Syzygies on Path Algebras

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Abstract: Let K be a field and KQ be a noetherian path algebra for the quiver Q. Given a left (resp. right) finitely generated ideal I of KQ, we propose a new idea for computing left (resp. right) Groebner bases on KQ. As application, we propose a method for computing the so called left (resp. right) syzygies, that is, given polynomials $f_1,...,f_s \in KQ \setminus\{0\}$ we propose a method for computing the set of all elements $(h_1,...,h_s) \in (KQ)^s$ such that $h_1f_1 + ... + h_sf_s = 0$ (resp. $f_1h_1 + ... + f_sh_s = 0$).

Key words: Groebner bases, path algebra, syzygies.

1. Introduction

The advent of Groebner bases made a lot of computation possible in different areas of mathematics especially in algebraic geometry and commutative algebra. The theory of Groebner bases is widely studied in the commutative as well as non-commutative case over free associative algebras (see [1]-[7]) with their applications in computing syzygies [8]-[10]. The goal of this work is to revisit the non-commutative Groebner bases over path algebras. We propose a new approach for computing left and right Groebner bases on path algebras. Beside the left and right S-polynomials, we introduce the notion of left and right extended S-polynomials which is suitable for path algebras. Our approach is a generalization of the idea used in [1] and [9] chapter 2 which enables to compute a left and right Groebner basis using only selected S−polynomials in the Buchberger's criterion. We generalize the Schreyer's theorem on path algebras and as application we propose a method for computing left and right syzygies modules on path algebras i.e given polynomails $f_1, ..., f_s \in KQ \setminus\{0\}$ we propose a method for computing the set of all elements $(h_1, ..., h_s) \in (KQ)^s$ such that $h_1f_1 + ... + h_sf_s = 0$ (resp. $f_1h_1 + ... + f_sh_s = 0$) called left and right syzygies. This result will be very useful for instance for those who wish to study the intersection of ideals since syzygies play a central role in finding a generating set of the intersection of two ideals.

2. Preliminaries

Definition 1.1. By a directed graph or a quiver we mean a quadruple $\Gamma = (\Gamma^0,\Gamma^1,r,s)$ where Γ^0 is the set of vertices, Γ^1 the set of edges and $r,s : \Gamma^1 \to \Gamma^0$ are maps. If $e \in \Gamma^1$ is an edge, then $s(e)$ the is called source of e and $r(e)$ is the range of e. A sequence of edges $\alpha = e_1...e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1,...,n-1$ is called path in Γ. In this case we denote $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$. The number of edges in the path α denoted $L(\alpha)$ is called the length of α. A vertex is regarded as a path of length zero. A closed path is a path α such that $s(\alpha) = r(\alpha)$. A cycle is a closed path α such that if e_i and e_j are edges occurring in α then $s(e_i) \neq s(e_j)$ $\forall$ $i \neq j$. We denote by path (Γ) the set of all paths in Γ. If α and β are paths, then we define the multiplication as follow: $\alpha\beta$ is the path adjoining α and β by concatenation if $r(\alpha) = s(\beta)$; otherwise we get zero.

Let K be a field and Γ be a quiver. The set of all linear combinations of paths in Γ with coefficients in K together with the above multiplication is called path K–algebra. In the other hand, if $\Gamma = (\Gamma^0,\Gamma^1,r,s)$ is a quiver, then the path K–algebra $K\Gamma$ is the free associative algebra $K <\Gamma^0 \cup \Gamma^1>$ generated by $\Gamma^0 \cup \Gamma^1$ satisfying to: $v_iv_j = \delta_{ij}v_i \, \forall \, v_i,v_j \in \Gamma^0$ and $s(e)e = er(e) = e \, \forall \, e \in \Gamma^1$.

A quiver Γ if row-finite if $\forall \, e \in \Gamma^1$, we have $s^{-1}(e) < \infty$.

Remark 1.2. If $f \in K\Gamma$ then $f = \sum_i a_i p_i$ where $a_i \in K$, $p_i \in$ path (Γ) and only finitely many $a_i = 0$. Elements of $K\Gamma$ will be called polynomials and paths will be called monomials. Since the path algebra is unital, then in this work we will always consider $1_{K\Gamma} = \sum_{v \in \Gamma^0} v$.

Definition 1.3. • A **well-ordering** in path (Γ) is a total ordering with the condition that every subset of path (Γ) has a least element.

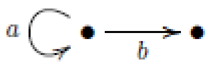
A well-ordering $<$ is called **left-admissible** in path (Γ) if $\forall \, p,q,r \in$ path(Γ), we have $p < q \implies rp < rq$ whenever rp and rq are both non-zeros.

A well-ordering $<$ is called **right-admissible** in path (Γ) if $\forall \, p,q,s \in$ path (Γ), we have $p < q \implies ps < qs$ whenever ps and qs are both non-zeros.

Example 1.4. - left-lexicographic order

Let $p = e_1 \cdots e_r$ and $q = f_1 \cdots f_l$ be two paths in Γ. We say that p is less than q with respect to the **left-lexicographic order** and we denote $p <_{llex} q$ if there exists a path m (otherwise we set $m = 1$) such that $p = me_k \cdots e_r$, $q = mf_s \cdots f_l$ and $e_k < f_s$.

Remark 1.5. The left lexicographic order is not a left-admissible ordering since it is not a well-ordering. For example, for the following graph



with $a < b$ we have $ab >_{llex} a^2b >_{llex}> a^3b >_{llex} \cdots$ then the subset $\{a^nb/n \in \mathbb{N}\backslash\{0\}\} \subset$ path (Γ) doesn't have a least element.

length left-lexicographic order

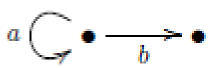
Let $p = e_1 \cdots e_r$ and $q = f_1 \cdots f_l$ be two paths in Γ. We say that p is less than q with respect to the **length left-lexicographic order** and we denote $p <_{Lex} q$ if $L(p) < L(q)$ or $L(p) = L(q)$ and $p <_{llex} q$.

Remark 1.6. The length left-lexicographic order is a left-admissible ordering.

right-lexicographic order

Let $p = e_1 \cdots e_r$ and $q = f_1 \cdots f_l$ be two paths in Γ. We say that p is less than q with respect to the **right-lexicographic order** and we denote $p <_{rlex} q$ if there exists a path m (otherwise we set $m = 1$) such that $p = e_1 \cdots e_k m$, $q = f_1 \cdots f_s m$ and $e_k < f_s$.

Remark 1.7. The right lexicographic order is not a right-admissible ordering since it is not a well-ordering. For example, for the following graph



with $a < b$ we have $ba >_{rlex} ba^2 >_{rlex}> ba^3 >_{rlex} \cdots$ then the subset $\{ba^n/n \in \mathbb{N}\backslash\{0\}\} \subset$ path(Γ) doesn't have a least element.

length right-lexicographic order

Let $p = e_1 \cdots e_r$ and $q = f_1 \cdots f_l$ be two paths in Γ. We say that p is less than q with respect to the **length right-lexicographic order** and we denote $p <_{rLex} q$ if $L(p) < L(q)$ or $L(p) = L(q)$ and $p <_{rlex} q$.

Remark 1.8. The length right-lexicographic order is a right-admissible ordering.

Definition 1.9. Let < be a left or right admissible ordering and $f \in K\Gamma \setminus \{0\}$. Then we call:

- the leading monomial of f denoted $Lm(f)$ to be the biggest monomial occurring to f with respect to <;
- the leading coecient of f denoted $Lc(f)$ to be the coecient of $Lm(f)$ in f;

the leading term of f denoted $Lt(f) = Lc(f)Lm(f)$.

In what follow we denote by $R = K\Gamma$.

Definition 1.10. (1) A subset of I_L of $K\Gamma$ is called left ideal if the following conditions hold:

- $0 \in I_L$;
- $f + g \in I_L \ \forall \ f, g \in I_L$;
- $fh \in I_L \ \forall \ f \in R$ and $h \in I_L$.

(2) A subset of I_R of $K\Gamma$ is called right ideal if the following conditions hold:

- $0 \in I_R$;
- $f + g \in I_R \ \forall \ f, g \in I_R$;
- $fh \in I_R \ \forall \ f \in R$ and $h \in I_R$.

Remark 1.11. If E is a subset of R, then we denote by $\langle E\rangle_L = \{\sum_i h_i g_i / h_i \in R \quad g_i \in E \ \forall \ i\}$ the left ideal generated by E and $<E>_R = \{\sum_i g_i f_i \ / g_i \in E$ and $f_i \in R \ \forall \ i\}$ the right ideal generated by E. We denote by $Lm(E)_L = <Lm(f) / f \in E \setminus \{0\}>_L$ and $Lm(E)_R = <Lm(f) / f \in E \setminus \{0\}>_R$.

Definition 1.12. A left (respectively right) ideal of a path K−algebra $K\Gamma$ is called left (respectively right) monomial if it is generated by monomials.

Remark 1.13. A left (respectively right) monomial ideal is also called left (respectively right) path ideal.

3. Left Groebner Bases on *KQ*

In this section we denote by Q a quiver and $R = KQ$ a path K−algebra. The goal of this section is to propose a method for computing a left Groebner basis of a left ideal of $R = KQ$. By division in this section we will always mean on the left i.e $a \mid b$ if there exists c such that $b = c \cdot a$.

Definition 2.1. Let I_L be a left ideal of R and G be a subset of I_L. We say that G is a left Groebner basis for I_L with respect to a given left-admissible ordering if for every $f \in I_L$, there exists $g \in G$ such that $Lm(g)$ divides $Lm(f)$ i.e there exists $h \in R$ such that $Lm(f) = Lm(h)Lm(g)$. In the other hand, G is a left Groebner basis for I_L if $Lm(I_L)_L = Lm(G)_L$.

Remark 2.2. A left Groebner basis is a generating set of a left ideal, since the path K−algebra R of a given graph is not necessarily left noetherian, for example for the graph

The corresponding path K−algebra is not left noetherian since the left ideal h$ba^n/n \in$ N\{0}i$_L$ is not finitely generated. a left Groebner basis may be innite and in this case the computation won't be interesting. In this work we will deal only with noetherian path K−algebras.

Definition 2.3. Let Q be a quiver. A cycle c in Q has an entry if there exists an edge in Q not occurring in c with its range in c. For example the cycle of the graph of the Remark 2.2 has an entry.

Proposition 2.4. The path K−algebra KQ is left noetherian if and only if no cycle in Q has an entry.

Proof. Assume that in Q there exists a cycle with an entry, we denote that cycle by c. Let p be a path in Q not occurring in c with $r(p)$ a vertex of c. Then the ideal $<pc, pc^2, pc^3, \cdots>_L$ is not finitely generated. Thus KQ is not left noetherian. Conversely, if no cycle has an entry then, we have two possibilities: either Q contains no cycle or it contains nitely many cycles (since the graph is row-nite) with no entry.

- If Q contains no cycle then we have a nite number of paths, thus any left ideal is nitely generated.
- Assume that Q contains nitely many cycles with no entry. Since Q contains only nitely many paths and

cycles, if there exists a left ideal which is not finitely generated, it must contains non-zero paths of the form qc^n where q is a path and c is a cycle with $n \in \mathbb{N}$. Such paths are defined if $r(q)$ occurs in c, that is c must have an entry. Contradicting the hypothesis.

In the rest of this section, $R = KQ$ will be considered to be a left noetherian path K–algebra.

Proposition 2.5. Let R be a left noetherian path K–algebra and $<$ be left a admissible ordering.

1) A path $p \in T = <q_i / q_i \in \text{path}(Q)\ \forall\ 1 \le i \le r>_L$ i there exists i_0 such that q_{i_0} divides p.

2) If $I_L \subset R$ is a left monomial (path) ideal then I_L has a unique finite minimal path generating set.

Proof. (1) Straightforward.

Let A be the set of all paths in I_L and let $M \subset A$ be the set of minimal (by division) paths i.e $M = \{p \in A/\text{if } q \in A \text{ divides } p \text{ then } q = p\}$. We want to prove that M is the unique nite minimal path generating set of I_L.

Since $M \subset A$ then $<M>_L \subseteq I_L$.

Let $p \in I_L$ be a path, then $p \in A$. Let B be the set of all paths that generates I_L i.e $I_L = <B>_L$. If p is not minimal (i.e $p \notin M$) then $p = ab$ where $a \in \text{path}(Q)$ and $b \in B$.

- If b is minimal (i.e $b \in M$) then $p \in <M>_L$.
- If b is not minimal (i.e $b \notin M$) then by induction b is multiple of an element of M, since $<$ is a well-ordering. Thus $p \in <M>_L$ and $I_L = <M>_L$. Since R is left noetherian then $I_L = <M>_L$ is finitely generated. Thus M is finite.

Let M' be a minimal finite paths generating set of I_L. Let us prove that $M = M'$. Since M' is minimal then $M' \subseteq M$. Let $m \in M \subset <M>_L = I_L = <M'>_L$ then there exists $m' \in M'$ such that m' divides m which means by definition of M that $m' = m$.

2.1. Left division's algorithm.

Let $F = \{f_1, \ldots, f_s\}$ be a set of elements of $R = KQ$ and $>$ be a left admissible ordering. Given element $g \in R \setminus \{0\}$, the following algorithm shows how to nd $q_1,\ldots,q_s,h \in KQ$ such that $g = q_1f_1 + \cdots + q_sf_s + h$ satisfying.

A1 if $h \neq 0$, then each term occurring in h is not divisible by any of $Lt(f_i)\ \forall\ 1 \le i \le s$.

A2 $Lm(g) \ge Lm(q_if_i)$ for each $i \in \{1,\ldots,s\}$ since q_i are obtained from g.

Input: $f, f_1,\ldots,f_s \in F$ and a left admissible ordering $>= (>_1,>_2)$.

Output: $q_1,\ldots,q_s,h \in F$ such that $f = q_1f_1 + \cdots + q_sf_s + h$.

Initialization: $q_1 := q_2 := \cdots := q_s := h := 0$ and $v := g$;

- While $v \neq 0$ do
  - $i := 1$;
  - while $i \le s$ do
    - if $Lt(f_i) \mid Lt(v)$ then
      - $q_i := q_i + \dfrac{Lt(v)}{Lt(f_i)}$;
      - $v := v - \dfrac{Lt(v)}{Lt(f_i)} f_i$;
    - Else
      - $i := i + 1$;
  - $h := h + Lt(v)$;
  - $v := v - Lt(v)$.

Proposition 2.6. Let $\{f_1,\ldots,f_r\} \subset R$ be a left Groebner basis for the ideal $I_L := <f_1, \ldots, f_r>_L \subset R$, $>$ be a left admissible ordering and $g \in R \setminus \{0\}$. $g \in I_L$ if and only if the remainder h of the division of g by $f_1,\ldots,f_r$ is zero.

Proof. If $h = 0$ then it is clear by the division's algorithm that $g \in I_L$. Conversely, if $g \in I_L$ then by the division's algorithm we can write $g = g_1f_1 + \ldots + g_rf_r + h$ for some $g_1,\ldots,g_r,h \in R$. Observe that $h \in I_L \Longrightarrow Lm(h) \in Lm(I_L) = <Lm(f_1),\ldots,Lm(f_r)>$ which is impossible by A1).

Definition 2.7. (Left useful paths)

Let $p,q \in$ path(Q) be paths in Q. Two paths u, v are called left useful for p and q with respect to the left admissible ordering <, if $up = vq \neq 0$ and for any other paths w_1 and w_2 satisfying $w_1p = w_2q \neq 0$, we have u divides w_1 and v divides w_2 (both on the left).

Definition 2.8. (Left S-polynomial)

Let f,g be non-zero polynomials and < be a left admissible ordering. Let u,v be left useful paths for $Lm(f)$ and $Lm(g)$ (i.e $uLm(f) = vLm(g)$), then the left S-polynomial $S_L(f,g)$ of f and g is dened as

$$S_L(f,g) = \frac{u}{Lc(f)}f - \frac{v}{Lc(g)}g.$$

Remark 2.9. If there exist no left useful paths for $Lm(f)$ and $Lm(g)$, we claim that $S_L(f, g) = 0$.

Definition 2.10. (Left extended S-Polynomial)

Let $f \in R$ be a non-zero polynomial and > be a left admissible ordering. We define the left extended S-polynomial $S_L(f)$ of f as $S_L(f) = p \cdot f$ where p is a path in KQ such that $p \cdot Lm(f) = 0$ and $p \cdot f \neq 0$.

Remark 2.11. Let p and q be two paths satisfying the conditions of the Definition 3.9, if q divides p then we choose $S_L(f) = q \cdot f$.

Definition 2.12. Let I_L be a left path ideal and $w \in$ path(Q). We define the left quotient ideal of I_L and w by $I_L : w = \{p \in \text{path}(Q) / \ p \cdot w \in I_L\}$.

Lemma 2.13. Let $I_L = <w_1,...,w_r>_L$ be a left path ideal of R, t be a monomial and > be a left admissible ordering. Then the quotient ideal $I_L : t = \{p$ monomial in $R / \ pt \in I_L\}$ can be written as

$$I_L : t = <v_1,\ldots,v_l>_L$$

where for each $1 \leq i \leq l \leq r$ there exists a monomial p_i such that $v_it = p_iw_i$ and v_i, p_i are left useful for t and w_i.

Proof. Let $x \in <v_1,...,v_r>_L$ then there exist $i \leq r$ and a path $y \in R$ such that $x = yv_i$. Our goal is to show that $xt \in I_L$. By hypothesis there is a path p_i such that $v_it = p_iw_i$ and v_i, p_i are left useful for t,w_i. Multiplying by y we get $yv_it = yp_iw_i$ i.e $xt = yp_iw_i \in I_L$ thus $x \in I_L : t$.

Conversely let $x \in I_L : t$ then $x \cdot t \in I_L$, that is, there exist $i \leq r$ and a path $y \in R$ such that $xt = yw_i$, this means there exist left useful paths x',y' for t,w_i such that $x' | x$ and $y' | y$. Set $v_i = x'$ then $x \in <v_1,...,v_r>$.

Definition 2.14. Let $R = KQ$ and $T = R^s$ be a let a left free R−module with basis $\{e_1,...,e_s\}$. By a monomial in T involving the component e_j we mean a monomial in R times e_j that is pe_j where p is a monomial in R.

Definition 2.15. (Syzygy)

Let $T = R^s$ be a left free R−module with basis $\{e_1,...,e_s\}$ and let $I_L = <f_1,...,f_s>_L$ be a left ideal of R. By a left syzygy we mean an element of the kernel of the R−module homomorphism

$$\psi : T = R^s \longrightarrow R$$

$$e_i \longmapsto f_i.$$

We call $\ker(\psi)$ the (first) left syzygy module on $f_1,...,f_s$ written $\text{syz}(f_1,...,f_s)_L = \ker(\psi)$.

Definition 2.16. (Induced ordering)

Let $G = \{f_1,...,f_s\}$ be a set of polynomials and > be a left admissible ordering. Let $T = R^s$ be a left free R−module with basis $\{e_1,...,e_s\}$. We define the module ordering $>_1$ induced by G and > as follow: if p,q are two paths in R then

$$pe_i >_1 qe_j \Longleftrightarrow pLm(f_i) > qLm(f_j) \text{ (whenever } pLm(f_i) \neq 0 \neq qLm(f_j)\text{) or } 0 \neq pLm(f_i) = qLm(f_j) \text{ and } i > j.$$

Theorem 2.17. (Left Buchberger's criterion)

Let $f_1,...,f_s \in R$ be non-zero polynomials and > be a left admissible ordering on R. Then $\{f_1,...,f_s\}$ form a left-Groebner basis for $I_L = <f_1,...,f_s>$ if and only if the following conditions are satisfied:

1) For each $i > j$, the remainder of $S_L(f_i,f_j)$ on division's algorithm by $\{f_1,...,f_s\}$ is zero.
2) For each i, the remainder of $S_L(f_i)$ on division's algorithm by $\{f_1,...,f_s\}$ is zero.

4. Proof of the Left Buchberger's Criterion

Proof. If $\{f_1,...,f_s\}$ form a left Groebner basis for $I_L = <f_1,...,f_s>$ then by the Proposition 2.6 and using the left division's algorithm we have for $k < i$, $S_L(f_i,f_k) = \sum_{i=1}^{s} g^i f_i + 0$ and for each i, $S_L(f_i) = \sum_{i=1}^{s} h^i f_i + 0$ since $S_L(f_i)$, $S_L(f_i,f_k) \in I_L$. and by proposition 2.6.

Conversely, suppose that for $1 \le k < i \le s$ all the remainders of $S_L(f_i,f_k)$ under the left division's algorithm by $\{f_1,...,f_s\}$ are zero and for each i, all the remainders of the $S_L(f_i)$ under the left division's algorithm by $\{f_1,...,f_s\}$ are zero. We have for

$$i > k, \ S_L(f_i, f_k) = \sum_{j=1}^{s} g_j^{ik} f_j \text{ with } Lm(S_L(f_i, f_k)) \geq Lm(g_j^{ik} f_j) \quad (**).$$

Applying the definition of the left S-polynomial we have

$$\frac{v}{Lc(f_i)} f_i - \frac{w}{Lc(f_k)} f_k = \sum_{j=1}^{s} g_j^{ik} f_j$$

where v,w are left useful paths for $Lm(f_i)$ and $Lm(f_k)$. Observe that

$-g_1^{ik} f_1 - \ldots - (g_k^{ik} + \frac{w}{Lc(f_k)}) f_k - \ldots - (g_i^{ik} - \frac{v}{Lc(f_i)}) f_i - \ldots - g_s^{ik} f_s = 0$, this means that

$(-g_1^{ik}, \ldots, -g_k^{ik} - \frac{w}{Lc(f_k)}, \ldots, -g_i^{ik} + \frac{v}{Lc(f_i)}, \ldots, -g_s^{ik})$ is a syzygy for $f_1,...,f_s$. Set

$G_{ik} = (-g_1^{ik}, \ldots, -g_k^{ik} - \frac{w}{Lc(f_k)}, \ldots, -g_i^{ik} + \frac{v}{Lc(f_i)}, \ldots, -g_s^{ik}) \quad (1).$

By (*) and the definition of the left S-polynomial we can notice that $Lm(g_j^{ik}f_j) \le Lm(S_L(f_i,f_k)) < vLm(f_i) = wLm(f_k)$, with respect to the induced ordering, this means that $ve_i > we_k$ since $i > k$, this leads us to

$Lm(G_{ik}) = ve_i$ (2).

In the other hand, we have for some i, $S_L(f_i) = p_i f_i$ where p_i is a vertex such that $p_i Lm(f_i) = 0$ and $p_i f_i \neq 0$. By the division's algorithm we have $S_L(f_i) = \sum_{j=1}^{s} h_j^i f_j = p_i f_i$, this means that

$$-h_1^i f_1 - \ldots - (-h_i^i - p_i) f_i - \ldots - h_s^i f_s = 0,$$

that is $H_i = (-h_1^i, \ldots, -(-h_i^i - p_i), \ldots, -h_s^i)$ is a left syzygy for $f_1,...,f_s$.

Observe that $Lm(h^i_j)Lm(f_j) \le Lm(S_L(f_i)) = Lm(p_i f_i) = p_i Lm(t_i)$ where t_i is a polynomial occurring in f_i such that $p_i f_i = p_i t_i$. By the Schreyer's ordering induced by > and $\{f_1,...,f_{i-1},t_i,f_{i+1},...,f_s\}$ we have $Lm(H_i) = p_i e_i$.

Let us now prove that $\{f_1,...,f_s\}$ form a left Groebner basis for I_L.

Let $g \in I_L \backslash \{0\}$ then $g = a_1 f_1 + \ldots + a_s f_s$ for some $a_1,...,a_s \in R$. Let $A = (a_1,...,a_s) \in R^s$, using the ordering induced by > and $\{f_1,...,f_s\}$, we can extend the left division's algorithm in R^s. Let $G =< g_1, \ldots, g_s > \in R^s$ be the remainder under the left division's algorithm of A by the set of all non-zero G_{ij} and H_i (listed in some order), then we have the expression.

A = $\sum_{G_{IJ}\neq 0} q_i G_{ij} + \sum_{H_i \neq 0} p_i H_i$+ G (C).

Let $F = (f_1,...,f_r)^t$, by multiplying (C) by F we get

AF = g = a1f1 + ... + asfs = g1f1 + ... + gsfs (D).

We transform (D) as follow:

$$g = \sum_{i=1}^{s} Lt(g_i) Lt(f_i) + \sum_{i=1}^{s} tail(g_i) Lt(f_i) + \sum_{i=1}^{s} g_i tail(f_i) \quad \text{Where } tail(f_i) = f_i - Lt(f_i)$$

1) If $Lm(g) \neq Lm(\sum_{i=1}^{s} g_i \text{tail}(f_i))$ then it is clear that

$$Lm(g) = Lm(\sum_{i=1}^{s} Lt(g_i)Lt(f_i) + \sum_{i=1}^{s} tail(g_i)Lt(f_i)$$

that is $Lm(g) \in \langle Lm(f_1),...,Lm(f_s)\rangle$ and by the Lemma 2.13 There exists $k \in\{1,...,s\}$ such that $Lm(f_k) \mid Lm(g)$.

2) If $\mathrm{Lm}(g) = \mathrm{Lm}(\sum_{i=1}^{s} g_i tail(f_i))$ then there exists $K \subseteq \{1,...,s\}$ such that

$$\sum_{i\in K} Lt(g_i)\, Lt(f_i) = 0 \qquad \text{and} \qquad \sum_{i\in K} Lt(g_i) tail(f_i) \neq 0.$$

If there exists $j \in K$ such that $Lm(g_j)Lm(f_j) \neq 0$ then without loss of generalities we can assume that $K = \{i_1,...,i_r\}$ and $j = i_r$, then

$$Lm(g_j)Lm(f_j) = -Lm(g_{i1})Lm(f_{i1}) - \ldots - Lm(g_{j-1})Lm(f_{j-1})$$

that is $Lm(g_j)Lm(f_j) \in \langle Lm(f_{i1}),...,Lm(f_{j-1})\rangle$ and $Lm(g_j) \in \langle Lm(f_{i1}),...,Lm(f_{j-1})\rangle : Lm(f_j)$, by the Lemma 2.13 we have $Lm(g_j) \in \langle v_{i1},...,v_{j-1}\rangle$ where for $i_1 \leq l \leq j-1$ we have $v_l Lm(f_j) = p_l Lm(f_l)$ and v_l, p_l are left useful monomials for $Lm(f_j)$ and $Lm(f_l)$. By the Lemma 2.13 there exists $k \in K$ such that $v_k \mid Lm(g_j)$ that is $Lm(G_{kj}) \mid Lm(g_j)e_j$ which contradict the fact that no monomial occurring in the remainder G is divisible by any of the $Lm(G_{ij})$.

Assume that $Lm(g_j)Lm(f_j) = 0\ \forall\, j \in K$ then $S_L(f_j) = p_j f_j$ where for each j, p_j is a vertex such that $p_j \neq s(Lm(f_j))$. Choose $p_j = r(Lm(g_j))$ then We have $Lm(g_j)e_j = Lm(g_j)p_j e_j = Lm(g_j)Lm(H_j)$, this means that $Lm(H_j)$ divides $Lm(g_j)e_j$ on the left, this is a contradiction since no monomial occurring in the remainder $G = (g_1,...,g_s)$ is divisible by any of $Lm(H_j)$.

Remark 2.18. To compute a left Groebner basis for $I_L = \langle f_1, \ldots, f_s \rangle_L$ like in the Theorem 3.16, there is no need to consider all the left S-polynomials $S_L(f_i, f_j) = \frac{v_{ji}}{Lc(f_i)} f_i - \frac{v_{ji}}{Lc(f_j)} f_j$ for some paths v_{ji}, v_{ij} left useful to $Lm(f_i)$ and $Lm(f_j)$. It is straightforward to see that

$$S_L(f_i,f_j) = -S_L(f_j,f_i)\ \forall\ i,j$$

so instead of considering all the left S-polynomials, we can just consider left the S-polynomials $S_L(f_i,f_j)$ with $j < i$. We can even do better by using the following technique.

For $i = 2,3,...,s$, consider the left path ideal

$M_i = \langle Lm(f_1),...,Lm(f_{i-1})\rangle : Lm(f_i) = \langle v_{1i},...,v_{i-1i}\rangle$ where $v_{ji}\ \forall\ 1 \leq j \leq i-1$, are like in the Proposition 2.13 i.e for each $v_{ji} \neq 0$, there exists a path v_{ij} such that v_{ji} and v_{ij} are left useful for $Lm(f_i), Lm(f_j)$. In the other words, each $v_{ji} \neq 0$ corresponds to a left S-polynomial $S_L(f_i, f_j) = \frac{v_{ji}}{Lc(f_i)} f_i - \frac{v_{ji}}{Lc(f_j)} f_j$.

As it turn out, there is no need to consider all the generators v_{ji} of M_i (i.e there is no need to consider all the corresponding S-polynomials of each v_{ji}) in the Buchberger's criterion. For each i, and for each minimal path generator v_{ji} of M_i, consider only the corresponding left S-polynomial $S_L(f_i,f_j)$, and compute the compute the remainder h_{ij} of the division of $S_L(f_i,f_j)$ by $f_1,...,f_s$.

Input: an ideal $I_L = \langle f_1,...,f_s\rangle \subset R$ and a left admissible order $>$ on R.

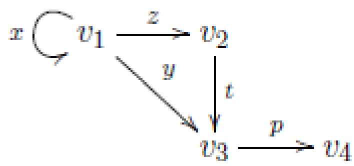
Output: a left Groebner basis for I_L.

A) Set $k = s$.
B) For $i = 2, \ldots, k$, and for each minimal path generator v_{ji} of

$$M_i = \langle Lm(f_1), \ldots, Lm(f_{i-1})\rangle_L : Lm(f_i) \subset R,$$

compute a remainder h_{ij} as described in the Remark 2.18 above.
C) If some h_{ij} is non-zero, set $k = k + 1$, $f_k = h_{ij}$, and go back to B).
D) Return $\{f_1, \ldots, f_k\}$.

Example 2.19. For the graph Γ



Let us compute the left Groebner bases for $F = \{f_1 = ztp + yp, f_2 = x^2zt + yp, f_3 = yp + zt\}$ in $R = \mathbb{Q}\Gamma$ with $x >_{llex} y >_{llex} z >_{llex} t >_{llex} p >_{llex} v1 >_{llex} v2 >_{llex} v3 >_{llex} v4$ where by $>_{llex}$ we mean the left length lexicographic ordering.

It is easy to see that $Lm(f_1) = ztp$, $Lm(f_2) = x^2zt$, $Lm(f_3) = yp$.

- For $i = 2$, it is easy to see that $<ztp>_L : x^2zt = 0$ i.e there are no useful monomials between $Lm(f_1)$ and $Lm(f_2)$, we claim that $S(f_2,f_1) = 0$.
- For $i = 3$, it is easy to see that $<ztp, x^2zt>_L : yp = 0$ i.e there are no useful monomials between $Lm(f_1)$ and $Lm(f_3)$, and between $Lm(f_2)$ and $Lm(f_3)$. We claim that $S_L(f_3,f_1) = S(f_3,f_2)_L = 0$. Observe that $S_L(f_1) = S_L(f_2) = S_L(f_3) = 0$.

Thus the set $F = \{f_1 = ztp + yp, f_2 = x^2zt + yp, f_3 = yp + zt\}$ is a left Groebner basis for $I_L = <F>_L$.

With notations as in the left Buchberger's criterions above, we have the following theorem.

Theorem 2.20. (Schreyer's theorem)

If $\{f_1,\ldots,f_s\}$ is a left Groebner basis for $I_L = <f_1,\ldots,f_s>_L$ w.r.t the an admissible ordering >, then the set $T = \{G_{ij}, H_i \in \text{syz}_L(f_1,\ldots,f_s) / G_{ij} \neq 0 \neq H_i\}$ considered in the proof of the left Buchberger's criterion form a left Groebner basis for $\text{syz}_L(f_1,\ldots,f_s)$ w.r.t the ordering $>_1$ induced by > and $\{f_1,\ldots,f_s\}$.

Proof. Let $A \in \text{Syz}_L(f_1,\ldots,f_s)$, we wish to prove that $\exists\, h \in T$ such that $Lt(h) \mid Lt(A)$.

By the left division's algorithm in R^s for A by T we have

$$A = \textstyle\sum_{ij} Q_{ij}G_{ij} + \sum_i p_iH_i + G \qquad (*)$$

where Q_{ij}, $H_i \in R$ and $G = (g_1,\ldots,g_s) \in R^s$. Let $F = (f_1,\ldots,f_s)^t \in R^s$, by multiplying $(*)$ by F we get

$$g_1f_1 + \ldots + g_sf_s = 0 \ (2*).$$

Assume that $G = (g_1,\ldots,g_s)$ is a non-trivial syzygy, then there exists $K = \{i_1,\ldots,i_r\} \subset \{1,\ldots,s\}$ such that

$$g_{i1}f_{i1} + \ldots + g_{ir}f_{ir} = 0 \ (3*).$$

We can transform (3∗) as follow

$$(4): 0 = \textstyle\sum_{i=i_1}^{i_r} Lt(g_i)Lt(f_i) + \sum_{i=i_1}^{i_r} tail(g_i)Lt(f_i) + \sum_{i=i_1}^{i_r} Lt(g_i)tail(f_i)$$

It is clear that $\sum_{i=i_1}^{i_r} Lt(g_i)Lt(f_i) = 0$

1) Assume that $\exists\ k \in K$ such that $Lm(g_k)Lm(f_k) \neq 0$. Without loss of generalities we can assume that $k = i_r$, in this case

$$Lt(g_{i_r})Lt(f_{i_r}) = -Lt(g_{i_1})Lt(f_{i_1}) - \cdots - Lt(g_{i_{r-1}})Lt(f_{i_{r-1}}),$$

that is

$$Lt(g_{i_r}) \in <Lt(f_{i_1}),\ldots,Lt(f_{i_{r-1}})> : Lt(f_{i_r}) = <v_{i_r i_1},\ldots, v_{i_r i_{r-1}}>$$

where for each v_{kj}, there exists p_{kj} such that v_{kj} and p_{kj} are left useful for $Lm(f_k)$ and $Lm(f_j)$. We have seen in the proof of the left Buchberger's criterion that for each j, $Lm(G_{kj}) = v_{ki}e_k$, according to the Lemma 2.13, there exists $j \in K \setminus \{i_r\}$ such that $Lt(G_{i_r j}) \mid Lt(g_{i_r})e_{i_r}$ which contradict the fact that no term occurring in G is divisible by any of the $Lt(G_{kj})$.

2) Assume that $Lm(g_i)Lm(f_i) = 0\ \forall\ i \in K$, then from (4) we have

$$(5) : \sum_{i=i_1}^{i_r} g_i^1 Lt(f_i) + \sum_{i=i_1}^{i_r} Lt(g_i) f_i = 0$$

where $g_i^1 = \text{tail}(g_i)$ and $\sum_{i=i_1}^{i_r} Lt(g_i) tail(f_i) = \sum_{i=i_1}^{i_r} Lt(g_i) f_i = \sum_{i=i_1}^{i_r} Lc(g_i) p_i S_L(f_i)$ where p_i is a path exactly like in the proof of the left Buchberger's criterion. Since $\{f_1,\ldots,f_s\}$ form a Groebner basis, by the left Buchberger's criterion we have $S_L(f_i) = \sum_{j=1}^{s} h_j^i f_j$ and $Lm(S_L(f_i)) = Lm(h_k^i)Lm(f_k)$ for some k.

We transform (5) as follow $g_{i_r}^1 Lt(f_r) = \sum_{i=i_1}^{i_{r-1}} g_i^1 Lt(f_i) - \sum_{i=i_1}^{i_r} \sum_{j=1}^{s} Lc(g_i)\, p_i h_j^i f_j$ therefore

$$Lt(g_{i_r}^1)Lt(f_r) \in <Lt(f_1),\ldots, Lt(f_s)>.$$

Using the Lemma 2.13 and the Proposition 2.5 we get a contradiction since no monomial occurring in the remainder is divisible by any of $Lm(G_{ij})$.

We give in the following example, an algorithm for computing a left Groebner basis for syz($f_1,\ldots,f_s$).

Input: left Groebner basis $G = \{f_1,\ldots, f_s\}$ for $I = <G>$ and an admissible ordering >.

Output: left Groebner basis T for syz($f_1,\ldots,f_s$) w.r.t $>_1$ induced by > and G.

(1) Set $T := \emptyset := H_1 := H_2$.
(2) For each $1 \leq k \leq s$, do
- $= S_L(f_k) = a_k f_k = \sum_{i=1}^{s} g_i^k f_i$;
- $r := -g_1^k e_1 - \cdots - (g_k^k - a_k)e_k - \cdots - g_s^k e_s$.
- $H_1 := H_1 \cup \{r\}$.

(3) For each $i = 2,\ldots,s$ and each minimal generator v_{ij} with $i > j$, do
- $= S(f_i, f_j) = \frac{v_{ji}}{Lc(f_i)} f_i - \frac{v_{ij}}{Lc(f_j)} f_j = \sum_{i=1}^{s} p_i^{ji} f_i$;
- $s := -p_1^{ji} e_1 - \cdots - (p_j^{ji} + \frac{v_{ij}}{Lc(f_j)})e_j - \cdots - (p_i^{ji} - \frac{v_{ji}}{Lc(f_i)})e_i - \cdots - p_s^{ji} e_s$;
- $H_2 := H_2 \cup \{s\}$;

(4) $T := H_1 \cup H_2$.
(5) Return T.

5. Right Groebner Bases on *KQ*

In this section we denote I_R as a right ideal of $R = KQ$ and $>$ a right admissible ordering. We will omit some proofs and details in this section since they follow from the previous section.

Definition 3.1. Let I_R be a right ideal of R and G be a subset of I_R. We say that G is a right Groebner basis for I_R with respect to a given right-admissible ordering if for every $f \in I_R$, there exists $g \in G$ such that $Lm(g)$ divides $Lm(f)$ i.e there exists $h \in R$ such that $Lm(f) = Lm(g)Lm(h)$. In the other hand, G is a right Groebner basis for I_R if $Lm(I_R)_L = Lm(G)_R$.

Definition 3.2. Let Q be a quiver. A cycle c in Q has an exit if there exists an edge in Q not occurring in c with its source in c. For example in the graph

$$Q: \quad a \circlearrowleft \bullet \xrightarrow{b} \bullet$$

the cycle has an exit.

Proposition 3.3. The path K−algebra KQ is right noetherian if and only if no cycle in Q has an exit.

Proof. Similar with the left side.

In this section $R = KQ$ will be considered as right noetherian path K−algebra.

Proposition 3.4. Let R be a right noetherian path K−algebra and $<$ be a right admissible order.

1) A path $p \in T = < q_i \;/ q_i \in \text{path}(Q)\ \forall\ 1 \leq i \leq r>_R$ if and only if there exists i_0 such that q_{i_0} divides p on the right.
2) If $I_R \subset R$ is a right path ideal then I_R has a unique finite minimal path generating set.

Proof. The proof is similar to the one of Proposition 2.5.

3.1. **Right division's algorithm**. Let $F = \{f_1,...,f_s\}$ be a set of elements of $R = KQ$ and $>$ be a right admissible ordering.

Given element $g \in R \setminus \{0\}$, the following algorithm shows how to nd $q_1,..., q_s, h \in KQ$ such that $g = f_1q_1 + \cdots + f_sq_s + h$ satisfying.

A1 if $h \neq 0$, then each term occurring in h is not divisible by any of $Lt(f_i)\ \forall\ 1 \leq i \leq s$.

A2 $Lm(g) \geq Lm(f_iq_i)$ for each $i \in \{1,...,s\}$ since q_i are obtained from g.

Input: $f, f_1,...,f_s \in F$ and a right admissible ordering $>= (>_1, >_2)$.

Output: $q_1,...,q_s, h \in F$ such that $f = f_1q_1 + \cdots + f_sq_s + h$.

Initialization: $q_1 := q_2 := \cdots := q_s := h := 0$ and $v := f$;

- While $v \neq 0$ do
  - $i := 1$;
  - while $i \leq s$ do
    * if $Lt(f_i) \mid Lt(v)$ then
      · $q_i := q_i + \dfrac{Lt(v)}{Lt(f_i)}$;
      · $v := v - f_i\dfrac{Lt(v)}{Lt(f_i)}$;
    * Else
      · $i := i + 1$;
  - $h := h + Lt(v)$;
  - $v := v - Lt(v)$.

Proposition 3.5. Let $\{f_1,...,f_r\} \subset R$ be a right Groebner basis for the ideal $I_R := <f_1, \ldots, f_r>_L \subset R$, $>$ be a right admissible ordering and $g \in R \setminus \{0\}$. $g \in I_L$ if and only if the remainder h of the division of g by $f_1,...,f_r$ is zero.

Definition 3.6. (Right useful paths)

Let $p,q \in \text{path}(Q)$ be paths in Q. Two paths u,v are called right useful for p and q with respect to the right admissible ordering $<$, if $up = vq \neq 0$ and for any other paths w_1 and w_2 satisfying $pw_1 = qw_2 \neq 0$, we have u divides w_1 and v divides w_2 (both on the right).

Definition 3.7. (Right S-polynomial)

Let f,g be non-zero polynomials and $<$ be a right admissible ordering. Let u,v be right useful paths for $Lm(f)$ and $Lm(g)$ (i.e $Lm(f) \cdot u = Lm(g) \cdot v$), then the right S-polynomial $S_R(f,g)$ of f and g is dened as

$$S_R(f,g) = f \cdot \frac{u}{Lc(f)} - g \cdot \frac{v}{Lc(g)}.$$

Remark 3.8. If there exist no right useful paths for $Lm(f)$ and $Lm(g)$, we claim that $S_R(f, g) = 0$.

Definition 3.9. (Right extended S-Polynomial)

Let $f \in R$ be a non-zero polynomial and $>$ be a right admissible ordering. We define the right extended S-polynomial $S_R(f)$ of f as $S_R(f) = f \cdot p$ where p is a path in KQ such that $Lm(f) \cdot p = 0$ and $f \cdot p \neq 0$.

Remark 3.10. Let p and q be two paths satisfying the conditions of the Definition 3.9, if q divides p then we choose $S_R(f) = f\cdot$.

Definition 3.11. Let I_R be a right path ideal and $w \in$ path (Q). We define the left quotient ideal of I_R and w by $I_R : w = \{p \in \text{path}(Q)/\ w \cdot p \in I_R\}$.

Lemma 3.12. Let $I_R = <w_1,...,w_r>_L$ be a right path ideal of R, t be a monomial and $>$ be a right admissible ordering. Then the quotient ideal $I_R : t$ can be written as

$$I_R : t = <v_1, \dots ,v_l>_L$$

where for each $1 \leq i \leq l \leq r$ there exists a monomial p_i such that $tv_i = w_ip_i$ and v_i,p_i are right useful for t and w_i.

Definition 3.13. Let $R = KQ$ and $T = R^s$ be a right free R−module with basis $\{e_1,...,e_s\}$. By a monomial in T involving the component e_j we mean a component e_j times a path in R.

Definition 3.14. (Right Syzygy)

Let $T = R^s$ be a right free R−module with basis $\{e_1,...,e_s\}$ and let $I_R = <f_1, \dots ,f_s>_R$ be a right ideal of R. By a right syzygy we mean an element of the kernel of the R−module homomorphism

$$\psi : T = R^s \longrightarrow R$$

$$e_i \longmapsto f_i.$$

We call $\ker(\psi)$ the (first) right syzygy module on $f_1,...,f_s$ written $\text{syz}(f_1, \dots ,f_s)_R = \ker(\psi)$.

Definition 3.15. (Induced ordering)

Let $G = \{f_1,...,f_s\}$ be a set of polynomials and $>$ be a right admissible ordering. Let $T = R^s$ be a right free R−module with basis $\{e_1,...,e_s\}$. We define the module ordering $>_1$ induced by G and $>$ as follow: if p,q are two paths in R then

$$e_ip >_1 e_jq \iff Lm(f_i)p > pLm(f_j)p \text{ (whenever } Lm(f_i)p \neq 0 \neq Lm(f_j))q \text{ or}$$

$$0 \neq Lm(f_i)p = Lm(f_j)q \text{ and } i > j.$$

Theorem 3.16. (Right Buchberger's criterion)

Let $f_1,...,f_s \in R$ be non-zero polynomials and $>$ be a right admissible ordering on R. Then $\{f_1,...,f_s\}$ form a right-Groebner basis for $I_R = <f_1,...,f_s>$ if and only if the following conditions are satisfied:

1) For each $i > j$, the remainder of $S_R(f_i, f_j)$ on right division's algorithm by $\{f_1, \dots ,f_s\}$ is zero.
2) For each i, the remainder of $S_R(f_i)$ on right division's algorithm by $\{f_1, \dots ,f_s\}$ is zero.

Input: give an ideal $I_R = <f_1, \dots, f_s>_R \subset R$ and a right admissible ordering $>= (>_1, >_2)$ on R.

Output: a right Groebner basis for I_R.

Set $k = s$;

Set $P := \{S_R(f_i, f_j)/\ 1 \leq j < i \leq k\}$ and $Q := \{S_R(f_i)/\ 1 \leq i \leq k\}$

A) Set $k = s$.
B) For $i = 2, \ldots, k$, and for each minimal path generator v_{ji} of

$$M_i = \langle Lm(f_1), \ldots, Lm(f_{i-1})\rangle_R : Lm(f_i) \subset R,$$

compute a remainder h_{ij} as described in the Remark 2.18 above.
C) If some h_{ij} is non-zero, set $k = k+1,\ f_k = h_{ij}$, and go back to B).
D) Return $\{f_1, \ldots, f_k\}$.

As in the previous section, we describe the right Schreyer's theorem

Theorem 3.17. (Schreyer's theorem)

If $\{f_1,\ldots,f_s\}$ is a right Groebner basis for $I_R = <f_1, \ldots ,f_s>_R$ w.r.t a right admissible ordering >, then the set $T = \{G_{ij}, H_i \in \mathrm{syz}_L(f_1, \ldots ,f_s)/\ G_{ij} \neq 0 \neq H_i\}$ of right syzygies form a right Groebner basis for $\mathrm{syz}_R(f_1,\ldots,f_s)$ w.r.t the ordering $>_1$ induced by > and $\{f_1, \ldots ,f_s\}$.

6. Computing Left and Right Syzygies

In this section we propose a method for computing left and right syzygies of given polynomials $f_1, \ldots ,f_s \in R \setminus\{0\}$.

4.1. **Left syzygies**. In this subsection, we denote by > a left admissible ordering and $f_1,\ldots,f_s \in R \setminus\{0\}$ polynomials. The goal of this subsection is to propose a method for computing left syzygies on $f_1,\ldots,f_s$.

Remark 4.1. Let $f_1,\ldots,f_s \in R \setminus\{0\}$ be polynomials and $G = \{f_1, \ldots ,f_s, \ldots ,f_{s'}\}$ be a left Groebner basis for $<f_1, \ldots ,f_s>_L$ w.r.t >. Let $T = \{G_{ij}, H_i/\ G_{ij}, H_i \in \mathrm{syz}(G)_L \setminus\{0\}\}$ be the set of left syzygies on the left Groebner bases G. Assume that we have t such G_{ij}, H_i arranged as follow:

By computing the left Groebner basis $G = \{f_1, \ldots ,f_s, \ldots ,f_{s'}\}$, we store each non-zero left syzygy G_{ij} and H_i such that those obtained from a left division leading to a new polynomial f_k are fist and those obtained from a left division with remainder zero are second. The G_{ij} fits as rows of the $t \times s'$ matrix

$$M'_{t\times s'} = \begin{pmatrix} g_1^1 & g_2^1 & \cdots & g_s^1 & 1 & 0 & 0 & \cdots & 0 \\ g_1^2 & g_2^2 & \cdots & g_s^2 & g_{s+1}^2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & & & \\ g_1^{s'-s} & g_2^{s'-s} & \cdots & g_s^{s'-s} & g_{s+1}^{s'-s} & g_{s+2}^{s'-s} & g_{s+3}^{s'-s} & \cdots & 1 \\ g_1^{s'-s+1} & g_2^{s'-s+1} & \cdots & g_s^{s'-s+1} & g_{s+1}^{s'-s+1} & g_{s+2}^{s'-s+1} & g_{s+3}^{s'-s+1} & \cdots & g_{s'}^{s'-s+1} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & & & \\ g_1^t & g_2^t & \cdots & g_s^t & g_{s+1}^t & g_{s+2}^t & g_{s'}^t & \cdots & g_{s'}^t \end{pmatrix}$$

Let

$$A'_{(s'-s)\times s} = \begin{pmatrix} g_1^1 & g_2^1 & \cdots & g_s^1 \\ g_1^2 & g_2^2 & \cdots & g_s^2 \\ \vdots & \vdots & \cdots & \vdots \\ g_1^{s'-s} & g_2^{s'-s} & \cdots & g_s^{s'-s} \end{pmatrix}, B'_{(t-(s'-s))\times s} = \begin{pmatrix} g_1^{s'-s+1} & g_2^{s'-s+1} & \cdots & g_s^{s'-s+1} \\ \vdots & \vdots & \cdots & \vdots \\ g_1^t & g_2^t & \cdots & g_s^t \end{pmatrix},$$

$$C'_{s'-s} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ g_{s+1}^2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{s+1}^{s'-s} & g_{s+2}^{s'-s} & g_{s+3}^{s'-s} & \cdots & 1 \end{pmatrix} \text{ and } D'_{(t-(s'-s))\times(s'-s)} = \begin{pmatrix} g_{s+1}^{s'-s+1} & g_{s+2}^{s'-s+1} & g_{s+3}^{s'-s+1} & \cdots & g_{s'}^{s'-s+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{s+1}^t & g_{s+2}^t & g_{s'}^t & \cdots & g_{s'}^t \end{pmatrix}$$

Then $M'_{t\times s'}$ can be regarded as the $t \times s'$ block matrix

$$M'_{t\times s'} = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}$$

where $A' = A'_{(s'-s)\times s}, B' = B'_{(t-(s'-s))\times s}, C' = C'_{s'-s}$ and $D' = D'_{(t-(s'-s))\times(s'-s)}$.

Theorem 4.2. With notations as above. Assume that the matrix B' is non-zero. The rows of the $(t - s' + s) \times s$ matrix $B' - D'C'^{-1}A'$

form non-zero left syzygies on $f_1,\dots,f_s$.

Proof. Set

$$F = \begin{pmatrix} f_1 \\ \\ f_s \end{pmatrix}, F' = \begin{pmatrix} f_{s+1} \\ \\ f_{s'} \end{pmatrix} \text{ and } \begin{pmatrix} F \\ F' \end{pmatrix} = \begin{pmatrix} f_1 \\ \\ f_s \\ \\ f_{s'} \end{pmatrix}.$$

We know by hypothesis that each row of the matrix

$$\begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}$$

form a left syzygy on $f_1,\dots,f_{s0}$. We can write

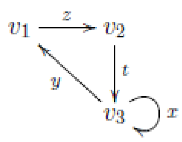
$$\begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} \begin{pmatrix} F \\ F' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then

$$\begin{cases} A'F + C'F' = 0 & (1) \\ B'F + D'F' = 0 & (2) \end{cases}$$

Observe that C' is invertible, by multiplying each side of (1) by C'^{-1} on the left, we get $C'^{-1}AF + F' = 0 \implies F' = -C'^{-1}AF$ (3). Replacing (3) in (2) we get $(B' - D'C'^{-1}A')F = 0$ (4). finalement

Example 4.3. For the quiver Γ



Let us compute a left Groebner basis for $F = \{f_1 = zt + yz, f_2 = tx + z, f_3 = x^2y + t\}$ in $R = \mathbb{Q}\Gamma$ with respect to the left length lexicographic ordering $t >_{llex} z >_{llex} z >_{llex} y >_{llex} x >_{llex} v_1 >_{llex} v_2 >_{llex} v_3$.

Observe that $S_L(f_2) = v_1f_2 = z = f_4$, $S_L(f_3) = v_2f_3 = f_5$ and $S_L(f_1) = v_3f_1 = yz = yf_4$. Observe that $S_L(f_2, f_1) =$ $S_L(f_3, f_1) = S_L(f_3, f_2) = S_L(f_4, f_1) = S_L(f_4, f_2) = S_L(f_4, f_3) = S_L(f_5, f_2) = S_L(f_5, f_3) = S_L(f_5, f_4) = 0$ by the lack of useful paths. We have $S(f_5, f_1) = zf_5 - v_1f_1 = zt - zt = 0$. The set $\{f_1, \dots ,f_5\}$ is then a left Groebner basis for the ideal $I_L = \langle f_1, f_2, f_3\rangle$. From Schreyer's theorem, the set $\{v_1e_2 - e_4, v_2e_3 - e_5, v_3e_1 - ye_4, -v_1e_1 + ze_5\}$ form a left Groebner basis for syz$(f_1, \dots ,f_5)$ w.r.t the ordering induced by > and f1, ... ,f5.

Let us compute the set of syzygies for f_1,f_2,f_3. We store each left syzygy of $f_1, \dots ,f_5$ as rows of the matrix

$$M = \begin{pmatrix} 0 & v_1 & 0 & -1 & 0 \\ 0 & 0 & v_2 & 0 & -1 \\ v_3 & 0 & 0 & -y & 0 \\ -v_1 & 0 & 0 & 0 & z \end{pmatrix}.$$

Set

$$A = \begin{pmatrix} 0 & v_1 & 0 \\ 0 & 0 & v_2 \end{pmatrix}, B = \begin{pmatrix} v_3 & 0 & 0 \\ -v_1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{and } D = \begin{pmatrix} -y & 0 \\ 0 & z \end{pmatrix}.$$

Observe that

$$B - DC^{-1}A = \begin{pmatrix} v_3 & -y & 0 \\ -v_1 & 0 & z \end{pmatrix},$$

then syz$(f_1, f_2, f_3)_L$ = <$(v_3, -y, 0), (-v_1, 0, z)$>.

4.2. **Right syzygies**. Let R be a right noetherian path algebra and $f_1, \dots, f_s \in R\backslash\{0\}$ be non-zero polynomials. The goal of this subsection is to propose a method for computing right syzygies on $f_1, \dots, f_s \in R$.

Remark 4.4. Let $f_1,\dots,f_s \in R\backslash\{0\}$ be non-zero polynomials and $G = \{f_1, \dots, f_s, \dots, f_{s'}\}$ be a right Groebner basis for $\langle f_1, \dots, f_s\rangle_R$ w.r.t the right admissible ordering <. Let G_{ij} be right syzygies on $f_1, \dots, f_s, \dots, f_{s'}$ for each pair (i, j) $i > j$. We have seen in the right Schreyer's theorem that syz$(f_1, \dots, f_{s'}) = \langle G_{ij} \neq 0 \neq H_i / \forall\, i > j\rangle_R$. Assume that we have t such G_{ij}, H_i arranged as follow:

By computing the right Groebner basis $G = \{f_1, \dots, f_s, \dots, f_{s'}\}$, we store each non-zero right syzygy G_{ij} and H_i such that those obtained from a right division leading to a new polynomial f_k are fist and those obtained from a right division with remainder zero are second. The G_{ij} fits as columns of the $s' \times t$ matrix

$$M_{s'\times t} = \begin{pmatrix}
g_1^1 & g_1^2 & \cdots & g_1^{s'-s} & g_1^{s'-s+1} & \cdots & g_1^t \\
g_2^1 & g_2^2 & \cdots & g_2^{s'-s} & g_2^{s'-s+1} & \cdots & g_2^t \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \\
g_s^1 & g_s^2 & \cdots & g_s^{s'-s} & g_s^{s'-s+1} & \cdots & g_s^t \\
1 & g_{s+1}^2 & \cdots & g_{s+1}^{s'-s} & g_{s+1}^{s'-s+1} & \cdots & g_{s+1}^t \\
0 & 1 & \cdots & g_{s+2}^{s'-s} & g_{s+2}^{s'-s+1} & \cdots & g_{s+2}^t \\
\vdots & 0 & \cdots & g_{s+3}^{s'-s} & g_{s+3}^{s'-s+1} & \cdots & g_{s+3}^t \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \\
0 & 0 & \cdots & 1 & g_{s'}^{s'-s+1} & \cdots & g_{s'}^t
\end{pmatrix}$$

Let

$$A_{s\times(s'-s)} = \begin{pmatrix}
g_1^1 & g_1^2 & \cdots & g_1^{s'-s} \\
g_2^1 & g_2^2 & \cdots & g_2^{s'-s} \\
 & & \cdots & \\
g_s^1 & g_s^2 & \cdots & g_s^{s'-s}
\end{pmatrix}, B_{s\times(t-(s'-s))} = \begin{pmatrix}
g_1^{s'-s+1} & \cdots & g_1^t \\
g_2^{s'-s+1} & \cdots & g_2^t \\
 & \cdots & \\
g_s^{s'-s+1} & \cdots & g_s^t
\end{pmatrix},$$

and

$$D_{(s'-s)\times(t-(s'-s))} = \begin{pmatrix}
g_{s+1}^{s'-s+1} & \cdots & g_{s+1}^t \\
g_{s+2}^{s'-s+1} & \cdots & g_{s+2}^t \\
 & \cdots & \\
g_{s'}^{s'-s+1} & \cdots & g_{s'}^t
\end{pmatrix}$$

$$C_{s'-s} = \begin{pmatrix} 1 & g_{s+1}^2 & \cdots & g_{s+1}^{s'-s} \\ 0 & 1 & \cdots & g_{s+2}^{s'-s} \\ \vdots & 0 & \cdots & g_{s+3}^{s'-s} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Then $M_{s'\times t}$ can be regarded as the $s'\times t$ block matrix

$$M_{s'\times t} = \begin{pmatrix} A_{s\times(s'-s)} & B_{s\times(t-(s'-s))} \\ C_{s'-s} & D_{(s'-s)\times(t-(s'-s))} \end{pmatrix}.$$

Notation 4.5. For simplicity, we denote A = $A_{s\times(s'-s)}$, B = $B_{s\times(t-(s'-s))}$, C =$C_{s'-s}$ and D =$D_{(s'-s)\times(t-(s'-s))}$.

Theorem 4.6. With notations as above. The columns of the $s \times (t - s' + s)$ matrix $B - AC^{-1}D$ are right syzygies on $f_1, \ldots, f_s$.

Proof. Set

$$F = (f_1, \ldots, f_s),\ F' = (f_{s+1}, \ldots, f_{s'}\ \ and\ \ (F\ \ F') = (f_1\ \ldots\ f_s\ \ldots\ f_{s'})\ \ .$$

We know by hypothesis that each column of the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

form a right syzygy on $f_1, \ldots, f_{s'}$. We can write

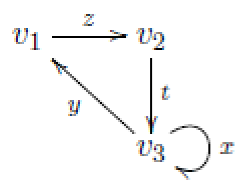
$$(F\ \ F') \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0_{1\times(s'-s)} & 0_{1\times(t-s'+s)} \end{pmatrix}$$

then

$$\begin{cases} FA + F'C = 0_{1\times(s'-s)} & (1) \\ FB + F'D = 0_{1\times(t-s'+s)} & (2) \end{cases}$$

Note that C is invertible, by multiplying each side of (1) by C^{-1} on the right, we get $FAC^{-1} + F' = 0_{1\times(s'-s)}$ (3) . By multiplying each side of (3) on the right by D, we get $FAC^{-1}D + F'D = 0_{1\times(t-s'+s)}$ (4). Subtracting (2) and (4) side by side we get $FB - FAC^{-1}D = F(B - AC^{-1}D) = 0_{1\times(t-s0+s)}$ (6). Thus, each column of the matrix $B - AC^{-1}D$ is a right syzygy on $f_1, \ldots, f_s$.

Example 4.7. For the quiver Γ



Let us compute a right Groebner basis for $F = \{f_1 = zt + yz, f_2 = tx + z, f_3 = x^2y + t\}$ in $R = \mathbb{Q}\Gamma$ with respect to the right length lexicographic ordering $t >_{rlex} z >_{rlex} z >_{rlex} y >_{rlex} x >_{rlex} v_1 >_{rlex} v_2 >_{rlex} v_3$.

Observe that $S_R(f_1) = f_2v_2 = yz = f_4$, $S_R(f_2) = f_2v_2 = z = f_5$ and $S_R(f_3) = f_3v_3 = t = f_6$.

By the lack of right useful paths, only the following right S-polynomials are computable:

$S_R(f_5, f_1) = f_5t - f_1v_3 = 0$ and $S_R(f_6, f_2) = f_6x - f_2v_3 = 0$, for the others, we claim them to be zero. The set $\{f_1,...,f_6\}$ is then a right Groebner basis for $<f_1, f_2, f_3>_R$.

Let us compute the set of syzygies for f_1, f_2, f_3. We store each right syzygy of $f_1, ..., f_6$ as column of the matrix

$$M = \begin{pmatrix} v_2 & 0 & 0 & -v_3 & 0 \\ 0 & v_2 & 0 & 0 & -v_3 \\ 0 & 0 & v_3 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & t & 0 \\ 0 & 0 & -1 & 0 & x \end{pmatrix}.$$

Set

$$A = \begin{pmatrix} v_2 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix}, B = \begin{pmatrix} -v_3 & 0 \\ 0 & -v_3 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 0 \\ t & 0 \\ 0 & x \end{pmatrix}.$$

Observe that

$$B - AC^{-1}D = \begin{pmatrix} -v_3 & 0 \\ t & -v_3 \\ 0 & x \end{pmatrix},$$

then $\mathrm{syz}(f_1, f_2, f_3)_R = <(-v_3, t, 0), (0, -v_3, x)>$.

7. Conlusion

This paper propose an algorithm for computing the left (resp. right) syzygies modules in a path K−algebra KQ, this gives necessary tools for working on an algorithm for computing a generating set of the intersection of left (resp. right) ideals in KQ. This paper also provides ideas that can be used to think of an algorithm for computing the two-sided syzygies modules. Since for any quiver Q we can define the associated Leavitt path K−algebra $L_K(Q)$ which is nothing but a path K−algebra on the extending quiver of Q, and satisfy the Cunz-Krieger relations, this paper can be the first step for studying the theory of Groebner bases and applications in $L_K(Q)$.

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