Extension of Some Common Fixed Point Theorems

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Abstract: The celebrated Banach fixed point theorem has been modified and developed in various aspects. It is not surprising all that this result admits several generalizations and improvement. Besides a good number of variations of the generalizations of the result are formed today. A few theorems dealing with existence and uniqueness of the fixed points are presented hear. This result not only extend the variation of the generalization of the Banach fixed point theorem but also improve the result. In addition the rate of convergence is found to be better.

Key words: Banach fixed point theorem, common fixedpoint, Hilbert space, uniqueness.

1. Introduction

Definition A: Fixed Point: Let E be a metric space and $T: E \to E$ be a mapping. A point x is said to be a fixed point of T if $T(x) = x$.

The well-known classical result of Banach's fixed point Theorem has been extended and generalized in various directions. In the year 1922 Banach obtained the existence and uniqueness of the fixed point in the following form.

Banach Fixed Point theorem: Let (X, d) be a complete metric space and $T : X \to X$, satisfying

$$d\big(T(x), T(y)\big) \leq k d(x, y)\ \forall x, y \in X.$$

and for some k with $0 < k < 1$, then T has a unique fixedpoint in X.

This result was generalized in some direction by Browder and Petryshyn [1].

Definition B: Common Fixed Point: Let H be a Hilbert space and $S, T : H \to H$ be two mappings. A point x is said to be a common fixed point of S and T if $T(x) = x = S(x)$.

Koparde and Waghmode [2] developed the result of Jungck [3] and Fisher [4] to common fixed points.

Theorem 1.1: Let f be a continuous mapping of a complete metric space (X, d) into itself. Then f has a fixed point in X if there exists $\alpha \in (0,1)$ and a mapping $g : X \to X$ which commutes with f and satisfies $g(X) \subset f(X)$ and

$$d\big(g(x), g(y)\big) \leq \alpha\, d\big(f(x), f(y)\big), \forall x, y \in X.$$

Indeed, f and g have a unique common fixed point.

Fisher [4] discussed the extension of above result in the form of

Theorem 1.2: Let $S$and T be continuous mapping of complete metric space (X, d) into itself. Then S and T have a common fixed point in X if f there exists a continuous mapping $A$of X into $SX \cap TX$, which commute with $S$and T and satisfies the inequality

$$d(Ax, Ay) \leq \alpha d(Sx, Ty), \forall x, y \in X.$$

where $0 < \alpha < 1$,Indeed, $S, T$and Athen have a uniquecommon fixed point.

Extending further, Koparde and Waghmode [2] extended and modified the following in the setting of Hilbert space.

Theorem 1.3: Let $S$and T be continuous mapping of a Hilbert space X into itself. Then S and T have a common fixed point in X if f there exists a continuous mapping A of X into $SX \cap TX$, which commute with S and T and satisfies the inequality

$$\|Ax - Ay\| \leq \alpha\|Ax - Sx\| + \beta\|Ay - Ty\| + \gamma\|Sx - Ty\|$$

for all x, y in X, where α, β, γ are non-negative reals with $0 < \alpha + \beta + \gamma < 1$ .Indeed, $S, T,$ and A then have a unique Common fixed point.

It is found that (Bisht and Joshi [5]) there exist maps that have a discontinuity in their domain but which admit fixed point, for instance, one can have a glance at Kannan [6], [7]. Bisht and Joshi [5] dealt with common fixed point theorems for a pair of weakly reciprocally continuous self-mappings satisfying generalized contraction or Lipschitz type conditions. This investigation provided enough scopes, which took the study of common fixed point theorems from the class of compatible continuous mappings to a wider class of mappings which included no compatible and discontinuous mappings. The main result of this discussion is contained in the following result.

Theorem 1.4: Let f and g be weakly reciprocally continuous self-mappings of a complete metric space (X, d) such that

1) $fX \subseteq gX$
2) $d(fx, fy) \leq ad(gx, gy) + b[d(fx, gx) + d(fy, gy)] + c[d(fx, gy) + d(fy, gx)]$

$$a, b, c \geq 0, \quad 0 \leq a + 2b + 2c < 1$$

If f and g are either compatible or g-compatible or f-compatible or compatible of type (P) then $f$and $g$have a unique common fixed point.

Introducing the concept of reciprocal continuity Pant [8] applied it to establish a situation where in a collection of mappings has a fixed point, which is a point of discontinuity of all mappings. Subsequently, a large number of papers dealt with the application of reciprocal continuity. Towards the generalization of reciprocal continuity, Pant, Bisht, Arora [9] introduced weak reciprocal continuity and deployed this new notion to obtain fixed point theorems. Indeed, this new notion was applicable to compatible mappings as well as non-compatible mappings. In this investigation the following result is proved.

Theorem 1.5: Let f and g be weakly reciprocally continuous self-mappings of a complete metric space(X, d) such that

1) $fX \subseteq gX$
2) $d(fx, fy) \leq ad(gx, gy) + bd(fx, gx) + cd(fy, gy) \qquad 0 \leq a, b, c < 1, \; 0 \leq a + b + c < 1$

If f and g are either compatible or R-weakly commuting of type (A_g) or R-weakly commuting of type (A_f) then $f$and g have a unique common fixed point.

Pant and Bisht [10] unified the approaches of reciprocal continuity, sub sequential continuity and conditional commutatively to generalize the notion of reciprocal continuity. Here, some common fixed point theorems in diverse settings were obtained as an application of the new notion introduced. Of particular importance is the following result from Pant and Bisht [10].

Theorem 1.6: Let f and g be conditionally reciprocally continuous self-mappings of a complete metric space (X, d) such that

1) $fX \subseteq gX$
2) $d(fx, fy) \leq kd(gx, gy), k \in [0,1)$

If f and g are either compatible or g-compatible or f-compatible then f and g have a unique common fixed point.

Takahashi [11], [12] introduced the notion of convexity in metric spaces, and discussed some fixed point theorems for nonexpansive mappings in such convex metric spaces. Mohammad Moosaei [13]-[15] obtained some fixed point theorems for self-mappings satisfying certain contraction principles on a convex complete metric space. In [16], a hybrid iteration method was employed and the strong convergence of the iteration scheme to a fixed point of nonself nonexpansive mapping was derived in Banach spaces. The main result dealing with the strong convergence of the Hybrid iteration scheme is the following.

Theorem 1.7: Let E be a real Banach space with a uniformly Gateaux differentiable norm and C be a nonempty weakly compact convex subset of E. Suppose that T is a nonnself nonexpansive mapping. $\{x_n\}$ (Defined as in [16]), $\{\alpha_n\}, \{\lambda_n\}$ are real number sequences in $[0, 1)$ satisfying the following conditions:

1) (C_1) And (C_2) (as in [14]);
2) $\sum_{n=2}^{\infty} \lambda_n < \infty$.

If $F = K_{mn} \cap F(PT) \neq \emptyset$, then $\{x_n\}$strongly converges to some point of F.

Gulnara Abduvalieva and Dmitry S. Kaliuzhnyi- Verbovetskyi [17] established a fixed point theorem for mappings of all sizes which respected the matrix sizes and direct sums of matrices. The main result reads as

Theorem 1.8: Let S be a set, and $\Omega \subseteq S_{nc}$ respect direct sums of matrices. Define supp

$$\Omega := \{n \in \mathbb{N}; \Omega_n \neq \emptyset\}. \text{ Let } f: \Omega \to \Omega \text{ satisfy}$$

1) $f(\Omega_n) \subseteq \Omega_n, n \in$supp Ω, $X, Y \in \Omega$;
2) f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$,
3) For every $n \in$suppΩ the mapping $f|\Omega_n$ has a unique fix point, X_{*n}.

Let d =gcd$\{n; n \in$supp$\Omega\}$. Then

4) there exist $X_* \in S^{d \times d}$ such that $X_{*n} = \oplus_{\alpha=1}^{\frac{n}{d}} X_*$; $n \in$suppΩ;
5) If moreover , $S = M$ is a bi-module over R, $O = 0 \in M$, and f is nc function, then there exit a nc set $\tilde{\Omega} \supseteq \Omega$ with supp $\tilde{\Omega} = \mathbb{N}d$ and nc function $\tilde{f}: \tilde{\Omega} \to \tilde{\Omega}$ such that
6) $\tilde{f}|\Omega = f$;
7) For every $n \in \mathbb{N}d$ the mapping $\tilde{f}|\tilde{\Omega}_n$ has a unique fixed point $X_{*n} = \oplus_{\alpha=1}^{\frac{n}{d}} X_*$;

In [11] Patel and Deheri made an attempt to prove an improved version of this result, which also strengthens a result of Jungck [3] and Fisher [4].

Patel and Deheri [13] established the importance of weak reciprocal continuity of the maps in deriving the common fixed point theorems. The rate of convergence was also discussed.

Recently, Patel and Deheri [14] dealt with the fixed point for a non-continuous map on a metric space, improving the rate of convergence. Besides, a variant of the well-known generalisation of the Banach Fixed Point Theorem, was proved.

An improvement of a result in Patel and Deheri [11] is presented below

2. Main Results

Theorem 2.1: Let S and T be two continuous mappings of Hilbert space X into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping $A$of Xinto SX ∩ TX , which commutes with S and T and satisfies the inequality,

$$\|Ax - Ay\|^2 \leq \alpha\|Ax - Sx\|^2 + \beta\|Ay - Ty\|^2 + \gamma\|Ax - Sy\|^2 + \delta\|Ax - Ty\|^2 + \lambda\|Ay - Sy\|^2$$

For all x, y in X; where $\alpha, \beta, \gamma, \delta, \lambda$ are non-negative reals with

$$0 < \alpha + \beta + \gamma + \delta + \lambda < 1.$$

Proof: First of all it is proved that the existence of such a mapping A is necessary. For this suppose that $Sz = z = Tz$, for some $z \in X$.

Define a mapping A of X in to$X$ by $Ax = z$ for all $x \in X$. Then clearly, A is a continuous mapping of Xinto $SX \cap TX$. Since, $Sx, Tx \in X$, for all $x \in X$ and$Ax = z$, for all $x \in X$, we gets

$$ASx = z, SAx = Sz = z, ATx = z, TAx = Tz = z.$$

Hence, A commutes with S and T. Now, for any $\alpha, \beta, \gamma, \delta, \lambda$ with
$0 < \alpha + \beta + \gamma + \delta + \lambda < 1$, it is noticed that

$$\|Ax - Ay\|^2 \leq \alpha\|Ax - Sx\|^2 + \beta\|Ay - Ty\|^2 + \gamma\|Ax - Sy\|^2$$
$$+\delta\|Ax - Ty\|^2 + \lambda\|Ay - Sy\|^2$$

which means

$$0 \leq \alpha\|z - Sx\|^2 + \beta\|z - Ty\|^2 + \gamma\|z - Sy\|^2 + \delta\|z - Ty\|^2 + \lambda\|z - Sy\|^2, \ \forall x, y \in X$$

This proves the existence of such a mapping A is necessary.

We now prove that the condition is sufficient. For this suppose that such a mapping A exists. Then we construct a sequence $\{x_n\}$ as follows. Let $x_0 \in X$ be an arbitrary point.

Since $AX \subset SX$ we choose a point $x_1 \in X$ such that $Sx_1 = Ax_0$.

Also, $AX \subset TX$ and hence we can choose $x_2 \in X$ such that $Tx_2 = Ax_1$. Continuing in this way, we get a sequence $\{x_n\}$ as follows:

$$Sx_{2n-1} = Ax_{2n-2}, \quad Tx_{2n} = Ax_{2n-1};\ n = 1,2,3, \ldots$$

We proceed to show that $\{Ax_n\}$ is Cauchy sequence. For this consider the inequality,

$$\begin{gathered}
\|Ax_{2n} - Ax_{2n-1}\|^2 = \|Ax_{2n-1} - Ax_{2n}\|^2 \\
\leq \alpha\|Ax_{2n-1} - Sx_{2n-1}\|^2 + \beta\|Ax_{2n} - Tx_{2n}\|^2 + \gamma\|Ax_{2n-1} - Sx_{2n}\|^2 \\
+\delta\|Ax_{2n-1} - Tx_{2n}\|^2 + \lambda\|Ax_{2n} - Sx_{2n}\|^2 \\
= \alpha\|Ax_{2n-1} - Ax_{2n-2}\|^2 + \beta\|Ax_{2n} - Ax_{2n-1}\|^2 \\
+\gamma\|Ax_{2n-1} - Ax_{2n-1}\|^2 + \delta\|Ax_{2n-1} - Ax_{2n-1}\|^2 \\
+\lambda\|Ax_{2n} - Ax_{2n-1}\|^2
\end{gathered}$$

This implies that

$$\|Ax_{2n} - Ax_{2n-1}\|^2 \leq \frac{\alpha}{1-(\beta+\lambda)} \|Ax_{2n-1} - Ax_{2n-2}\|^2 \quad (1)$$

Further, it is seen that

$$\begin{gathered}\|Ax_{2n+1} - Ax_{2n}\|^2 \leq \alpha\|Ax_{2n} - Ax_{2n-1}\|^2 + \beta\|Ax_{2n+1} - Ax_{2n}\|^2 \\ +\gamma\|Ax_{2n} - Ax_{2n}\|^2 + \delta\|Ax_{2n} - Ax_{2n}\|^2 \\ +\lambda\|Ax_{2n+1} - Ax_{2n}\|^2\end{gathered}$$

equivalently,

$$\|Ax_{2n+1} - Ax_{2n}\|^2 \leq \frac{\alpha}{1-(\beta+\lambda)} \|Ax_{2n} - Ax_{2n-1}\|^2 \quad (2)$$

Since $0 < \alpha + \beta + \gamma + \delta + \lambda < 1$, one finds that $\mu = \frac{\alpha}{1-(\beta+\lambda)} \in (0,1)$then $0 < \mu < 1$.

Therefore, from (1) and (2) one concludes that

$$\begin{gathered}\|Ax_{n+1} - Ax_n\|^2 \leq \mu\|Ax_n - Ax_{n-1}\|^2 \\ \leq \mu^2\|Ax_{n-1} - Ax_{n-2}\|^2 \\ \dots\dots\dots\dots \\ \leq \mu^n\|Ax_1 - Ax_0\|^2\end{gathered}$$

Now, it follows that $\{Ax_n\}$ is a Cauchy sequence and so it has a limit point x in X. Since sequences $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ are sub sequence of$\{Ax_n\}$, they have the same limit z. As, S and A are commuting mapping, one can have

$$Sz = \lim_{n\to\infty} SAx_{2n+1} = \lim_{n\to\infty} ASx_{2n+1} = Az$$

Similarly, one gets

$$Tz = Az$$

This gives,

$$Tz = Az = Sz \quad (3)$$

Now it follows that,

$$\begin{gathered}\|Az - AAz\|^2 \leq \alpha\|Az - Sz\|^2 + \beta\|AAz - TAz\|^2 + \gamma\|Az - SAz\|^2 \\ +\delta\|Az - TAz\|^2 + \lambda\| AAz - SAz\|^2\end{gathered}$$

In view of (3) and commutativity of A, S and T, one finds that

$$\|AAz - TAz\| = \|AAz - ATz\| = \|AAz - AAz\| = 0$$

and

$$\|AAz - ASz\| = \|Az - AAz\|$$

Also, it is clear that

$$\|AAz - ATz\| = \|Az - AAz\|$$

Making use of these, one arrives at

$$\|Az - AAz\|^2 \leq (\gamma + \delta)\|Az - AAz\|^2$$

Science, $\gamma + \delta < 1$, we must have, $Az = AAz$
Finally, putting $Az = z_1$, one obtains$Az_1 = AAz = Az = z_1$
Similarly,

$$Tz_1 = TAz = ATz = AAz = Az = z_1, \text{ and } Sz_1 = z_1$$

So z_1 is a fixed point of S ,T and A.
Next to show the uniqueness of this common fixed point, let us suppose that z_2 is also a common fixed point of S,T and A other then z_1. Then, it becomes clear that

$$Sz_2 = z_2, Tz_2 = z_2,\ Az_2 = z_2.$$

Thus, it follows that

$$\|z_1 - z_2\|^2 \leq \alpha\|z_1 - z_1\|^2 + \beta\|z_2 - z_2\|^2 + \gamma\|z_1 - Z_2\|^2$$
$$+\delta\|z_1 - z_2\|^2 + \lambda\|z_2 - z_2\|^2$$
$$= (\gamma + \delta)\|z_1 - z_2\|^2$$

which implies that

$$\|z_1 - z_2\|^2 \leq (\gamma + \delta)\|z_1 - z_2\|^2.$$

Since, $\gamma + \delta < 1$, this gives $z_1 = z_2$.
Next to follow is the variant;

Theorem 2.2: Let S and T be two continuous mappings of Hilbert space X into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into SX ∩ TX, which commutes with S and T and satisfies the inequality,

$$\|Ax - Ay\|^2 \leq \alpha\|Ax - Sx\|\|Ay - Sy\| + \beta\|Ax - Tx\|\|Ay - Ty\|$$
$$+\gamma\|Ax - Sx\|\|Ay - Sx\| + \delta\|x - y\|^2$$

for all x, y in X; where $\alpha, \beta, \gamma, \delta$, are non-negative reals with

$$0 < \alpha + \beta + \gamma + \delta + < 1.$$

Proof: First of all it is proved that the existence of such a mapping A is necessary, for this suppose that

$$Sz = z = Tz, \text{for some } z \in X.$$

Define a mapping A of X into X by $Ax = z$ for all $x \in X$. Then clearly, A is a continuous mapping of X in to$SX \cap TX$. Since,$Sx, Tx \in X$, for all $x \in X$ and$Ax = z$, for all $x \in X$, one gets

$$ASx = z, SAx = Sz = z, ATx = z, TAx = Tz = z.$$

Hence, A commutes with S and T. Now, for any $\alpha, \beta, \gamma, \delta$, with $0 < \alpha + \beta + \gamma + \delta < 1$, it is noticed that

$$\|A_x - A_y\|^2 \leq \alpha\|Ax - Sx\|\|Ay - Sy\| + \beta\|Ax - Tx\|\|Ay - Ty\| \\ +\gamma\|Ax - Sx\|\|Ay - Sx\| + \delta\|x - y\|^2$$

which yields,

$$0 \leq \alpha\|Ax - Sx\|\|Ay - Sy\| + \beta\|Ax - Tx\|\|Ay - Ty\| \\ +\gamma\|Ax - Sx\|\|Ay - Sx\| + \delta\|x - y\|^2$$

for all$x, y \in X$.This proves the existence of such a mapping A is necessary.

To prove that the condition is sufficient, we construct a sequence $\{x_n\}$as follows.

Let $x_0 \in X$ be an arbitrary point. Since $AX \subset SX$ we choose a point $x_1 \in X$ such that $Sx_1 = Ax_0$.

Also, $AX \subset TX$ and hence we can select $x_2 \in X$ such that $Tx_2 = Ax_1$.

Continuing in this way, we get a sequence $\{x_n\}$ as follows:

$$Sx_{2n-1} = Ax_{2n-2}, \quad Tx_{2n} = Ax_{2n-1};\ n = 1,2,3, \ldots$$

We proceed to show that $\{Ax_n\}$ is Cauchy sequence. For this one avails the inequality; $\|Ax_{2n+1} - Ax_{2n}\|^2 \leq \alpha\|Ax_{2n+1} - Ax_{2n}\|\|Ax_{2n} - Ax_{2n-1}\|$

$$+\beta\|Ax_{2n+1} - Ax_{2n}\|\|Ax_{2n} - Ax_{2n-1}\| \\ +\gamma\|Ax_{2n+1} - Ax_{2n}\|\|Ax_{2n} - Ax_{2n}\| + \delta\|x_{2n+1} - x_{2n}\|^2 \\ \leq (\alpha + \beta)\left[\frac{\|Ax_{2n+1} - Ax_{2n}\|^2}{2} + \frac{\|Ax_{2n} - Ax_{2n-1}\|^2}{2}\right] + \delta\|x_{2n+1} - x_{2n}\|^2$$

which gives

$$2\|Ax_{2n+1} - Ax_{2n}\|^2 \leq (\alpha + \beta)[\|Ax_{2n+1} - Ax_{2n}\|^2 + \|Ax_{2n} - Ax_{2n-1}\|^2] \\ +2\delta\|x_{2n+1} - x_{2n}\|^2$$

Therefore, one arrives at

$$\|Ax_{2n+1} - Ax_{2n}\|^2 \leq \left(\frac{\alpha+\beta}{2-\alpha-\beta}\right)\|Ax_{2n} - Ax_{2n-1}\|^2 + \left(\frac{2\delta}{2-\alpha-\beta}\right)\|x_{2n+1} - x_{2n}\|^2 \tag{4}$$

Further, one can observe that

$$\|Ax_{2n} - Ax_{2n-1}\|^2 \leq \alpha\|Ax_{2n} - Sx_{2n}\|\|Ax_{2n-1} - Sx_{2n-1}\| \\ +\beta\|Ax_{2n} - Tx_{2n}\|\|Ax_{2n-1} - Tx_{2n-1}\| \\ +\gamma\|Ax_{2n} - Sx_{2n}\|\|Ax_{2n-1} - Sx_{2n}\| \\ +\delta\|x_{2n} - x_{2n-1}\|^2$$

Simplifying this by using Young's inequality, one obtains

$$\|Ax_{2n} - Ax_{2n-1}\|^2 \leq \left(\frac{\alpha+\beta}{2-\alpha-\beta}\right)\|Ax_{2n-1} - Ax_{2n-2}\|^2 + \left(\frac{2\delta}{2-\alpha-\beta}\right)\|x_{2n} - x_{2n-1}\|^2 \tag{5}$$

Since

$$0 < \alpha + \beta + \gamma + \delta + < 1$$

we find that

$$\left(\frac{\alpha+\beta}{2-\alpha-\beta}\right) \in (0,1) \ ,\left(\frac{2\delta}{2-\alpha-\beta}\right) \in (0,1)$$

Suppose $\lambda = max\left\{\frac{\alpha+\beta}{2-\alpha-\beta}, \frac{2\delta}{2-\alpha-\beta}\right\}$. Then $0 < \lambda < 1$.

From (4) and (5) one concludes that

$$\begin{gathered}\|Ax_{n+1} - Ax_n\|^2 \leq \lambda [\ \|Ax_n - Ax_{n-1}\|^2 + \|Ax_{n+1} - Ax_n\|^2 \\ \leq \lambda^2 [\|Ax_{n-1} - Ax_{n-2}\|^2 + \|Ax_n - Ax_{n-1}\|^2] \\ +\lambda\|Ax_{n+1} - Ax_n\|^2 \\ \ldots\ldots\ldots.. \\ \leq \lambda^n [\|Ax_1 - Ax_0\|^2 + \|Ax_2 - Ax_1\|^2] + \epsilon \ ,\end{gathered}$$

for large n.

Now, it can be seen that $\{Ax_n\}$ is a Cauchy sequence and so ithas a limit point $x$in X. Since $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ are sub sequences of $\{Ax_n\}$, they have thesame limit z. As, S and A are commuting mapping, one comes across

$$Sz = \lim_{n\to\infty} SAx_{2n+1} = \lim_{n\to\infty} \mathrm{A}Sx_{2n+1} = Az$$

Likewise, one finds $Tz = Az$,

Leading to

$$Tz = Az = Sz \tag{6}$$

It turns out that

$$\begin{gathered}\|Az - AAz\|^2 \leq \ \alpha\|Az - Sz\|\|AAz - SAz\| + \beta\|Az - Tz\|\|AAz - TAz\| \\ +\gamma\|Az - Sz\|\|AAz - Sz\| + \delta\|z - Az\|^2\end{gathered}$$

The commutativity of A and S and T, yield the following in view of (6)

$$\begin{gathered}\|AAz - TAz\| = \|AAz - ATz\| = \|AAz - AAz\| = 0 \\ \|AAz - SAz\| = \|AAz - AAz\| = 0\end{gathered}$$

and

$$\|\ Az - TAz\| = \|Az - AAz\|,$$

Making use of these, one arrives at

$$\|Az - AAz\|^2 \leq \delta\|z - Az\|^2,$$

Since $\delta < 1$, it is clear that $Az = AAz$

Finally, putting $Az = z_1$, we have

$$Az_1 = AAz = Az = z_1$$

Similarly, one finds that

$$Tz_1 = TAz = ATz = AAz = Az = z_1, \text{ and } \quad Sz_1 = z_1$$

So z_1 is a fixed point of S , T and A.

Next to show uniqueness of this common fixed point, let us suppose that z_2 is also a common fixed point of S and T and A other thanz_1. Then

$$Sz_2 = z_2, Tz_2 = z_2, \; Az_2 = z_2, \text{ with } \quad \|z_1 - z_2\| \neq 0$$

Hence, the inequality given in the hypothesis

$$\|z_1 - z_2\|^2 \leq \alpha\|Az - Sz\|\|AAz - SAz\| + \beta\|Az - Tz\|\|AAz - TAz\| + \gamma\|Az - Sz\|\|AAz - Sz\| + \delta\|z - Az\|^2$$

This leads to

$$\|z_1 - z_2\|^2 \leq \delta\|z_1 - z_2\|^2$$

But $\delta < 1$,and hence uniqueness occurs.

Theorem 2.3: let S and T be two continuous mappings of Hilbert space X into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping$A$ of X into SX ∩ TX , which commutes with S and T and satisfies the inequality,

$$\|Ax - Ay\|^{1/n} \leq \alpha\|Ax - Sx\|^{1\backslash n} + \beta\|Ay - Ty\|^{1\backslash n} + \gamma\|Ax - Sy\|^{1\backslash n} + \delta\|Ax - Ty\|^{1\backslash n} + \lambda\|Ax - Tx\|^{1\backslash n} + \varepsilon\|Ay - Sx\|^{1\backslash n} + \theta\|Ay - Sy\|^{1\backslash n} + \vartheta\|Ay - Tx\|^{1\backslash n}$$

For all *x*, *y* in *X*; where $\alpha, \beta, \gamma, \delta\ \lambda, \varepsilon, , \theta, \vartheta$ are non-negative reals with

$$0 < \alpha + \beta + \gamma + \delta + \lambda + \epsilon + \theta + \vartheta < 1.$$

Proof: First of all we prove that the existence of such a mapping A is necessary. For this assume,

$$Sz = z = Tz \text{ , for some } z \in X.$$

Define a mapping A of X in to X by $Ax = z$ for all $x \in X$. Then clearly, A is a continuous mapping of X into$SX \cap TX$. Since,$Sx, Tx \in X$, for all $x \in X$ and$Ax = z$, for all $x \in X$, one gets

$$ASx = z, SAx = Sz = z, ATx = z, TAx = Tz = z.$$

Hence, A commute with S and T. Now for any $\alpha, \beta, \gamma, \delta, \lambda, \epsilon, \theta, \vartheta$,with

$$0 < \alpha + \beta + \gamma + \delta + \lambda + \epsilon + \theta + \vartheta < 1.$$

it is noticed that

$$0 \leq \alpha\|Ax - Sx\|^{1\backslash n} + \beta\|Ay - Ty\|^{1\backslash n} + \gamma\|Ax - Sy\|^{1\backslash n} + \delta\|Ax - Ty\|^{1\backslash n} + \lambda\|Ax - Tx\|^{1\backslash n} + \varepsilon\|Ay - Sx\|^{1\backslash n} + \theta\|Ay - Sy\|^{1\backslash n} + \vartheta\|Ay - Tx\|^{1\backslash n}$$

for all $x, y \in X$. This proves the existence of such a mapping A is necessary.

To prove the sufficient part, a sequence $\{x_n\}$ is constructed as follows. Let $x_0 \in X$ be an arbitrary point.
Since $AX \subset SX$, we choose a point x_1 in x such that $Sx_1 = Ax_0$
Also, $AX \subset TX$ and hence we can select $x_2 \in X$ such that $Tx_2 = Ax_1$.
Continuing in this way, one obtains a sequence $\{x_n\}$ as follows:

$$Sx_{2n-1} = Ax_{2n-2}, \quad Tx_{2n} = Ax_{2n-1}, \quad n = 1,2,3 \ldots$$

To show that $\{Ax_n\}$ is a Cauchy sequence, one derives the inequality,

$$\begin{aligned}\|Ax_{2n+1} - Ax_{2n}\|^{1\backslash n} \leq\ & \alpha\|Ax_{2n+1} - Sx_{2n+1}\|^{1\backslash n} + \beta\|Ax_{2n} - Tx_{2n}\|^{1\backslash n} \\ & + \gamma\|Ax_{2n+1} - Sx_{2n}\|^{1\backslash n} + \delta\|Ax_{2n+1} - Tx_{2n}\|^{1\backslash n} \\ & + \lambda\|Ax_{2n+1} - Tx_{2n+1}\|^{1\backslash n} + \epsilon\|Ax_{2n} - Sx_{2n+1}\|^{1\backslash n} \\ & + \theta\|Ax_{2n} - Sx_{2n}\|^{1\backslash n} + \vartheta\|Ax_{2n} - Tx_{2n+1}\|^{1\backslash n} \\ =\ & (\alpha + \lambda)\|Ax_{2n+1} - Ax_{2n}\|^{1\backslash n} + (\beta + \theta)\|Ax_{2n} - Ax_{2n-1}\|^{1\backslash n} \\ & + (\gamma + \delta)\|Ax_{2n+1} - Ax_{2n-1}\|^{1\backslash n}\end{aligned}$$

which yields

$$\begin{aligned}\|Ax_{2n+1} - Ax_{2n}\|^{1\backslash n} \leq\ & (\alpha + \lambda)\|Ax_{2n+1} - Ax_{2n}\|^{1\backslash n} + (\beta + \theta)\|Ax_{2n} - Ax_{2n-1}\|^{1\backslash n} \\ & + (\gamma + \delta)2^{1\backslash n}[\|Ax_{2n+1} - Ax_{2n}\|^{1\backslash n} + \|Ax_{2n} - Ax_{2n-1}\|^{1\backslash n}]\end{aligned}$$

In view of the Yong's inequality (P.123, [18])
This leads to

$$\|Ax_{2n+1} - Ax_{2n}\|^{1\backslash n} \leq \frac{[(\beta+\theta)+(\gamma+\delta)2^{1\backslash n}]}{(1-[(\alpha+\lambda)+(\gamma+\delta)2^{1\backslash n}])}\|Ax_{2n} - Ax_{2n-1}\|^{1\backslash n} \tag{7}$$

Similarly, one derives that

$$\|Ax_{2n} - Ax_{2n-1}\|^{1\backslash n} \leq \frac{[(\beta+\theta)+(\gamma+\delta)2^{1\backslash n}]}{(1-[(\alpha+\lambda)+(\gamma+\delta)2^{1\backslash n}])}\|Ax_{2n-1} - Ax_{2n-2}\|^{1\backslash n} \tag{8}$$

Hence, one concludes that

$$\|Ax_{n+1} - Ax_n\|^{1\backslash n} \leq \frac{[(\beta+\theta)+(\gamma+\delta)2^{1\backslash n}]}{(1-[(\alpha+\lambda)+(\gamma+\delta)2^{1\backslash n}])}\|Ax_n - Ax_{n-1}\|^{1\backslash n}$$

which implies that

$$\|Ax_{n+1} - Ax_n\| \leq \left(\frac{[(\beta+\theta)+(\gamma+\delta)2^{1\backslash n}]}{(1-[(\alpha+\lambda)+(\gamma+\delta)2^{1\backslash n}])}\right)^n \|Ax_n - Ax_{n-1}\|$$

If

$$c = \left(\frac{[(\beta+\theta)+(\gamma+\delta)2^{1\backslash n}]}{(1-[(\alpha+\lambda)+(\gamma+\delta)2^{1\backslash n}])}\right)^n$$

then, $0 < c < 1$. As we have

$$\left(1 - [(\alpha+\lambda)+(\gamma+\delta)2^{1\backslash n}]\right) < 1\,.$$

from (7) and (8), one infers that

$$\|Ax_{n+1} - Ax_n\| \le c\,\|Ax_n - Ax_{n-1}\|$$
$$\le c^2\|Ax_{n-1} - Ax_{n-2}\|$$
$$\ldots\ldots\ldots\ldots\ldots..$$
$$\le c^n\|Ax_1 - Ax_0\| \text{ , for large n.}$$

Now, it follows that $\{Ax_n\}$ is a Cauchy sequence and so it has a limit point x in X. Since sequences $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ are sub sequence of $\{Ax_n\}$, they have the same limit z. As, S and A are commuting mapping, one gets

$$Sz = \lim_{n\to\infty} SAx_{2n+1} = \lim_{n\to\infty} \mathrm{A}Sx_{2n+1} = Az$$

Similarly, one finds that $Tz = Az$
which gives us,

$$Tz = Az = Sz \qquad (9)$$

It now, follows that

$$\|Az - AAz\|^{1\backslash n} \le \alpha\|Az - Sz\|^{1\backslash n} + \beta\|AAz - TAz\|^{1\backslash n} + \gamma\|Az - SAz\|^{1\backslash n}$$
$$+\delta\|Az - TAz\|^{1\backslash n} + \lambda\|Az - Tz\|^{1\backslash n} + \varepsilon\|AAz - Sz\|^{1\backslash n}$$
$$+\theta\|AAz - SAz\|^{1\backslash n} + \vartheta\|AAz - Tz\|^{1\backslash n}$$

The commutativity of A and S and T present the following inequality in view of (9)

$$\|AAz - TAz\| = \|AAz - ATz\| = \|AAz - AAz\| = 0,$$

$$\|AAz - SAz\| = \|AAz - ASz\| = \|AAz - AAz\| = 0,$$

$$\| Az - SAz\| = \|Az - AAz\|, \text{ And } \|Az - TAz\| = \|Az - AAz\|.$$

Making use of these, one arrives at

$$\|Az - AAz\|^{1\backslash n} \le (\gamma + \delta + \varepsilon + \vartheta)\|Az - AAz\|^{1\backslash n}$$

Since $\gamma + \delta + \varepsilon + \vartheta < 1$, we must have, $Az = AAz$.
Finally, putting $Az = z_1$, we have $Az_1 = AAz = Az = z_1$
Similarly, one concludes that

$$Tz_1 = TAz = ATz = AAz = Az = z_1, \text{ and } \quad Sz_1 = z_1$$

So z_1 is a fixed point of S , T and A.
Next to show that uniqueness of this common fixed point, let us suppose that z_2 is also a common fixed point of S, T and A other then z_1.Then

$$Sz_2 = z_2, Tz_2 = z_2, \; Az_2 = z_2, \text{ with} \|z_1 - z_2\| \ne 0$$

Hence, one arrives at the inequality

$$\|z_1 - z_2\|^{1\backslash n} \le \alpha\|z_1 - z_1\|^{1\backslash n} + \beta\|z_2 - z_2\|^{1\backslash n} + \gamma\|z_1 - z_2\|^{1\backslash n}$$

$$+\delta\|z_1 - z_2\|^{1\backslash n} + \lambda\|z_1 - z_1\|^{1\backslash n} + \varepsilon\|z_2 - z_1\|^{1\backslash n}$$
$$+\theta\|z_2 - z_2\|^{1\backslash n} + \vartheta\|z_2 - z_1\|^{1\backslash n}$$

Thus,

$$\|z_1 - z_2\|^{1\backslash n} \leq (\gamma + \delta + \varepsilon + \vartheta)\|z_1 - z_2\|^{1\backslash n}$$

But $\gamma + \delta + \varepsilon + \vartheta < 1$,

And hence the uniqueness is in place.

3. Conclusion

The methods adopted in the proofs of fixed point theorems reveal that yet there are various directions in which the Banach's fixed point theorem can be refined and extended retaining the convergence. It is strongly felt that some of the results presented here can be generalized and modified from rate of convergence point of view.

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