L_0 Stable Trigonometrically Fitted Block Backward Differentiation Formula of Adams Type for Autonomous Oscillatory Problems

Solomon A. Okunuga* and R. I. Abdulganiy
Department of Mathematics, University of Lagos, Lagos, Nigeria.

* Corresponding author. Tel.: +2348023244422; email: waleokunuga@gmail.com
Manuscript submitted April 6, 2016; accepted November 30, 2016.
doi: 10.17706/ijapm.2017.7.2.128-133

Abstract: In this paper, a L_0 Stable Second Derivative Trigonometrically Fitted Block Backward Differentiation Formula of Adams Type (SDTFF) of algebraic order 4 is presented for the solution of autonomous oscillatory problems. A Continuous Second Derivative Trigonometrically Fitted (CSDTF) whose coefficients depend on the frequency and step size is constructed using trigonometric basis function. The CSDTF is used to generate the main method and one additional method which are combined and applied in block form as simultaneous numerical integrators. The stability properties of the method are investigated using boundary locus plot. It is found that the method is zero stable, consistent and hence converges. The method is applied on some numerical examples and the result show that the method is accurate and efficient.

Key words: Autonomous oscillatory problems, backward differentiation formula, continuous scheme, trigonometrically fitted methods.

1. Introduction

An important and interesting class of initial value problems which arise in practice include the differential equations whose solutions are known to oscillate with a fitting frequency. Such problems arise frequently in area such as Biological Science, Economics, Chemical Kinetics, Theoretical Chemistry, Medical Science to mention but a few.

The form and structure of the oscillating problems is highly application dependent [1]. They also noted that the best numerical method to use is strongly dependent on the application. Numerical methods used to treat oscillatory problems differ depending on the formulation of the problem, the knowledge of certain characteristic of the solution and the objective of the computation [1].

A number of numerical methods based on the use of polynomial function have been developed for solving this class of problems by various researchers such as [2]-[5]. Other methods based on exponential fitting techniques which takes advantage of the special properties of the solution that may be known in advance have also been proposed to solve this class of IVP (see [6], [7]).

In order to solve differential equations whose solutions are known to oscillate, methods based on trigonometric polynomials have been proposed (see [8]-[13]). However, little attentions have been paid to the Block Differentiation Formula using the trigonometric polynomial as the basis function for solving IVP whose solution oscillate. Hence the motivation for this paper.

2. Derivation of the Method

Let us consider the system of first order

$$y' = f(x,y),\ y(x_0) = y_0\ ,\ x \in [x_0, b] \tag{1}$$

where f satisfies the Lipschitz theorem

The proposed $k-$step Second Derivative Trigonometrically Fitted Block Backward Differentiation Formula of Adams Type (SDTFF) is of the form

$$y_{n+k} = y_{n+k+1} + h\sum_{j=0}^{k} \beta_j(u)\, f_{n+j} + h^2\gamma_k g_{n+k} \tag{2}$$

where $u = \omega h$, ω is the frequency, $y_{n+k} = y(x_n + kh)$, $y'_{n+k} = f_{n+j}$, $g_{n+k} = \frac{df}{dx}\Big|_{y_{n+j}}^{x_{n+j}}$

$\beta_j, j = 0(1)k$ and γ_k are parameters to be obtained from multistep collocation techniques [2], [4], [10], [14].

In order to obtain "(2)" for $k = 2$, we proceed by seeking to approximate the exact solution $y(x)$ in the interval of integration by the interpolating function

$$I(x) = \sum_{j=0}^{2} a_j x^j + a_3 \sin(\omega x) + a_4 \cos(\omega x) \tag{3}$$

where $a_j, j = 0(1)4$ are coefficients to be determined uniquely. The following conditions are imposed.

$$I(x_{n+j}) = y_{n+j}\quad ,\qquad J = 1 \tag{4}$$

$$I'(x_{n+j}) = f_{n+j}\quad ,\qquad J = 0(1)\,2 \tag{5}$$

$$I''(x_{n+j}) = g_{n+j}\quad ,\qquad J = 2 \tag{6}$$

Equations $(4) - (6)$ lead to system of 5 equations which are solved simultaneously with the aid of Maple 2015.1 package to obtain the coefficients a_j. The continuous form (CSDTF) is obtained by substituting the values of a_j into "(3)". After some correct manipulations, the CSDTF is expressed in the form

$$I(x) = y_{n+k-1} + h\sum_{j=0}^{2} \beta_j(u)\, f_{n+j} + h^2\gamma_2 g_{n+2} \tag{7}$$

On evaluating "(7)" at the points $x = \{x_{n+2}, x_n\}$, we obtain the main method and additional method as follows

$$\begin{aligned} y_{n+2} - y_{n+1} = {} & h\left\{\frac{u^2\cos u - 4u\sin u - 4u\cos u + u^2 + 4}{4u^2\cos u - 2u^2\cos 2u - 4u\sin u + 2u\sin 2u - 2u^2}\right\} f_n \\ & + h\left\{\frac{-u^2\cos 2u + 4u\sin u + 2u\sin 2u + 2\cos u - 3u^2 - 2}{4u^2\cos u - 2u^2\cos 2u - 4u\sin u + 2u\sin 2u - 2u^2}\right\} f_{n+1} + \\ & h\left\{\frac{3u^2\cos u - u^2\cos 2u - 4u\sin u + 4\cos u - 2\cos 2u - 2}{4u^2\cos u - 2u^2\cos 2u - 4u\sin u + 2u\sin 2u - 2u^2}\right\} f_{n+2} + \\ & h^2\left\{\frac{-2u\sin u + u\sin 2u - 8\cos u + 2\cos 2u + 6}{4u^2\cos u - 2u^2\cos 2u - 4u\sin u + 2u\sin 2u - 2u^2}\right\} g_{n+2} \end{aligned} \tag{8}$$

$$y_n - y_{n+1} = h\left\{\frac{-3u^2\cos u + 2u\sin 2u - 4\cos u + 2\cos 2u + u^2 + 2}{4u^2\cos u - 2u^2\cos 2u - 4u\sin u + 2u\sin 2u - 2u^2}\right\} f_n + h\left\{\frac{3u^2\cos u - 6u\sin 2u + 4u\sin u + 2\cos 2u + u^2 + 2}{4u^2\cos u - 2u^2\cos 2u - 4u\sin u + 2u\sin 2u - 2u^2}\right\} f_{n+1} + h\left\{\frac{-u^2\cos u - u^2\cos 2u + 2u\sin 2u + 4\cos u - 4}{4u^2\cos u - 2u^2\cos 2u - 4u\sin u + 2u\sin 2u - 2u^2}\right\} f_{n+2} + h^2\left\{\frac{2u\sin u + u\sin 2u - 8\cos u + 2\cos 2u + 6}{4u^2\cos u - 2u^2\cos 2u - 4u\sin u + 2u\sin 2u - 2u^2}\right\} g_{n+2} \quad (9)$$

2.1. Local Truncation Error

Following [5], the local truncation errors of "(8)" and "(9)" are better obtained using their series expansion. Thus Local Truncation Error (LTE) of "(8)" and "(9)" are respectively as obtained.

$$LTE\ (Main) = \frac{7h^5}{1440}\left(y^{(5)}(x_n) + \omega^2 y^{(3)}(x_n) + O(h^6)\right)$$

$$LTE\ (Additional) = \frac{23h^5}{1440}\left(y^{(5)}(x_n) + \omega^2 y^{(3)}(x_n) + O(h^4)\right)$$

In spirit of [4] and [5], we remark that our method is of order 4 and hence it is consistent.

2.2. Stability

Following [2], [3] and [14], the block method can be rearranged and rewritten as a matrix difference equation of the form

$$A^{(1)}Y_{w+1} = A^{(0)}Y_w + hB^{(1)}F_w + hB^{(0)}F_{w-1} + D^{(1)}G_{w+1} \quad (10)$$

where

$$Y_{w+1} = (y_{n+1}, y_{n+2}, \dots, y_{n+k})^T \qquad Y_w = (y_{n-k+1}, \dots, y_{n-1}, y_n)^T$$

$$F_w = (f_{n+1}, f_{n+2}, \dots, f_{n+k})^T \quad , \quad F_{w-1} = (f_{n-k+1}, \dots, f_{n-1}, f_{n+k})^T$$

$$G_{w+1} = (g_{n+1}, g_{n+2}, \dots, g_{n+k})^T$$

From our method, setting $u = 10$, we have

$$A^{(1)} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \qquad A^{(0)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad D^{(1)} = \begin{bmatrix} 0 & 5.81\times10^{-2} \\ 0 & 5.81\times10^{-2} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} 2.55\times10^{-1} & 9.35\times10^{-2} \\ -5.98\times10^{-1} & -4.80\times10^{-2} \end{bmatrix}, \quad B^{(0)} = \begin{bmatrix} 0 & 6.52\times10^{-1} \\ 0 & 7.83\times10^{-2} \end{bmatrix}$$

2.2.1. Zero stability

In the spirit of [3], the block method "(10)" is zero stable if the roots of the first characteristic polynomial have modulus less than or equal to one and those of modulus one are simple. i.e.

$$\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] = 0 \text{ and } \qquad |R_i| \le 1$$

Our method is zero stable since $-R(R+1) = 0$

$$\Longrightarrow |R| = (0,1)$$

Since our method is of order 4 and also zero stable, then it converges in the spirit of [4] and [5].

2.2.2. Linear stability

The block method "(10)" applied to the test equations $y^{'} = \lambda y$ and $y^{''} = \lambda^2 y$ yields

$$Y_{w+1} = M(z)Y_w$$

where

$$M(z) = \frac{A^{(1)} - ZB^{(1)} - Z^2 D^{(1)}}{A^{(0)} + ZB^{(0)}} \tag{11}$$

$$Z = \lambda h$$

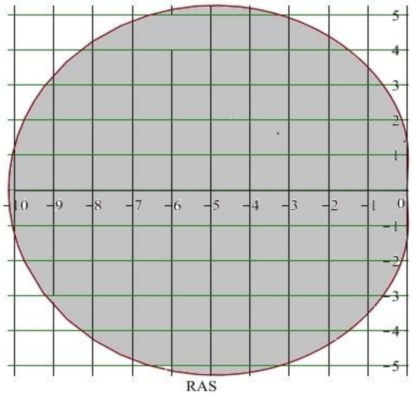

Fig. 1. Region of absolute stability.

The matrix "(11)" has eigenvalues $(\mu_1, \mu_2) = (0, \mu_2)$, where

$$\mu_2 = \frac{-2.10\times10^3 Z - 3.41\times10^4 Z^2 \times 8.33\times10^4}{6.90\times10^4 Z + 5.52\times10^3 Z^2 \times 8.33\times10^4 - 4.13\times10^3 Z^3} \tag{12}$$

Equation (12) is referred to as the stability function. By employing boundary locus technique, the region of absolute stability of our method is as shown in figure 1

It is obvious from the RAS that our method is A_0 stable. Also since $\lim_{z\to\infty} \mu_2 = 0$

This shows that our method is L_0 stable.

3. Numerical Examples

In this section, we provide numerical examples both linear and nonlinear autonomous systems to illustrate the accuracy of our method. All computations are carried out by written codes with the aid of MAPLE 2015.1 software package. An error of the form $P \times 10^{-s}$ is written as $P(-s)$.

Example 1

We consider the following linear homogeneous Autonomous systems by Sanugi and Evans [8]

$$y^{'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y \ , \ \ y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

whose exact solution is

$$y = \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$$

The problem is solve for $h = 0.1$, $\omega = 1$ in the interval $0 \le x \le 1$.

Table 1, show that comparison of error and it is found that our method SDTFF is better than that of Sanugi and Evans.

Table 1. Comparison of Error

x	Sanugi and Evans [8]		SDTFF	
	Err y_1	Err y_2	Err y_1	Err y_2
1.0	9.29(−10)	4.40(−09)	8.19(−24)	5.27(−24)
2.0	2.14(−10)	5.02(−09)	1.77(−23)	8.09(−24)
3.0	2.69(−09)	3.61(−09)	4.14(−24)	2.89(−23)
4.0	2.48(−09)	1.12(−09)	2.95(−23)	2.55(−23)
5.0	8.27(−10)	5.23(−10)	4.67(−23)	1.38(−23)
6.0	2.86(−09)	2.09(−09)	1.64(−23)	5.61(−23)
7.0	2.44(−09)	4.97(−09)	4.47(−23)	5.14(−23)
8.0	1.65(−10)	2.77(−09)	7.71(−23)	1.1.3(−23)
9.0	2.33(−09)	3.09(−09)	3.62(−23)	7.98(−23)
10.0	1.71(−09)	1.95(−09)	5.29(−23)	8.17(−23)

Example 2

We consider the following nonlinear Autonomous system by Neta [9]

$$Y'(t) = F(t, Y), \quad Y(0) = (0, 1, 1, 0)^T$$

where $Y = (y_1, y_2, y_3, y_4)^T$, $F = \left(y_2, -\frac{y_1}{r^3}, y_4, -\frac{y_3}{r^3}\right)$, $r^2 = y_1^2 + y_2^2$

whose exact solution is

$$Y_e = (\sin t\, , \cos t\, , \cos t\, , -\sin t)$$

Table 2 shows the comparison of L_2 Norm of the error at $t = 12\pi$ using $h = \frac{\pi}{60}$ with $\omega = 0.90, 0.95, 1.00, 1.05$ and 1.10. It can be seen that the SDTFF performs better than that of Neta.

Table 2: Comparison of the L_2 norm of error

ω	Neta	SDTFF
0.9	3.23(−02)	8.75(−07)
0.95	1.66(−02)	4.49(−07)
1.00	2.02(−08)	5.00(−19)
1.05	1.74(−02)	4.27(−07)
1.10	3.56(−02)	9.67(−07)

4. Conclusion

In this paper, a Second Derivative Block Backward Differentiation Formula of Adams Type using trigonometric basis for solving autonomous oscillating problems is proposed. The method is zero stable, consistent and produced good results on autonomous oscillating IVPs. The method has advantage of being self-starting and efficient.

Acknowledgment

The authors would like to thank the referees whose useful suggestions greatly improve the quality of this paper.

References

[1] Petzold, L. R., Jeny, L. O., & Yen, J. (1997). Numerical solution of highly oscillatory ordinary differential

equation. *Acta Numerica*, *20*, 437-483.

[2] Akinfenwa, O. A., Jator, S. N., & Yao, N. (2012). On the stability of continuous block backward differentiation formula for solving stiff ordinary differential equation. *Journal of Mod. Meth. In. Numer. Maths*, *3(2)*, 50-58.

[3] Akinfenwa, O. A., & Akinnukawe, B. (2013). Second derivative hybrid block method with two off-step points for solution of initial value problems for solution of initial value problems. *Journal of Nig. Assoc. Math. Phys*., *25(2)*, 387-392.

[4] Fatunla, S. O. (1988). *Numerical Methods for Initial Value Problems in Ordinary Differential Equation*. United Kingdom: Academic Press.

[5] Lambert, J. D. (1973). *Computational Methods in Ordinary Differential Equations*. London: John Wiley.

[6] Psihoyios, G., & Simos, T. E. (2003). Exponentially and trigonometrically fitted explicit advanced step-point (EAS) methods for initial value problems with oscillating solutions. *International Journal of Modern Physics C*, *14(2)*, 175-184.

[7] Simos, T. E. (2003). Exponentially-fitted and trigonometrically-fitted symmetric linear multistep methods for the numerical integration of orbital problems. *Physics Letters A*, *315*, 437-446.

[8] Sanugi, B. B., & Evans, D. J. (1989). The numerical solution of oscillatory problems. *International Journal of Computer Mathematics*, *31*, 237-255.

[9] Neta, B. (1986) Families of backward differentiation methods based on trigonometric polynomials. *International Journal of Computer Mathematics*, *20*, 67-75.

[10] Ngwane, F. F., & Jator, S. N. (2014). Trigonometrically-fitted second derivative method for oscillatory problems. *Springer Plus*, *3(304)*, 1-11.

[11] Zhang, Y., Che, H., Fang, Y., & You, X. A new trigonometrically fitted two-derivative runge-kutta method for the numerical solution of the schrodinger equation and related problems.

[12] Neta, B., & Ford, C. H. (1984). Families of methods for ordinary differential equations based on trigonometric polynomials. *J. Comp. Appl. Math*., *10*, 33-38.

[13] Jator, S. N., Swindell, S., & French, R. (2013). Trigonometrically fitted block numerov type method for $y'' = f(x, y, y')$, *Numer Algor*, *62*, 13-26.

[14] Ehigie, E. O., Okunuga S. A., & Sofoluwe, A. B. (2011). A generalized 2-step continuous hybrid implicit linear multistep methods of IVPs. *Journal of the Nigerian Mathematical Society*, *30*, 145-162.

Solomon A. Okunuga is a professor of mathematics with keen focus on numerical analysis and computational mathematics in University of Lagos, Nigeria. He has supervised successfully three (3) Ph.D. students. He is currently the director of Academic Planning in University of Lagos, Nigeria.

Prof. Okunuga is a member of Nigerian Mathematical Society (NMS) and International Association of Engineers (IAENG). He has published several papers in both national and international journals.

R. I. Abdulganiy is currently a Ph.D. student in the Department of Mathematics University of Lagos, Nigeria under the supervision of Prof. S. A. Okunnuga. He is a member of Nigerian Mathematical Society (NMS).