c) If \( D=10 \), it is shown in the main conclusion [16].
   d) If \( D=12 \), it is shown in the main conclusion [17]-[19].
   e) If \( D=30 \), it is shown in the main conclusion [20].

3) When \( k=1 \) and \( m=25 \),
   a) If \( D=23 \), the situation of the system is discussed in [21].

However, the pell equations

\[
\begin{aligned}
&x^2 - k(k + 1)y^2 = k \\
y^2 - Dz^2 = 4
\end{aligned}
\]

\((k \in \mathbb{N}^*)\),

is one of the kind of the equations (1). When \( k=2 \), it is shown in [11]-[15], when \( k=3 \), it is shown in the main conclusion [17]-[19]. In this paper, we deal with the case of \( k=4 \), namely

\[
\begin{aligned}
x^2 - 20y^2 = 1 \\
y^2 - Dz^2 = 4
\end{aligned}
\]

(2)

And the following conclusions are obtained:

**Theorem** If \( D=2^i p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r} \), where \( \alpha_s = 0 \) or \( 1, p_s (1 \leq s \leq 4) \) are distinct odd primes, \( t \) is a positive integer, and the solution of the indefinite system (1) is as follows:

a) \( D=2 \times 7 \times 23 \), the system (2) has non-trivial solutions \((x, y, z) = (\pm 2889, \pm 646, \pm 36)\);

b) \( D=2^3 \times 7 \times 23 \), the system (2) has non-trivial solutions \((x, y, z) = (\pm 2889, \pm 646, \pm 18)\);

c) \( D=2^5 \times 7 \times 23 \), the system (2) has non-trivial solutions \((x, y, z) = (\pm 2889, \pm 646, \pm 9)\).

d) When \( t \neq 1, 3, 5 \), the system (2) only has trivial solutions \((x, y, z) = (\pm 9, \pm 2, 0)\).

2. Preliminaries

**Lemma 1** [18] If \( p \) is an odd prime number, then the diophantine equation \( x^4 - py^2 = 1 \) has no other positive integer solution except \( p=5, x=3, y=4 \) and \( p=29, x=99, y=1820 \).

**Lemma 2** [18] If \( a \) is a square number and \( a > 1 \), the equation \( ax^4 - by^2 = 1 \) has only one positive integer solution.

**Lemma 3** [18] If \( D \) is a non-square positive integer, then \( x^4 - Dy^4 = 1 \) has at most two positive integer solutions. And the sufficient and necessary condition for the equation to have two groups of solutions is that \( D=1785 \) or \( D=28550 \), or that \( 2x_0 \) and \( 2y_0 \) are squares, where \((x_0, y_0)\) is the fundamental solution of the equation.

**Lemma 4** If \( x_n, y_n \) is any integer solution of Pell equation \( x^2 - 104y^2 = 1 \), then \( x_n, y_n \) has the following properties:
Lemma 5 If \((x_1, y_1)\) is the fundamental solution of Pell equation \(x^2 - 20y^2 = 1\), and all integer solutions are \((x_n, y_n), n \in \mathbb{Z}\). For any \((x_n, y_n)\), it has the following properties:

a) \(x_n\) is square if and only if \(n = 0\);

b) \(\frac{x_n}{5}\) is square if and only if \(n = \pm 1\);

c) \(\frac{y_n}{2}\) is square if and only if \(n = 0, 1\).

3. Proof of Theorem

Proof: Since the fundamental solution of Pell equation \(x^2 - 20y^2 = 1\) is \((x_1, y_1) = (9, 2)\), all integer solutions of the pell equation are \(x_n + y_n \sqrt{20} = (9 + 2\sqrt{20})^n, n \in \mathbb{Z}\). Thus:

If \((x, y, z) = (x_n, y_n, z)\) is the integer solution to (2), then \(\forall n \in \mathbb{Z}\),

\[
y_n^2 - 4 = y_n^2 - 4 \left( x_n^2 - 20y_n^2 \right) = 81y_n^2 - 4x_n^2 = (9y_n + 2x_n)(9y_n - 2x_n) = y_{n+1}y_{n-1}
\]

By (2) \(Dz^2 = y_n^2 - 4\)

Then

\[
Dz^2 = y_{n+1}y_{n-1}
\]

case1 Let \(n\) be odd, might as well \(n = 2m - 1 (m \in \mathbb{Z})\). At this point, equation (4) becomes:

\[
Dz^2 = y_{n-1}y_{n+1} = y_{2m-2}y_{2m} = 4x_{m-1}y_{m-1}x_my_m
\]

case1.1 Let \(m\) be odd, might as well \(m = 2r (r \in \mathbb{N}^*)\). At this point, equation (5) becomes:

\[
Dz^2 = 4x_{2r-1}y_{2r-1}x_{2r}y_{2r} = 8x_{2r-1}y_{2r-1}x_ry_r
\]

case 1.1.1 Let r be odd, might as well \(r = 2u - 1 (u \in \mathbb{Z})\), At this point, equation (5) becomes:
\[ \Delta^2 = \begin{cases} 8 & \text{if } r = 1, \\ 2^3 & \text{if } r = 0. \end{cases} \]

From Lemma 5, if \( \frac{x_{4u-1}}{9}, \frac{y_{4u-1}}{9}, \frac{y_{4v}}{2}, \frac{x_{4v}}{2}, \frac{x_{4u-2}}{9}, \frac{y_{4u-3}}{9}, \frac{x_{4v-1}}{2}, \frac{y_{4v-1}}{2}, \frac{x_{4v-2}}{9}, \frac{y_{4v-2}}{9} \) are two relatively prime, and \( \frac{y_{4u-1}}{2}, \frac{y_{4v-1}}{2} \) are odd, \( \frac{x_{4u-1}}{9}, \frac{x_{4v-1}}{9}, \frac{x_{4u-2}}{9}, \frac{x_{4v-2}}{9} \) are odd, namely \( \frac{x_{4u-1}}{9}, \frac{x_{4v-1}}{9}, \frac{y_{4u-1}}{2}, \frac{y_{4v-1}}{2}, \frac{x_{4u-2}}{9}, \frac{x_{4v-2}}{9} \) are two relatively odd prime.

From Lemma 5, that if, and only if \( u=0 \), \( \frac{x_{2u-1}}{9} \) is a square, and if and only if \( \frac{u}{2} \) all are a square number. So if \( u \neq 0,1 \), \( \frac{x_{2u-1}}{9}, \frac{x_{4u-3}}{9}, \frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}, \frac{x_{4v-2}}{9}, \frac{x_{4v-2}}{9}, \frac{y_{4v-2}}{9}, \frac{y_{4v-2}}{9} \), does not equal 0,1, it's not a square number.

When \( u=0 \), equation (7) is

\[ \Delta^2 = 2^3 \cdot 9^2 \cdot x_2 \cdot \frac{x_1}{9} \cdot \frac{y_1}{2} \]

However, \( x_2 = 161 = 7 \times 23, \frac{x_1}{9} = \frac{2889}{9} = 3 \times 107, \frac{y_1}{2} = \frac{646}{2} = 17 \times 19 \)

Therefore, the right hand side of (8) contains six different odd prime Numbers, so formula (8) does not hold, and the system (2) has no solution.

When \( u = 1 \),

\[ \Delta^2 = 8x_1y_1x_2y_2 = 2^3 \times 9^2 \times 2^2 \times 161 = 2^3 \times 3^3 \times 7 \times 23 = 2 \times 7 \times 23 \times (2^3 \times 3^2)^2 = 2^3 \times 7 \times 23 \times (2 \times 3^2)^2 = 2^3 \times 7 \times 23 \times 3^4. \]

So when \( D = 2 \times 7 \times 23 \), the system (2) has a nontrivial solutions \( (x, y, z) = (\pm 2889, \pm 646, \pm 36) \); \( D = 3^2 \times 7 \times 23 \), (2) has a nontrivial solution \( (x, y, z) = (\pm 2889, \pm 646, \pm 18) \), when \( D = 2^3 \times 7 \times 23 \), (2) has a nontrivial solution \( (x, y, z) = (\pm 2889, \pm 646, \pm 9) \).

**Case 1.1.2** If \( r \) is even, let \( r = 2v (v \in \mathbb{Z}) \), then equation (5) can be written into

\[ \Delta^2 = 8x_{4v-1}y_{4v-1}x_{4v}y_{4v} = 16x_{4v-1}y_{4v-1}x_{4v}y_{4v} \]

From Lemma 5, when \( v \) is even, \( \frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}, \frac{y_{4v}}{2}, \frac{x_{4v}}{9}, \frac{x_4}{2}, \frac{x_2}{9}, \frac{y_1}{18} \) are two relatively prime, when \( v \) is odd \( \frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}, \frac{x_{4v}}{9}, \frac{x_4}{2}, \frac{x_2}{9}, \frac{y_1}{18} \) are two relatively prime. And when \( v \) is odd, \( \frac{y_{4v-1}}{2}, \frac{y_{4v}}{2}, \frac{x_4}{9}, \frac{x_2}{9}, \frac{x_4}{2}, \frac{x_2}{9} \) all are odd; when \( v \) is even, \( \frac{y_{4v-1}}{2}, \frac{y_{4v}}{2}, \frac{x_4}{9}, \frac{x_2}{9}, \frac{x_4}{2}, \frac{x_2}{9} \) all are odd.

From Lemma 5, if and only if \( v=0 \), \( \frac{x_{4v-1}}{9}, \frac{x_4}{9}, \frac{x_2}{9}, \frac{y_{4v-1}}{2}, \frac{y_{4v}}{2}, \frac{x_4}{9} \) are squares, and if and only if \( v=\pm 1 \), \( \frac{x_4}{9} \) is a square; For any \( v \in \mathbb{Z} \), \( \frac{x_{4v-1}}{2} \) is not square. If and only if \( v=0,1 \), \( \frac{y_1}{2} \) is a square. So if \( v \neq 0 \) and \( v \) is even.
\[ x_{4s}, x_{2s}, x_r, \frac{x_{4s-1}}{9}, \frac{y_{4s-1}}{2} \] are not squares. At this point, they have at least five different odd prime numbers, so formula (9) is not true, so when \( u \neq 0,1 \), the system (2) has no solution.

When \( v \neq \pm 1 \) and \( v \) is odd, \( x_r, x_{2s}, \frac{x_{4s-1}}{9}, \frac{y_{4s-1}}{2} \) are not squares. At this point, they have at least five different odd prime numbers, so formula (9) is not true, so when \( u \neq 0,1 \), the system (2) has no solution. So when \( v \neq 0, v \neq \pm 1 \) and \( v \) is even, the system (2) has no solution.

When \( v = 0 \), (9) can be written into \( Dz^2 = 16 \cdot x_0^2 \cdot y_0 \cdot x_1 \cdot y_1 = 0 \), thus \( z = 0 \). At this point, the system (2), only has ordinary solutions \((x, y, z) = (\pm 9, \pm 2, 0)\).

When \( v = 1 \), (9) can be written into
\[
Dz^2 = 16 \cdot x_0 \cdot y_0 \cdot x_1 \cdot y_1 = 16 \cdot 3^3 \cdot 107 \cdot 2 \cdot 17 \cdot 19 \cdot 47 \cdot 1103 \cdot 7 \cdot 23 \cdot 92,
\]
\[
= 2^6 \cdot 3^3 \cdot 7 \cdot 17 \cdot 19 \cdot 23 \cdot 47 \cdot 107 \cdot 1103
\]
The right hand side of the above equation contains eight odd prime numbers, so the above formula is impossible. Therefore when \( v = 1 \), the system (2) has no common solution.

When \( v = -1 \),
\[
Dz^2 = 16 \cdot x_0 \cdot y_0 \cdot x_1 \cdot y_1 = 16 \cdot 9^2 \cdot 2^2 \cdot x_2 \cdot y_2 \cdot \frac{x_3}{9} \cdot \frac{y_3}{2}
\]
\[
= 16 \cdot 9^2 \cdot 2^2 \cdot 161 \cdot 51841 \cdot \frac{930249}{9} \cdot \frac{208010}{2}
\]
\[
= 2^6 \cdot 3^4 \cdot 7 \cdot 23 \cdot 47 \cdot 1103 \cdot 41 \cdot 2521 \cdot 5 \cdot 3 \cdot 61
\]
\[
= 2^6 \cdot 3^4 \cdot 7 \cdot 23 \cdot 47 \cdot 1103 \cdot 2521
\]
Therefore, the right hand side of the above equation contains ten odd prime numbers, so the above formula is impossible. Therefore when \( v = -1 \), the system (2) has no common solution.

\textbf{case 1.2} If \( m \) is odd, modelled on the case 1.1, it can be proved that the equation (2) is only the common solution \((x, y, z) = (\pm 9, \pm 2, 0)\).

\textbf{case 2} If \( n \) is even, by lemma 4, \( y_{n \pm 1} = y_{n \mp 1} = 1 \text{ (mod 2)} \), the right-hand side of equation (4) is odd, while the left-hand side is even in the form of \( D \), so the system (2) has no common solution.

\textbf{4. Summary and Prospect}

In this paper, we have gotten the solutions of the following equations
\[
\begin{align*}
&x^2 - k(k + 1)y^2 = k \\
&y^2 - Dz^2 = 4
\end{align*}
\]
When \( k = 4 \) and \( D = 2^m p_2^{a_2} p_3^{a_3} p_5^{a_5} p_s^{a_s}, \) where \( a_s = 0 \) or 1 \((1 \leq s \leq 4)\). And we can go on and talk about the solutions to this system when \( k > 4 \), and \( D \) is some other form. So there is a lot of room for studying these kinds of equations.

Due to the diverse forms of such equations, many scholars have done more studies on the smaller values of \( k \) and \( m \), and the main conclusions are focused on the estimation of solutions under some special forms of \( D_1 \) and the specific values of \( D \). We need more powerful ways of finding common solutions to more forms of equations.
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References


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