PBW Basis of Non-standard Quantum Groups $X_q(A_n)$

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Abstract: A kind of non-standard quantum group $X_q(A_n)$ is studied in the paper. Root vectors of $X_q(A_n)$ and their commutation relation are described. Then we establish the PBW basis of $X_q(A_n)$.

Key words: Non-standard quantum group, PBW basis, root vector.

1. Introduction

Throughout the paper, we always assume that the base field is the complex number field $\mathbb{C}$ and $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$. Let the non-zero parameter $q \in \mathbb{C}$ and not be a root of unity.

Quantum groups always are the hot research topic in mathematics and physics after they were introduced by Drinfeld and Jimbo in 1980s. Ge et al. [1] constructed a new quantum group by solving exotic solution of quantum Yang-Baxter equation, which is also called the non-standard quantum group. Jing et al. [2] derived a new quantum group $X_q(2)$ and described all finite dimensional irreducible representations of $X_q(2)$. Aghamohammadi et al. [3] obtained a non-standard quantum group $X_q(A_{n-1})$ corresponding to type A_{n-1}. It is noted that $X_q(A_1)$ is just quantum algebra $X_q(2)$. In 1994, Aghamohammadi et al. [4] constructed the non-standard quantum group $X_q(B_n)$ corresponding to the series B_n. Cheng and Yang [5] construct a weak Hopf algebra $wX_q(A_1)$ corresponding to non-standard quantum group $X_q(A_1)$, and describe the PBW basis of $wX_q(A_1)$.

In this paper, we describe the PBW basis of a particular class of $X_q(A_n)$. The paper is arranged as follows. In Section 2, we rewrite definition of the Hopf algebra $X_q(A_n)$ referred to the quantum algebra $X_q(A_n)$ [3]. In Section 3, we establish the root vectors and investigate commutation relations of $X_q(A_n)$ in the case of $q_i = q\ (0 \leq i \leq n)$ and $q_{n+1} = -q^{-1}$. In Section 4, we construct the PBW basis of $X_q(A_n)$ as described in Section 3.

2. Preliminaries

In first, we have quantum group $X_q(A_n)$ by replacing some generators in [3].

Definition 2. 1. $X_q(A_n)$ is an associative algebra over the field $\mathbb{C}$ with 1 generated by

$K_1^{\pm 1}, K_2^{\pm 1}, ..., K_n^{\pm 1}$,

$K_{n+1}^{\pm 1}, E_1, ..., E_n, F_1, ..., F_n$ with the following relations:

(R1) $K_iK_i^{-1} = K_i^{-1}K_i = 1, K_iK_j = K_jK_i$, (R2) $K_iE_j = E_jK_i, K_iF_j = F_jK_i, i \neq j, j+1$,

(R3) $K_iE_i = q_i^{-1}E_iK_i, K_iF_i = q_iF_iK_i$, (R4) $K_{i+1}E_i = q_{i+1}E_iK_{i+1}, K_{i+1}F_i = q_{i+1}^{-1}F_iK_{i+1}$,

(R5) $E_iF_j - F_jE_i = \delta_{i,j}\dfrac{K_{i+1}K_i^{-1} - K_iK_{i+1}^{-1}}{q - q^{-1}}$, (R6) $(q_i - q_{i+1})E_i^2 = (q_i - q_{i+1})F_i^2 = 0$,

(R7) $q_iE_i^2E_{i\pm 1} - (1 + q_iq_{i+1})E_iE_{i\pm 1}E_i + q_{i+1}E_{i\pm 1}E_i^2 = 0$,

(R8) $q_iF_i^2F_{i\pm 1} - (1 + q_iq_{i+1})F_iF_{i\pm 1}F_i + q_{i+1}F_{i\pm 1}F_i^2 = 0$,

where $q_i = \text{q}$ or $-q^{-1}$. If all $q_i = q$ $(0 \leq i \leq n+1)$, then $X_q(A_n)$ is similar to $U_q(sl_{n+1})$. If $q_i \neq q_{i+1}$ for some $1 \leq i \leq n$, then $E_i^2 = F_i^2 = 0$. The relations is different from $U_q(sl_{n+1})$.

Proposition 2. 2. Keeping notations as above. Then $X_q(A_n)$ is a Hopf algebra with comultiplication Δ, counit ε and antipode S, which are defined as following

$\Delta: X_q(A_n) \to X_q(A_n) \otimes X_q(A_n)$, $\Delta(K_i) = K_i \otimes K_i, \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1}$,

$\Delta(E_i) = K_iK_{i+1}^{-1} \otimes E_i + E_i \otimes 1, i \neq n$, $\Delta(F_i) = 1 \otimes F_i + F_i \otimes (K_i^{-1}K_{i+1}), i \neq n$,

S: $X_q(A_n) \to X_q(A_n)$, $S(K_i) = K_i^{-1}, S(K_i^{-1}) = K_i$, $S(E_i) = -K_{i+1}K_i^{-1}E_i, i \neq n$,

$S(F_i) = -F_iK_iK_{i+1}^{-1}, i \neq n, S(F_n) = -F_nK_n$,

$\varepsilon: X_q(A_n) \to \mathbb{C}, \varepsilon(K_i) = \varepsilon(K_i^{-1}) = 1, \varepsilon(E_i) = \varepsilon(F_i) = 0$.

Proof. The proof is more or less the same as that in [6, Proposition VII.1.1].

3. Commutation relations between root vectors

In first, we recall the definition of operator ad. Assume that H is a Hopf algebra and $x \in H$, we can define the adjoint operator $\text{ad}_x: H \to H$, associated to x as $\text{ad}_x(y) = \sum x_{(1)}yS(x_{(2)})$ for all $y \in H$, where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$.

Let $V = \mathbb{R}^{n+1}$ be an $n+1$-dimensional Euclidean space with $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \varepsilon_{n+1}$ as an orthogonal basis of V. Each positive root α can be written as $\alpha = \varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}, 1 \leq i, j \leq n+1$. We define root vectors by $E_{i,j} := \text{ad}_{E_i}(E_{i+1,j}), 1 \leq i, j \leq n$, corresponding to the root $\varepsilon_i - \varepsilon_j$, specially, $E_{i,i+1} := E_i$. Define the order of all positive roots by

$$\alpha_1 > \alpha_1 + \alpha_2 > \ldots > \alpha_1 + \alpha_2 + \ldots + \alpha_n > \alpha_2 > \alpha_2 + \alpha_3 > \ldots > \alpha_2 + \ldots + \alpha_n > \ldots > \alpha_{n-1} > \alpha_{n-1} + \alpha_n > \alpha_n.$$

The algebra $X_q(A_n)$ is graded via $\deg E_i = \alpha_i, \deg F_i = -\alpha_i, \deg K_i = \deg K_i^{-1} = 0$.

Set ξ (resp. η) be root vector corresponding to positive root β (resp. γ). We also can say that ξ (resp. $\eta \in X_q(A_n)$ is of degree β(resp. γ). We denote $K_iK_{i+1}^{-1}\xi = t_{i\beta}\xi$. Then we have the following formula.

Proposition 3. 1 We have $ad_{E_i}(\xi\eta) = ad_{E_i}(\xi)\eta + t_{i\beta}\xi ad_{E_i}(\eta)$, where

$$t_{i,\beta}=\begin{cases}1, & \text{if } \alpha_{i-1},\alpha_i,\alpha_{i+1}\in\beta, \text{or } \alpha_{i-1},\alpha_i,\alpha_{i+1}\notin\beta;\\ q_{i+1}, & \text{if } \alpha_{i+1}\in\beta, \text{and } \alpha_{i-1},\alpha_i\notin\beta;\\ q_i^{-1}q_{i+1}^{-1}q_{i+1}=q_i^{-1}, & \text{if } \alpha_i,\alpha_{i+1}\in\beta, \text{and } \alpha_{i-1}\notin\beta;\\ q_i^{-1}q_{i+1}^{-1}, & \text{if } \beta=\alpha_i;\\ q_i, & \text{if } \alpha_{i-1}\in\beta, \text{and } \alpha_i\alpha_{i+1}\notin\beta;\\ q_{i+1}^{-1}, & \text{if } \alpha_{i-1},\alpha_i\in\beta, \text{and } \alpha_{i+1}\notin\beta.\end{cases}$$

Proof. Note that $t_{i(\beta+\gamma)}=t_{i\beta}t_{i\gamma}$, we have $\mathrm{ad}_{E_i}(\xi\eta)=E_i\xi\eta+t_{i(\beta+\gamma)}(\xi\eta)E_i$, $\mathrm{ad}_{E_i}(\xi)\eta=E_i\xi\eta-t_{i(\beta)}\xi E_i\eta$, $\xi ad_{E_i}(\eta)=\xi E_i\eta-t_{i(\gamma)}\xi ad_{E_i}(\eta)$, $ad_{E_i}(\xi\eta)=ad_{E_i}(\xi)\eta+t_{i\beta}\xi ad_{E_i}(\eta)$. The proof is finished.

Next we investigate commutation relations of $X_q(A_n)$. We consider $X_q(A_n)$ in the case of $q_i=q\ (0\le i\le n)$, $q_{n+1}=-q^{-1}$ in the sequel. We have (R6') $E_n^2=F_n^2=0$,

(R7') $E_i^2E_{i\pm1}-(q^{-1}+q)E_iE_{i\pm1}E_i+E_{i\pm1}E_i^2=0\,(i\ne n)$, (R8') $F_i^2F_{i\pm1}-(q^{-1}+q)F_iF_{i\pm1}F_i+F_{i\pm1}F_i^2=0\,(i\ne n)$.

Now, we describe the commutation relations between E_k and $E_{\varepsilon_i-\varepsilon_j}$.

Lemma 3. 2. For all $0\le i,j\le n+1$,

(1) If $k=i-1$, then $E_kE_{i,j+1}-q_iE_{i,j+1}E_k=E_{i-1,j+1}$.

(2) If $k=j+1\ne n$, then $E_{i,j+1}E_{j+1}-q_{j+1}E_{j+1}E_{i,j+1}=E_{i,j+2}$.

(3) If $k\le i-2$ or $k\ge j+2$, then $E_kE_{i,j+1}=E_{i,j+1}E_k$.

(4) If $k=i$, then $E_kE_{i,j+1}=q_i^{-1}E_{i,j+1}E_k$.

(5) If $i+1\le k\le j-1$, then $E_kE_{i,j+1}=E_{i,j+1}E_k$.

(6) If $k=j$, then $E_jE_{j-1,j+1}=q_{j+1}E_{j-1,j+1}E_j$ (if $E_j^2\ne0$), $E_jE_{j-1,j+1}=q_j^{-1}E_{j-1,j+1}E_j$ (if $E_j^2=0$).

Proof. (1) If $k=i-1$, by definition we have $ad_{E_k}(E_{i,j+1})=E_kE_{i,j+1}-q_iE_{i,j+1}E_k=E_{\varepsilon_{i-1}-\varepsilon_{j+1}}=E_{i-1,j+1}$.

(2) If $k=j+1\ne n$, then we have $E_{j+1}E_{i,j+1}=ad_{E_i}ad_{E_i+1}\dots ad_{E_{j-1}}(E_j)=ad_{E_i}ad_{E_i+1}\dots ad_{E_{j-1}}(E_{j+1}E_j)$,

$E_{i,j+1}E_{j+1}=ad_{E_i}ad_{E_i+1}\dots ad_{E_{j-1}}(E_jE_{j+1})$, and $ad_{E_j}(E_{j+1})=-q_{j+1}E_{j+1}E_j+E_jE_{j+1}$.

So $E_{i,j+1}E_{j+1}-q_{j+1}E_{j+1}E_{i,j+1}=E_{i,j+2}$.

(3) $k\le i-2$ or $k\ge j+2$, it is easy to see that $E_kE_{i,j+1}=E_{i,j+1}E_k$.

(4) $k=i$, if $E_i^2\ne0$, as the proof in [7], we have

$$ad_{E_i}(E_{i,j+1})=ad_{E_i}ad_{E_i}ad_{E_{i+1}}(E_{i+2,j+1})=(q_i^{-1}+q_{i+1})ad_{E_i}ad_{E_{i+1}}ad_{E_i}-q_i^{-1}q_{i+1}ad_{E_i}^2(E_{i+2,j+1})=0.$$

If $E_i^2=0, q_{i+1}=-q_i^{-1}$,

$$ad_{E_i}(E_{i,j+1})=E_iE_iE_{i+1,j+1}-q_{i+1}E_iE_{i+1,j+1}E_i+(K_iK_{i+1}^{-1}E_iE_{i+1,j+1}S(E_i)-q_{i+1}K_iK_{i+1}^{-1}E_{i+1,j+1}E_iS(E_i))$$

$$=-q_{i+1}E_iE_{i+1,j+1}E_i-K_iK_{i+1}^{-1}E_iE_{i+1,j+1}K_{i+1}K_i^{-1}=(-q_{i+1}-q_i^{-1})E_iE_{i+1,j+1}E_i=0.$$

Also $ad_{E_i}(E_{i,j+1})=E_kE_{i,j+1}-q_i^{-1}E_{i,j+1}E_k$, So $E_kE_{i,j+1}=q_i^{-1}E_{i,j+1}E_k$.

(5) If $i+1 \le k \le j-1$, , it is similar to the proof in [7]. We have $ad_{E_k}(E_{i,j+1})E_k E_{i,j+1} - E_{i,j+1}E_k ad_{E_i} \dots ad_{E_{k-2}} ad_{E_k} ad_{E_{k-1}} ad_{E_k}(E_{k+1,j+1}) = 0.$

(6) If $k = j$, then $E_j E_{i,j+1} = ad_{E_i} ad_{E_{i+1}} \dots ad_{E_{j-2}}(E_j E_{j-1,j+1})$,

$E_{i,j+1}E_j = ad_{E_i} ad_{E_{i+1}} \dots ad_{E_{j-2}}(E_{j-1,j+1}E_j)$, $E_{j-1,j+1} = ad_{j-1}(E_j) = -q_j E_j E_{j-1} + E_{j-1}E_j$,

$E_j E_{j-1,j+1} = -q_j E_j^2 E_{j-1} + E_j E_{j-1} E_j$, $E_{j-1,j+1}E_j = -q_j E_j E_{j-1} E_j + E_{j-1}E_j^2$. So we have

$E_j E_{j-1,j+1} - q_{j+1} E_{j-1,j+1} E_j = 0 (E_j^2 \ne 0)$, $q_j E_j E_{j-1,j+1} - E_{j-1,j+1}E_j = 0 (E_j^2 = 0)$.

The proof is finished.

Remark 3. 3. It is noted that Lemma 3.2 (1)(2)(3)(4)(6) also hold when $X_q(A_n)$ is in the case of $q_i \ne q_{i+1} (i \ne n)$.

Let $\alpha = \alpha_i + \ldots + \alpha_p, \beta = \alpha_j + \ldots + \alpha_k$. Next we consider the relations between E_α and E_β.

Lemma 3. 4. (1) If $j \ge p+2$, then $E_\alpha E_\beta = E_\beta E_\alpha$.

(2) If $j = p+1$, then $E_\alpha E_\beta - q_{p+1} E_\beta E_\alpha = E_{\alpha+\beta}$.

(3) If $j \le p$, then $E_\alpha E_\beta - E_\beta E_\alpha = (q_j^{-1} - q_j) E_{\alpha+\beta-\gamma} E_\gamma$.

Proof. (1) $j \ge p+2$, it is easy to see $E_\alpha E_\beta = E_\beta E_\alpha$.

(2) If $j = p+1$, we have $E_\alpha = ad_{E_i} ad_{E_{i+1}} \ldots ad_{E_{p-2}}(-q_p E_p E_{p-1} + E_{p-1}E_p) = E_{i,p}E_p - q_p E_p E_{i,p}$.

$$E_\alpha E_\beta = E_{i,p}E_p E_\beta - q_p E_p E_{i,p} E_\beta = E_{i,p} E_p E_\beta - q_p E_p E_\beta E_{i,p},$$

$$E_\beta E_\alpha = E_\beta E_{i,p} E_p - q_p E_\beta E_p E_{i,p} = E_{i,p} E_\beta E_p - q_p E_\beta E_p E_{i,p},$$

Also $E_p E_\beta - q_{p+1} E_\beta E_p = E_{p+\beta}$. Then $E_\alpha E_\beta - q_{p+1} E_\beta E_\alpha = E_{\alpha+\beta}$.

(3) If $j \le p$, we assume that $\gamma = \alpha_j + \alpha_{j+1} + \ldots + \alpha_p, \alpha = \alpha_i + \ldots + \alpha_{j-1} + \gamma, \beta = \gamma + \alpha_{p+1} + \ldots + \alpha_k$.

So we have $E_\alpha E_\beta = ad_{E_i} \ldots ad_{E_{j-2}}(E_{\alpha_{j-1}+\gamma} E_\beta)$, $E_\beta E_\alpha = ad_{E_i} \ldots ad_{E_{j-2}}(E_\beta E_{\alpha_{j-1}+\gamma})$

$$E_{\alpha_{j-1}+\gamma} E_\beta = ad_{E_{j-1}}(E_\gamma E_\beta) - q_j E_\gamma E_{\alpha_{j-1}+\beta}, E_\beta E_{\alpha_{j-1}+\gamma} = q_j^{-1}\left(ad_{E_{j-1}}(E_\beta E_\gamma) - E_{\alpha_{j-1}+\beta} E_\gamma\right).$$

We also know that $E_j, E_{j+1}, \ldots, E_p$ are commutative with $E_{\alpha_{j-1}+\beta}$, then $E_{\alpha_{j-1}+\beta} E_\gamma = E_\gamma E_{\alpha_{j-1}+\beta}$. And set $\gamma = \alpha_j + \gamma'$, also we have $E_\beta E_{\gamma'} = E_{\gamma'} E_\beta$. Then $E_\gamma E_\beta = ad_{E_j}(E_{\gamma'} E_\beta)$,

$$E_\beta E_\gamma = E_\beta ad_{E_j}(E_{\gamma'}) = q_j ad_{E_j}(E_\beta E_{\gamma'}), E_\gamma E_\beta = q_j^{-1} E_\beta E_\gamma.$$

Then we get $E_{\alpha_{j-1}+\gamma} E_\beta - E_\beta E_{\alpha_{j-1}+\gamma} = -q_j E_\gamma E_{\alpha_{j-1}+\gamma} + q_j^{-1} E_{\alpha_{j-1}+\gamma} E_\gamma = (q_j^{-1} - q_j) E_{\alpha_{j-1}+\gamma} E_\gamma$.

Hence, $E_\alpha E_\beta - E_\beta E_\alpha = ad_{E_i} \ldots ad_{E_{j-2}}\left((q_j^{-1} - q_j) E_{\alpha_{j-1}+\gamma} E_\gamma\right) = (q_j^{-1} - q_j) E_{\alpha+\beta-\gamma} E_\gamma$.

The proof is finished.

Lemma 3. 5. $\Delta(E_{i,j+1}) = K_i K_{j+1}^{-1} \otimes E_{i,j+1} + E_{i,j+1} \otimes 1 + \sum_{k=i}^{j-1}(1-q_{k+1}^2)E_{i,k+1}K_{k+1}K_{j+1}^{-1} \otimes E_{k+1,j+1}.$

Proof. We use induction on $E_{i,j+1}$ to prove the formula. Assume that

$$\Delta(E_{i+1,j+1}) = K_{i+1}K_{j+1}^{-1} \otimes E_{i+1,j+1} + E_{i+1,j+1} \otimes 1 + \sum_{k=i+1}^{j-1}(1-q_{k+1}^2)E_{i+1,k+1}K_{k+1}K_{j+1}^{-1} \otimes E_{k+1,j+1}$$

holds, then

$$\Delta(E_{i,j+1}) = \Delta(E_i E_{i+1,j+1} - q_{i+1}E_{i+1,j+1}E_i) = (K_i K_{i+1}^{-1} \otimes E_i + E_i \otimes 1)\Delta(E_{i+1,j+1}) - q_{i+1}\Delta(E_{i+1,j+1})(K_i K_{i+1}^{-1} \otimes E_i + E_i \otimes 1)$$

$$= K_i K_{i+1}^{-1} \otimes E_i E_{i+1,j+1} + K_i K_{i+1}^{-1}E_{i+1,j+1} \otimes E_i + \sum_{k=i+1}^{j-1}(1-q_{k+1}^2)K_i K_{i+1}^{-1}E_{i+1,k+1}K_{k+1}K_{j+1}^{-1} \otimes E_i E_{k+1,j+1}$$

$$+E_i K_{i+1}K_{j+1}^{-1} \otimes E_{i+1,j+1} + E_i E_{i+1,j+1} \otimes 1 + \sum_{k=i+1}^{j-1}(1-q_{k+1}^2)E_i E_{i+1,k+1}K_{k+1}K_{j+1}^{-1} \otimes E_{k+1,j+1}$$

$$-q_{i+1}(K_i K_{j+1}^{-1} \otimes E_{i+1,j+1}E_i + K_{i+1}K_{j+1}^{-1}E_i \otimes E_{i+1,j+1} + E_{i+1,j+1}K_i K_{j+1}^{-1} \otimes E_i + E_{i+1,j+1}E_i \otimes 1$$

$$+\sum_{k=i+1}^{j-1}(1-q_{k+1}^2)E_{i+1,k+1}K_{k+1}K_{j+1}^{-1}K_i K_{i+1}^{-1} \otimes E_{k+1,j+1}E_i + \sum_{k=i+1}^{j-1}(1-q_{k+1}^2)E_{i+1,k+1}K_{k+1}K_{j+1}^{-1}E_i \otimes E_{k+1,j+1}$$

$$= K_i K_{j+1}^{-1} \otimes E_{i,j+1} + E_{i,j+1} \otimes 1 + E_i K_{i+1}K_{j+1}^{-1} \otimes E_{i+1,j+1} - q_{i+1}^2 E_i K_{i+1}K_{j+1}^{-1} \otimes E_{i+1,j+1} + \sum_{k=i+1}^{j-1}(1-q_{k+1}^2)E_{i,k+1}K_{k+1}K_{j+1}^{-1} \otimes E_{k+1,j+1}$$

$$= K_i K_{j+1}^{-1} \otimes E_{i,j+1} + E_{i,j+1} \otimes 1 + \sum_{k=i}^{j-1}(1-q_{k+1}^2)E_{i,k+1}K_{k+1}K_{j+1}^{-1} \otimes E_{k+1,j+1}.$$

The proof is finished.

4. PBW Basis of Nonstandard Quantum Group $X_q(A_n)$

To describe the PBW basis of $X_q(A_n)$, we need to prove the following equality

Lemma 4.1. $q_i E_i^2 E_{i+1,n+1} - (1+q_i q_{i+1})E_i E_{i+1,n+1}E_i + q_{i+1}E_{i+1,n+1}E_i^2 = 0 \ (0 \le i \le n-1).$

Proof. $q_i E_i^2 E_{i+1,n+1} - (1+q_i q_{i+1})E_i E_{i+1,n+1}E_i + q_{i+1}E_{i+1,n+1}E_i^2$

$$= q_i E_i^2(E_{i+1}E_{i+2,n+1} - q_{i+2}E_{i+2,n+1}E_{i+1}) - (1+q_i q_{i+1})E_i(E_{i+1}E_{i+2,n+1} - q_{i+2}E_{i+2,n+1}E_{i+1})E_i + q_{i+1}(E_{i+1}E_{i+2,n+1} - q_{i+2}E_{i+2,n+1}E_{i+1})E_i^2$$

$$= -q_{i+2}E_{i+2,n+1}(q_i E_i^2 E_{i+1} - (1+q_{i+1}q_i)E_i E_{i+1}E_i + q_{i+1}E_{i+1}E_i^2) + (q_i E_i^2 E_{i+1} - (1+q_{i+1}q_i)E_i E_{i+1}E_i + q_{i+1}E_{i+1}E_i^2)E_{i+2,n+1}$$

$$= 0.$$

The proof is finished.

Lemma 4.2. Keeping notations as above, we have $E_{i,n+1}^2 = 0 \ (1 \le i \le n),$ where $E_{i,n+1}$ is a root vector which contain α_n.

Proof. We verify the equality by induction on the root.

$$E_{\alpha_{n-1}+\alpha_n}^2 = E_{n-1,n+1}^2 = (E_{n-1}E_n - q_n E_n E_{n-1}) = q_n^2(E_n E_{n-1})^2 + (E_{n-1}E_n)^2 - q_n E_n E_{n-1^2} E_n$$

$$= q_n^2(E_n E_{n-1})^2 + (E_{n-1}E_n)^2 - q_n E_n((q_{n-1}^{-1} + q_n)E_{n-1}E_n E_{n-1} - q_{n-1}^{-1}q_n E_n E_{n-1}^2)$$

$$= q_n^2(E_n E_{n-1})^2 + (E_{n-1}E_n)^2 - q_n(q_{n-1}^{-1} + q_n)(E_n E_{n-1})^2 = (E_{n-1}E_n)^2 - q_n q_{n-1}^{-1}(E_n E_{n-1})^2.$$

Also $(q_{n-1}E_{n-1}^2 E_n - (1+q_{n-1}q_n)E_{n-1}E_n E_{n-1} + q_n E_n E_{n-1}^2)E_n = -(1+q_{n-1}q_n)(E_{n-1}E_n)^2 + q_n E_n E_{n-1}^2 E_n = 0$

and $E_n(q_{n-1}E_{n-1}^2 E_n - (1+q_{n-1}q_n)E_{n-1}E_n E_{n-1} + q_n E_n E_{n-1}^2) = q_{n-1}E_n E_{n-1}^2 E_n - (1+q_{n-1}q_n)(E_n E_{n-1})^2 = 0.$

Since $q_n = q_{n-1}$, then we have $(E_{n-1}E_n)^2 = (E_n E_{n-1})^2$. So $E_{\alpha_{n-1}+\alpha_n}^2 = 0.$

Next we assume that if $i=j+1,\ E^2_{j+1,n+1}=E^2_{\alpha_{j+1}+\alpha j+2+\ldots+\alpha_n}=0$ set up. By Lemma 4.1, then

$$E^2_{j,n+1}=(adE_j(E_{\alpha_{j+1}+\ldots+\alpha_n}))^2=(E_jE_{\alpha_{j+1}+\ldots+\alpha_n}-q_{j+1}E_{\alpha_{j+1}+\ldots+\alpha_n}E_j)^2$$
$$=(E_jE_{\alpha_{j+1}+\ldots+\alpha_n})^2+q^2_{j+1}(E_{\alpha_{j+1}+\ldots+\alpha_n}E_j)^2-q_{j+1}E_{\alpha_{j+1}+\ldots+\alpha_n}E^2_jE_{\alpha_{j+1}+\ldots+\alpha_n}$$
$$=(E_jE_{\alpha_{j+1}+\ldots+\alpha_n})^2+q^2_{j+1}(E_{\alpha_{j+1}+\ldots+\alpha_n}E_j)^2-(q_{j+1}q_j^{-1}+q^2_{j+1})(E_{\alpha_{j+1}+\ldots+\alpha_n}E_j)^2,$$

Also

$$E_{\alpha_{j+1}+\ldots+\alpha_n}(q_jE^2_jE_{j+1,n+1}-(1+q_jq_{j+1})E_jE_{j+1,n+1}E_j+q_{j+1}E_{j+1,n+1}E^2_j)$$
$$=(q_jE^2_jE_{j+1,n+1}-(1+q_jq_{j+1})E_jE_{j+1,n+1}E_i+q_{j+1}E_{j+1,n+1}E^2_j)E_{\alpha_{j+1}+\ldots+\alpha_n}=0$$

Then we get $(E_{\alpha_{j+1}+\ldots+\alpha_n}E_j)^2=(E_jE_{\alpha_{j+1}+\ldots+\alpha_n})^2.$ Since $q_j=q_{j+1}=q(j\le n-1),$ we get

$$E^2_{j,n+1}=\left((1+q^2_{j+1})-(q_{j+1}q_j^{-1}+q^2_{j+1})\right)(E_jE_{\alpha_{j+1}+\ldots+\alpha_n})^2=0.$$

By induction, we have $E^2_{i,n+1}=0(1\le i\le n).$ The proof is finished.

Now we prove that $X_q(A_n)$ has a triangular decomposition. Let $\tilde{X}_q(A_n)$ be an algebra generated by $K_1^{\pm1},K_2^{\pm1},\ldots,K_n^{\pm1},K_{n+1}^{\pm1},E_1,\ldots,E_n,F_1,\ldots,F_n$ which satisfies $(\mathrm{R1})$-$(\mathrm{R5})$. The left hand side of quantum Serre relation $(\mathrm{R7'})$ (resp. $(\mathrm{R8'})$) is denoted by $u_s^+(\text{resp. } u_s^-)$. Let $L=(\beta_1,\beta_2,\ldots,\beta_r)(\beta_i=\alpha_j,1\le j\le n)$ be a finite sequence of simple roots and $E_L=E_{\beta_1}E_{\beta_2}\ldots E_{\beta_r}$. Similar to the proof of [8, 4.12-4.19], the elements $F_IK_1^{l_1}K_2^{l_2}\ldots K_{n+1}^{l_{n+1}}E_J(l_i\in\mathbb{Z})$ form a basis of $\tilde{X}_q(A_n)$, where I,J are finite sequences of simple roots.

Lemma 4. 3. Let m be the multiplication map of $\tilde{X}_q(A_n)$, then the image of $m(I\otimes m)$ of $\tilde{X}_q^-(A_n)\otimes\tilde{X}_q^0(A_n)\otimes I^+$, where I^+ is the two-sided ideal of $\tilde{X}_q^+(A_n)$ generated by u_s^+ and E_n^2, is just a two-sided ideal of $\tilde{X}_q(A_n)$.

Proof. Let $V=m(I\otimes m)\left(\tilde{X}_q^-(A_n)\otimes\tilde{X}_q^0(A_n)\otimes I^+\right)$, then it is spanned by $uu_s^+E_n^2E_L$, with all sequences L, where $u\in\tilde{X}_q(A_n)$. It is obvious that V is a left ideal in $\tilde{X}_q(A_n)$. We have to show that V is a right ideal in $\tilde{X}_q(A_n)$. Let Φ be the root system corresponding to the same type Lie algebra and π be a basis of Φ. Since $E_n^2F_i=F_iE_n^2,E_n^2F_i=F_iE_n^2$, it implies that for all $\gamma\in\pi$, we have $E_n^2F_\gamma=F_\gamma E_n^2,[E_n^2,F_\gamma]=0.$

$$u_s^+F_i=E_i^2F_iE_{i+1}-(q^{-1}+q)E_iE_{i+1}(F_iE_i+\frac{K_{i+1}K_i^{-1}-K_iK_{i+1}^{-1}}{q-q^{-1}})+E_{i+1}E_i(F_iE_i+\frac{K_{i+1}K_i^{-1}-K_iK_{i+1}^{-1}}{q-q^{-1}})=F_iu_s^+.$$

Similarly, we have $u_s^+F_{i+1}=F_{i+1}u_s^+,u_s^+F_{i-1}=F_{i-1}u_s^+$. It implies that $u_s^+F_\gamma=F_\gamma u_s^+,[u_s^+,F_\gamma]=0.$

Therefore, we have $[u_s^+E_n^2,F_\gamma]=u_s^+[E_n^2,F_\gamma]+[u_s^+,F_\gamma]E_n^2=0.\ \ u[u_s^+E_n^2E_L,F_\gamma]=uu_s^+E_n^2[E_L,F_\gamma].$

Note that $[E_L,F_\gamma]$ can be written as the linear combinations of the terms $F_{I'}K_1^{l_1}K_2^{l_2}\ldots K_{n+1}^{l_{n+1}}E_{J'}(l_i\in\mathbb{Z})$, we have $uu_s^+E_n^2E_LF_\gamma=uF_\gamma u_s^+E_n^2E_L+uu_s^+E_n^2[E_L,F_\gamma]\in V$. The proof is completed.

In a similar way, we get

Lemma 4. 4. The image of $I^{-}\otimes\tilde{X}_q^0(A_n)\otimes\tilde{X}_q^{+}(A_n)$ under the map $m(m\otimes I)$ is just the ideal of $\tilde{X}_q(A_n)$ generated by u_s^{-} and F_n^2.

Proposition 4. 5. There exist an isomorphism of $\mathbb{C}$ -linear spaces $X_q(A_n)\cong X_q^{-}(A_n)\otimes X_q^0(A_n)\otimes X_q^{+}(A_n)$.

Proof. The proof is similar to [8, Theorem 4.21].

Let $\gamma_1<\gamma_2<\ldots<\gamma_N$, where N the number of positive roots and γ_i runs over all positive roots.

Lemma 4. 6. $\mathbf{B}^{+}=\{E_{\gamma_1}^{n_1}E_{\gamma_2}^{n_2}\ldots E_{\gamma_N}^{n_N}\mid n_i\in\mathbb{Z}_2 \text{ if } E_{\gamma_i}=E_{j,k}(k=n+1), n_i\in\mathbb{Z}_{\geq 0} \text{ if } E_{\gamma_i}=E_{j,k}(k\leq n)\}$ form a basis of $X_q^{+}(A_n)$.

Proof. We show that any element of $X_q^{+}(A_n)$ is a linear combinations of $E_{\gamma_1}^{n_1}E_{\gamma_2}^{n_2}\ldots E_{\gamma_N}^{n_N}$ and $E_{\gamma_1}^{n_1}E_{\gamma_2}^{n_2}\ldots E_{\gamma_N}^{n_N}$ are linearly independent with all $E_{\gamma_1}^{n_1}E_{\gamma_2}^{n_2}\ldots E_{\gamma_N}^{n_N}\neq 0$.

(1) It is noted that $E_1,\ldots,E_N$ generate $X_q^{+}(A_n)$. It is suffices to show that $E_{\gamma_{i_1}}E_{\gamma_{i_2}}\ldots E_{\gamma_{i_k}}$ is a linear combination of $E_{\gamma_1}^{n_1}E_{\gamma_2}^{n_2}\ldots E_{\gamma_N}^{n_N}$ with $n_1+n_2+\ldots+n_N\leq k$. Using induction on k, given an integer i, if $k=1$, then the case is clear. Suppose the assertion holds for k. If $i_1=1$, by the induction on k to the element $E_{\gamma_{i_2}}\ldots E_{\gamma_{i_{k+1}}}$, then we get the result. If $i_1>1$, by the induction it is easy to see that $E_{\gamma_{i_1}}E_{\gamma_{i_2}}\ldots E_{\gamma_{i_{k+1}}}$ is a linear combination of $E_{\gamma_{i_1}}E_{\gamma_j}^{n_j}\ldots E_{\gamma_N}^{n_N}$ with $n_j+n_2+\ldots+n_N\leq k$.

a) If $i_1\leq j$, then it is straightforward to see.

b) If $i_1>j$, $E_{\gamma_{i_1}}E_{\gamma_j}^{n_j}\ldots E_{\gamma_N}^{n_N}=E_{\gamma_{i_1}}E_{\gamma_j}E_{\gamma_j}^{n_j-1}\ldots E_{\gamma_N}^{n_N}$, by lemma 3.4, for some non-zero coefficients λ,μ,

$$E_{\gamma_{i_1}}E_{\gamma_j}-\lambda E_{\gamma_j}E_{\gamma_{i_1}}=\begin{cases}0,\\ \mu E_{\gamma_{i_1}+\gamma_j},\\ \mu E_{\gamma_{i_1}+\gamma_j-\gamma'}E_{\gamma'}, \quad \text{where } \gamma_{i_1}=\alpha'+\gamma', \gamma_j=\gamma'+\beta',\end{cases}$$

Then

$$E_{\gamma_{i_1}}E_{\gamma_j}E_{\gamma_j}^{n_j-1}E_{\gamma_{j+1}}^{n_{j+1}}\ldots E_{\gamma_N}^{n_N}=\begin{cases}\lambda E_{\gamma_j}E_{\gamma_{i_1}}E_{\gamma_j}^{n_j-1}E_{\gamma_{j+1}}^{n_{j+1}}\ldots E_{\gamma_N}^{n_N},\\ \lambda E_{\gamma_j}E_{\gamma_{i_1}}E_{\gamma_j}^{n_j-1}E_{\gamma_{j+1}}^{n_{j+1}}\ldots E_{\gamma_N}^{n_N}+\mu E_{\gamma_{i_1}+\gamma_j}\lambda E_{\gamma_j}E_{\gamma_{i_1}}E_{\gamma_j}^{n_j-1}E_{\gamma_{j+1}}^{n_{j+1}}\ldots E_{\gamma_N}^{n_N},\\ \lambda E_{\gamma_j}E_{\gamma_{i_1}}E_{\gamma_j}^{n_j-1}E_{\gamma_{j+1}}^{n_{j+1}}\ldots E_{\gamma_N}^{n_N}+\mu E_{\gamma_{i_1}+\gamma_j-\gamma'}E_{\gamma'}\lambda E_{\gamma_j}E_{\gamma_{i_1}}E_{\gamma_j}^{n_j-1}E_{\gamma_{j+1}}^{n_{j+1}}\ldots E_{\gamma_N}^{n_N}.\end{cases}$$

Similar to the proof in [9, Proposition 5.3] and [7], we get that $E_{\gamma_{i_1}}E_{\gamma_{i_2}}\ldots E_{\gamma_{i_k}}$ is a linear combination of $E_{\gamma_1}^{n_1}E_{\gamma_2}^{n_2}\ldots E_{\gamma_N}^{n_N}$ with $n_1+n_2+\ldots+n_N\leq k$.

(2) We now show that all $E_{\gamma_1}^{n_1}E_{\gamma_2}^{n_2}\ldots E_{\gamma_N}^{n_N}\neq 0$ and $E_{\gamma_1}^{n_1}E_{\gamma_2}^{n_2}\ldots E_{\gamma_N}^{n_N}$ are linearly independent by comultiplication Δ. Firstly, one can prove that

$$\left[E_{i,j+1},F_{i,j+1}\right]=\frac{(-q)^{j-i}(K_i^{-1}K_{j+1}-K_iK_{j+1}^{-1})}{q-q^{-1}}$$

by induction on $j-i$, we get that $E_{i,n+1}{}^2=0\,(i\le n)$ and $E^l_{i,j+1}\ne 0\,(1\le i\le j<n)$ for all l.

The root lattice Q-gradation of $X_q^+(A_n)$ implies that E_{γ_j} 'are independent. Note that

$\Delta: X_q^+(A_n)\to X_q^{\ge 0}(A_n)\otimes X_q^+(A_n)$ is a graded algebra homomorphism and $\Delta(E_\beta)$ has a component of bidegree $(\alpha_i,\beta-\alpha_i)$ if and only if $\beta=\alpha_i+(\beta-\alpha_i)$. It follows that for the positive root $\beta=\varepsilon_i-\varepsilon_j\ (i<j<n+1)$ the component of bidegree $(l\alpha_i, l(\beta-\alpha_i))$ of $\Delta(E_\beta)^l$ is proportional to $E_i^l K^l_{\beta-\alpha_i}\otimes E^l_{\beta-\alpha_i}$ for any l, and for the positive root $\beta=\varepsilon_i-\varepsilon_{n+1}\ (i<n+1)$ the component of bidegree $(\alpha_i,\beta-\alpha_i)$ of $\Delta(E_\beta)$ is proportional to $E_i\,K_{\beta-\alpha_i}\otimes E_{\beta-\alpha_i}$. The rest is similar to the proof in [7] and [9, Proposition 5.3]. By Lemma 3.5, we get that $E^{n_1}_{\gamma_1}E^{n_2}_{\gamma_2}\ldots E^{n_N}_{\gamma_N}$ are linearly independent. The proof is finished.

Similarly, we have

Lemma 4. 7. $\mathbf{B}^-=\{F^{m_1}_{\gamma_1}F^{m_2}_{\gamma_2}\ldots F^{m_N}_{\gamma_N}\mid m_i\in\mathbb{Z}_2 \text{ if } F_{\gamma_i}=F_{j,k}(k=n+1), m_i\in\mathbb{Z}_{\ge 0} \text{ if } F_{\gamma_i}=F_{j,k}(k\le n)\}$ form a basis of $X_q^-(A_n)$.

Let $E^s=E^{n_1}_{\gamma_1}E^{n_2}_{\gamma_2}\ldots E^{n_N}_{\gamma_N}, F^t=F^{m_1}_{\gamma_1}F^{m_2}_{\gamma_2}\ldots F^{m_N}_{\gamma_N}$, where $s=(n_1,n_2,\ldots,n_N), t=(m_1,m_2,\ldots,m_N)$. If $E_{\gamma_i}=E_{j,n+1}$ (resp. $F_{\gamma_i}=F_{j,n+1}$), then $n_i\in\mathbb{Z}_2$ (resp. $m_i\in\mathbb{Z}_2$); if $E_{\gamma_i}=E_{j,k}(k\le n)$ (resp. $F_{\gamma_i}=F_{j,k}(k\le n)$), then $n_i\in\mathbb{Z}_{\ge 0}$ (resp. $m_i\in\mathbb{Z}_{\ge 0}$). The elements in $\mathbf{B}^0=\{K_1^{l_1}K_2^{l_2}\ldots K_{n+1}^{l_{n+1}}\mid l_i\in\mathbb{Z}\}$ is denoted by K^l, we have

Theorem 4. 8. $\mathbf{B}=\{F^tK^lE^s\mid E^s\in\mathbf{B}^+, F^t\in\mathbf{B}^-, K^l\in\mathbf{B}^0\}$ form a basis of $X_q(A_n)$.

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