Research on the Imaginary Relationship and Rational Relationship between π and e

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Abstract: The authors of this paper propose the new formula $\pi=\frac{1}{2}e^{\theta}$ to describe the rational relationship between π and e, where the constant θ is the linear combination of the Euler constant γ and the new constant μ, namely $\theta=1+\gamma+2\mu$. Two advantages can be revealed from the rational relationship. One is that the constants in the new formula can be computed, the other is that the constants such as π and e, μ and γ in the formula can be converted to each other. There exists two methods to compute the constants μ using computers in this paper. The significance of conversion between π and e lies in the fact that the use of $e^{\frac{\theta}{2}}$ to substitute $\sqrt{2\pi}$ can simplify the formulas' computation in probability and statistics.

Key words: The ratio of the circumference of a circle to its diameter, the logarithm of the natural constant, the Euler's constant, the new constant, the new formula.

1. Introduction

The Euler's formula $e^{\pi i}=-1$ reveals the imaginary relationship among π, e and the imaginary unit i. The mathematician Benjamin Peirce comments on the formula: "The Euler formula is definitely correct but also strange which is difficult to comprehend. As it has been proved, we know it's definitely correct." So Euler formula is also called 'God's formula' in some books. The authors of this paper propose a new formula to describe the rational relationship between π and e, as a supplement to the Euler's formula.

2. The Sketch of the New Constants μ、θ and the New Formula $\pi=\frac{1}{2}e^{\theta}$

As the detailed proof of the new constants μ、θ and the new formula $\pi=\frac{1}{2}e^{\theta}$ can be found in the references [1] and [2], we just make a brief description here.

The sum of the harmonic series is [3]:

$$\sum_{k=1}^{n}\frac{1}{k}=\gamma+\ln n+\frac{1}{2n}-\sum_{k=2}^{\infty}\frac{A_k}{n(n+1)\cdots(n+k-1)} \tag{1}$$

where $A_k = \frac{1}{k}\int_0^1 x\left(1-x\right)\left(2-x\right)\cdots\left(k-1-x\right)\mathrm{d}x$. We denote the tail term as:

$$\varepsilon_n = \sum_{k=2}^{\infty}\frac{A_k}{n\left(n+1\right)\cdots\left(n+k-1\right)} \tag{2}$$

The references [1] and [2] prove that the sum of the tail term $\sum_{n=1}^{\infty}\varepsilon_n$ converges to a constant, which is denoted by μ

$$\mu = \sum_{n=1}^{\infty}\varepsilon_n \tag{3}$$

With the use of Abel's formula:

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} S_k \Delta b_k + S_n b_n \tag{4}$$

where we denote $a_k = \frac{1}{k}$, $b_k = k$, hence $S_k = \sum_{m=1}^{k} a_m = \sum_{m=1}^{k}\frac{1}{m}$, $\Delta b_k = b_k - b_{k+1} = -1$. Finally we obtain the formula:

$$n = \left(n+1\right)\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\sum_{m=1}^{k}\frac{1}{m} \tag{5}$$

We obtain the new formula by the summation to the harmonic series formula (1), where the tail term is substituted with ε_n:

$$\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{1}{m} = n\gamma + \ln\left(n!\right) + \frac{1}{2}\left(\gamma + \ln n + \frac{1}{2n} - \varepsilon_n\right) - \sum_{k=1}^{n}\varepsilon_k \tag{6}$$

After combining (6) with (5), we get the formula:

$$n = \frac{1+\gamma}{2} + \mu + \ln n^{n+1} - \ln\left(n!\right) - \ln n^{\frac{1}{2}} + \eta_n \tag{7}$$

where $\eta_n = \frac{1}{4n} - n\varepsilon_n - \frac{\varepsilon_n}{2} - u_n$ and $u_n = \mu - \sum_{k=1}^{n}\varepsilon_k$.

Let

$$\theta = 1 + \gamma + 2\mu \ , \tag{8}$$

The references [1] and [2] prove that

$\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty} \eta_n =$

After the exponential transformation on the both sides of the formula (7), we get a new approximate formula of $n!$, which is different from the Stirling formula:

$$n! = e^{\frac{1}{2}\theta} n^{n+\frac{1}{2}} e^{-n} e^{\eta_n} \tag{9}$$

The references [1] and [2] prove the existence of the limits:

$$\lim_{n\to\infty} \frac{n!e^n}{n^n\sqrt{n}} = e^{\frac{1}{2}\theta} \tag{10}$$

According to the theorem of limit existence and uniqueness and the Stirling formula's limit,

$$\lim_{n\to\infty} \frac{n!e^n}{n^n\sqrt{n}} = \sqrt{2\pi}\ , \tag{11}$$

we obtain the equation $\sqrt{2\pi} = e^{\frac{1}{2}\theta}$, a new formula is obtained from the above equation :

$$\pi = \frac{1}{2}e^{\theta} \tag{12}$$

The new formula gives a description of the rational relationship between the circumference ratio π and Natural constant e, which is very different from the Euler formula which reveals the imaginary relationship between π and e.

3. Comprehensions about the Imaginary and Rational Relationships between π and e

The Euler's formula $e^{\pi i} = -1$ reveals the imaginary relationship among π, e and the imaginary unit i [4]. The mathematician Benjamin Peirce comments on the formula: "The Euler formula is definitely correct but also strange which is difficult to comprehend. As it has been proved, we know it's definitely correct." So Euler formula is also called 'God's formula' in some books.

The new formula $\pi = \frac{1}{2}e^{\theta}$ not only reveals the tight rational relationships among π, e and θ, but also conserves the conciseness of Euler formula as well.

Both the formulas are concise and subtle, but the essences are quite different. One is described for imaginary relationship and the other is for rational relationship. The new rational relationship brings us more benefits. Specifically, the advantages can be summarized as follows:

1) The constants in the formula can be easily computed. Although the Euler constant γ is difficult to compute, we can make use of the value μ to cope with.
2) The constants in the formula can be converted to each other.

We can make use of $e^{\frac{\theta}{2}}$ to substitute $\sqrt{2\pi}$ for the simplification of the formulas in probability and statistics.

4. μ Is the Latent Constant Concerning about the Euler Constant γ

Now let's analyze the relationship between μ and γ. The formula (1) can be rewritten as follows:

$$\sum_{k=1}^{n}\frac{1}{k}=\gamma+\ln n+\frac{1}{2n}-\varepsilon_n$$

Namely,

$$\varepsilon_n=\gamma+\frac{1}{2n}-\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right) \text{ and } \mu=\sum_{n=1}^{\infty}\varepsilon_n .$$

Here we draw three curves to describe m-curve ($\mu=\sum_{n=1}^{\infty}\varepsilon_n$), b-curve ($\gamma+\frac{1}{2n}$) and c-curve ($\sum_{k=1}^{n}\frac{1}{k}-\ln n$). The variations of their first 100 items can be described in Fig. 1.

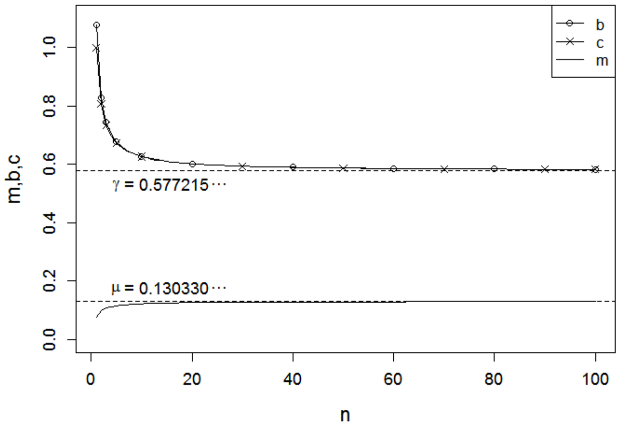

Fig. 1. The comparison between the three curves (the first 100 items).

In Fig. 1, the b-curve and c-curve are quite close, nearly in the same line. The numeric Table 1 is as follows.

Table 1. The Numeric of the b-curve and c-curve

n	m-curve ($\mu=\sum_{n=1}^{\infty}\varepsilon_n$)	b-curve ($\gamma+\frac{1}{2n}$)	c-curve $\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)$
1	0.0772156649015328	1.07721566490153	1
10	0.122402555424179	0.627215664901533	0.626383160974208
30	0.127598802804225	0.5938823315682	0.593789749258235
50	0.128680611213722	0.587215664901533	0.587182332901277
70	0.129148695400236	0.58435852204439	0.584341515588712
100	0.129501522934988	0.582215664901533	0.582207331651529

Owing to

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)=\gamma \quad , \tag{13}$$

it can be seen that ε_n is the difference between $\gamma+\frac{1}{2n}$ and $\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)$. In the table above, the difference value between b-curve and c-curve are quite tiny, which is also obvious in Fig 1. By accumulating the difference value between $\gamma+\frac{1}{2n}$ and $\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)$, we can obtain the constant μ, which is the latent constant behind γ. When we accumulate the first 20000 items of the series in this method, the value of the constant μ can be estimated as μ =0.130326534636424….

As μ and γ are both concerning the difference between the sum of harmonic series and $\ln n$, we can define a new constant, $\theta=1+\gamma+2\mu$, by combining μ and γ. Through the new formula $\pi=\frac{1}{2}e^{\theta}$, the new constant θ makes a close connection between π and e. It is obvious that θ is the perfect constant combined with μ and γ.

5. Computing Euler Constant with the Use of New Constant μ and New Formula $\pi=\frac{1}{2}e^{\theta}$

5.1. The Algorithm for Computing the New Constant μ on Computer

The references [1] and [2] give another formula for computing the new constant μ:

$$\mu=\sum_{k=1}^{\infty}\frac{A_{k+1}}{k\cdot k!} \tag{14}$$

where

$$\frac{A_{k+1}}{k\cdot k!}=\frac{1}{k(k+1)k!}\int_0^1 x(1-x)(2-x)\cdots(k-x)dx$$
$$=\frac{1}{k(k+1)}\int_0^1 x\frac{(1-x)}{1}\cdot\frac{(2-x)}{2}\cdots\frac{(k-x)}{k}dx=\frac{1}{k(k+1)}\int_0^1 x(1-x)\left(1-\frac{x}{2}\right)\cdots\left(1-\frac{x}{k}\right)dx$$

Then

$$\mu=\sum_{k=1}^{\infty}\frac{A_{k+1}}{k\cdot k!}=\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\int_0^1 x(1-x)(1-\frac{x}{2})\cdots(1-\frac{x}{k})\ dx \tag{15}$$

Now we discuss the calculation item of the formula. Let

$$S_k=\int_0^1 f_k(x)\,\mathrm{d}x=\int_0^1 x(1-x)(1-\frac{x}{2})\cdots\left(1-\frac{x}{k}\right)\mathrm{d}x \tag{16}$$

The integrand is a polynomial function $f_k(x)$. We denote the polynomial function $f_k(x)=b_1^{(k)}x+b_2^{(k)}x^2+\cdots+b_{k+1}^{(k)}x^{k+1}$, the following result is obtained by integration:

$$S_k=\int_0^1 x(1-x)(1-\frac{x}{2})\cdots\left(1-\frac{x}{k}\right)dx=\int_0^1 f_k(x)dx=\int_0^1(b_1^{(k)}x+b_2^{(k)}x^2+\cdots+b_{k+1}^{(k)}x^{k+1})dx$$
$$=\frac{1}{2}b_1^{(k)}+\frac{1}{3}b_2^{k}+\cdots+\frac{1}{k+2}b_{k+1}^{(k)}=\sum_{j=2}^{k+2}\frac{1}{j}b_{j-1}^{(k)} \tag{17}$$

The coefficients of the polynomial's expansion by k items can be represented as the following Table 2 triangle series:

Table 2. The Coefficients of the Polynomial's Expansion by k Items

	x	x^2	x^3	x^4	x^5	x^6	X^7	…	X^{k+1}
k=0	1								
k=1	1	-1							
k=2	1	-3/2	1/2						
k=3	1	-11/6	6/6	-1/6					
k=4	1	-50/24	35/24	-10/24	1/24				
k=5	1	-274/120	225/120	-85/120	15/120	-1/120			
k=6	1	-1764/720	1624/720	-735/720	175/720	-21/720	1/720		
k	1	$b_2^{(k)}$	$b_3^{(k)}$	$b_4^{(k)}$	$b_5^{(k)}$	$b_6^{(k)}$	$b_7^{(k)}$	…	$b_{k+1}^{(k)}=1/k!$

Now we find the recursion relation between the coefficient $b^{(k-1)}$ and $b^{(k)}$:

$$
\begin{aligned}
&f_k(x)=b_1^{(k)}x+b_2^{(k)}x^2+\cdots+b_k^{(k)}x^k+b_{k+1}^{(k)}x^{k+1} = f_{k-1}(x)(1-\frac{x}{k})= \\
&b_1^{(k-1)}x+b_2^{(k-1)}x^2+\cdots+b_k^{(k-1)}x^k-\frac{x}{k}(b_1^{(k-1)}x+b_2^{(k-1)}x^2+\cdots+b_k^{(k-1)}x^k) \\
&=b_1^{(k-1)}x+\left(b_2^{(k-1)}-b_1^{(k-1)}\big/k\right)x^2+\cdots+\left(b_k^{(k-1)}-b_{k-1}^{(k-1)}\big/k\right)x^k-\left(b_k^{(k-1)}\big/k\right)x^{k+1}
\end{aligned}
\tag{18}
$$

Comparing the coefficients of x -term, we get the recursion formulas as follows:

$$
b_1^{(k)}=b_1^{(k-1)},\ b_j^{(k)}=b_j^{(k-1)}-b_{j-1}^{(k-1)}\big/k,\quad b_{k+1}^{(k)}=-b_k^{(k-1)}\big/k\quad ,\ j=2,3,\cdots k \tag{19}
$$

In the above formulas, we add the equation $b_0^{(k-1)}=0$ to the first formula and then get $b_0^{(k-1)}/k=0$, which is combined with the first formula. In this way, we can get the new first formula with the index j = 1, consistent to the second formula. By the same trick, we add $b_{k+1}^{(k-1)}=0$ to the third formula to get the new third formula consistent to the second formula. The formula is equivalent to $j = k+1$.

Then three formulas are merged into the middle formula:

$$
b_j^{(k)}=b_j^{(k-1)}-b_{j-1}^{(k-1)}\big/k \qquad j=1,2,3,\cdots k,k+1 \tag{20}
$$

This general formula requires $b_0^{(k-1)}=0$ and $b_{k+1}^{(k-1)}=0$.

The formula indicates that $f_{k-1}(x)$'s coefficients $b_j^{(k-1)}$ can be derived from $f_k(x)$'s coefficients $b_j^{(k-1)}$. For example,

$$
f_1(x)=x(1-x)=x-x^2,\ b_1^{(1)}=1,\ b_2^{(1)}=-1.
$$

$$
f_2(x)=x(1-x)(1-\frac{x}{2})=x-\frac{3}{2}x^2+\frac{1}{2}x^3,\ b_1^{(2)}=1,\ b_2^{(2)}=-\frac{3}{2},b_3^{(2)}=\frac{1}{2}.
$$

The following is similar.

With the help of the recursive formula, we can program on computer to compute the approximate value of μ. The concrete process is as follows:

1) We create a one dimension array b[j] with k+2 numbers （0，1，...k+1）for storing the coefficients of the function $f_k(x)$ to be calculated. b[0] is given specially with the value 0 and b[1] equals 1. The rest of b[j] are all 0. Just as the Table 3 in the following：

Table 3. The Rest of b[j] Are All 0

0	1	0	0	0					0

b[0] b[1] b[2] b[3] b[4] ...... b[k+1]

2) By the use of the j-th and (j-1)-th value in the (k-1)-layer, we can compute the value of b[j] in the k-layer according to the formula（20）. By the trick of back propagation, we can compute b[j] in the k-layer after getting the value of b[j] in the (k+1)-layer. In this way, we can compute all of the b[k], until k equals 1.

By the trick, we just need to keep the value of the b[j] in one layer and the array is enough for representation. We can make use of the coefficient of the (k-1)-th item to compute the coefficient of the k-th item, then the coefficient of the (k-1) item can be erased.

1) After getting the value of b[j]，we can make use of the formula (17) to sum the coefficients and get the calculation S_k.
2) Combining the calculation S_k with the formula (15), we get the value which can be accumulated to approximate the μ.

By the use of series sum formula with polynomial calculation, the value of which is described as s-curve , the result described as m-curve is as follows:

n	m	s
1	0.0833333333333333	0.166666666666667
10	0.126068470948011	0.0651646205357144
100	0.13010143728929	0.0296574697457057
1000	0.130317582972294	0.016014987048576
2000	0.1303250362332	0.0136648069931553
10000	0.130329867745623	0.00981667421982024
20000	0.130330331383179	0.00863287370418853

For n<=100, the variation can be depicted as follows Fig. 2:

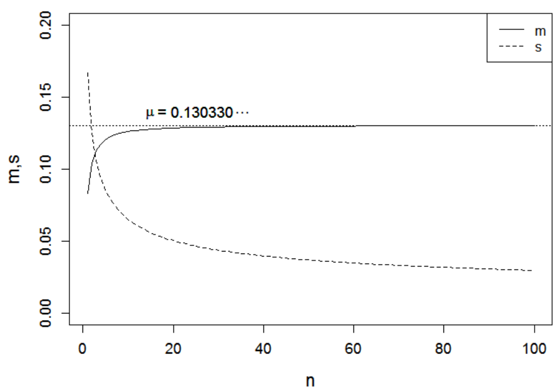

Fig. 2. For n<=100, the variation can be depicted.

Though computers have word-digit limit leading to error, the result still well approximates the true μ. By programming in C#, we compute the sum of the first 2000 items in the series and get the approximate value of the new constant μ as follows:

$$\mu=0.130330331383179\cdots.$$

We can also get the numeric calculation described in graph as s-curve which tends to be a constant. The approximate value is as follows:

$$S_{20000}=\int_0^1 x(1-x)(1-\frac{x}{2})\cdots\left(1-\frac{x}{20000}\right)dx=0.008632873704\cdots$$

5.2. The Use of New Constant μ to Compute the Euler Constant γ

The Euler constant γ is the limit hard to compute, while the constant μ is easily computed. It is easier to compute γ by the use of μ.

The new formula $\pi=\frac{1}{2}e^{\theta}$ can be written as $ln\ 2\pi=\theta$. Combing the formula $\theta=1+\gamma+2\mu$, we have the new formula: $\gamma=\ln 2\pi-1-2\mu$.

By the use of the formula, we compute the Euler constant γ as: $\gamma=0.577215\cdots$.

6. The Use of New Formula to Simplify Normal Distribution Function

By the use of the new formula, we can simplify the following formulas in common use: $\sqrt{2\pi}$ replaced by $e^{\frac{1}{2}\theta}$ can be naturally merged with the item in the right as they have the same e's exponential form.

1) normal distribution function

$p(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ simplified as $p(x)=e^{-\frac{1}{2}(x^2+\theta)}$.

2) normal distribution formula

$\phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt$ simplified as $\phi(x)=\int_{-\infty}^{x}e^{-(\frac{t^2}{2}+\theta)}dt$.

3) Euler-Possion formula

$\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx=\sqrt{2\pi}$ simplified as $\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left(x^2+\theta\right)}dx=1$.

4) Fourier transformation and its inverse transformation

$F(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt$ simplified as $F(\lambda)=\int_{-\infty}^{\infty}f(t)e^{-(i\lambda t+\frac{\theta}{2})}dt$.

$\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx=\sqrt{2\pi}$ simplified as $f(t)=\int_{-\infty}^{\infty}F(\lambda)e^{(i\lambda t-\frac{\theta}{2})}d\lambda$.

Some other formulas containing $\sqrt{2\pi}$ can also be simplified in the same way.

7. Conclusions

The new constants μ, θ and the new formula $\pi=\frac{1}{2}e^{\theta}$ bring us a new understanding about the relation between π and e. Euler formula gives the imaginary relation between π and e, while the new formula establishes the rational relation between them. The rational relation which is the useful transformation can be widely applied in the numeric computation field.

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