Positive Solutions for a System of Second-Order Differential Equations with Integral Boundary Conditions

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Abstract: In this paper, we study a system of nonlinear second-order ordinary differential equations with Riemann-Stieltjes integral boundary conditions which contain some positive constants. By using the Schauder fixed point theorem and some properties of the associated Green's functions, we show that this problem has at least one positive solution for sufficiently small constants. Then, we give sufficient conditions for the nonexistence of positive solutions. Similar results for other three boundary value problems are also presented.

Key words: Existence, integral boundary conditions, nonexistence, positive solutions, second-order differential equations.

1. Introduction

Boundary value problems with positive solutions describe many phenomena in the applied sciences such as the nonlinear diffusion generated by nonlinear sources, thermal ignition of gases and concentration in chemical or biological problems (see [1]-[6]). Problems with integral boundary conditions arise in thermal conduction problems ([7]), semiconductor problems ([8]) and hydrodynamic problems ([9]). In the last decades, many authors investigated differential equations or systems of differential equations with integral boundary conditions, for which they prove the existence, multiplicity and nonexistence of positive solutions by using various methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder type, fixed point index theory and coincidence degree theory (see for example [10]-[17]).

We consider the system of nonlinear second-order ordinary differential equations

$$(S) \qquad \begin{cases} u''(t) + p(t)f\big(v(t)\big) = 0, & t \in (0,1), \\ v''(t) + q(t)g\big(u(t)\big) = 0, & t \in (0,1), \end{cases}$$

with the integral boundary conditions

$$(BC) \qquad \begin{cases} \alpha u(0) - \beta u'(0) = \int_0^1 u(s)\, dH_1(s), \quad \gamma u(1) + \delta u'(1) = \int_0^1 u(s)\, dH_2(s) + a_0, \\ \tilde{\alpha} v(0) - \tilde{\beta} v'(0) = \int_0^1 v(s)\, dK_1(s), \quad \tilde{\gamma} v(1) + \tilde{\delta} v'(1) = \int_0^1 v(s)\, dK_2(s) + b_0, \end{cases}$$

where the above integrals are Riemann-Stieltjes integrals, and a_0 and b_0 are positive constants. The boundary conditions (BC) include multi-point and integral boundary conditions and sum of these in a single framework.

By using the Schauder fixed point theorem, we shall prove the existence of positive solutions of problem $(S)-(BC)$. By a positive solution of $(S)-(BC)$ we mean a pair of functions $(u,v) \in (C^2([0,1],[0,\infty)))^2$ satisfying (S) and (BC), with $u(t)>0$ and $v(t)>0$ for all $t\in(0,1]$. We shall also give sufficient conditions for the nonexistence of the positive solutions for this problem. The particular case of the above problem when $\gamma=\tilde{\gamma}=1$, $\delta=\tilde{\delta}=0$, the functions H_1 and K_1 are constant, and H_2 and K_2 are step functions has been investigated in [18]. We also mention pape [19], where the authors studied the existence and nonexistence of positive solutions for the m-point boundary value problem on time scales $u^{\Delta\nabla}(t)+a(t)f\big(u(t)\big)=0,\ t\in(0,T),\ \beta u(0)-\gamma u^{\Delta}(0)=0,\ u(T)-\sum_{i=1}^{m-2} a_i u(\xi_i)=b,\ m\geq 3$ and $b>0$.

We present the assumptions that we shall use in the sequel.

(J1) $\alpha,\beta,\gamma,\delta,\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}\ \in[0,\infty)$.

(J2) $H_1,\ H_2,\ K_1,\ K_2\colon[0,1]\to\mathbb{R}$ are nondecreasing functions.

(J3) $\alpha-\int_0^1 dH_1(\tau)>0,\ \gamma-\int_0^1 dH_2(\tau)>0,\ \tilde{\alpha}-\int_0^1 dK_1(\tau)>0,\ \tilde{\gamma}-\int_0^1 dK_2(\tau)>0.$

(J4) The functions $p,q\colon[0,1]\to[0,\infty)$ are continuous and there exist $t_1,t_2\in(0,1)$such that $p(t_1)>0$, $q(t_2)>0$.

(J5) $f,g\colon[0,\infty)\to[0,\infty)$ are continuous functions and there exists $c_0>0$ such that $f(u)<\frac{c_0}{L}$, $g(u)<\frac{c_0}{L}$ for all $u\in[0,c_0]$, where $L=\max\left\{\int_0^1 p(s)J_1(s)ds,\int_0^1 q(s)J_2(s)ds\right\}$ and $J_1,\ J_2$ are defined in Section 2.

(J6) $f,g\colon[0,\infty)\to[0,\infty)$ are continuous functions and satisfy the conditions $\lim_{u\to\infty}\frac{f(u)}{u}=\infty,\ \lim_{u\to\infty}\frac{g(u)}{u}=\infty.$

In the proof of our main existence result, we shall use the Schauder fixed point theorem which we present now.

Theorem 1.1 Let X be a Banach space and $Y\subset X$ a nonempty, bounded, convex and closed subset. If the operator $A\colon Y\to Y$ is completely continuous, then A has at least one fixed point.

The paper is organized as follows. Section 2 contains some auxiliary results. The main theorems are presented in Section 3, and finally in Section 4 an example is given to support the new results.

2. Auxiliary Results

We consider the second-order differential equations with integral boundary conditions

$$u''(t)+y(t)=0,\ \ t\in(0,1), \tag{1}$$

$$\alpha u(0)-\beta u'(0)=\int_0^1 u(s)\,dH_1(s),\quad \gamma u(1)+\delta u'(1)=\int_0^1 u(s)\,dH_2(s), \tag{2}$$

and

$$v''(t)+\tilde{y}(t)=0,\ \ t\in(0,1), \tag{3}$$

$$\tilde{\alpha} v(0)-\tilde{\beta} v'(0)=\int_0^1 v(s)\,dK_1(s),\quad \tilde{\gamma} v(1)+\tilde{\delta} v'(1)=\int_0^1 v(s)\,dK_2(s). \tag{4}$$

For $\alpha,\beta,\gamma,\delta\in\mathbb{R}, |\alpha|+|\beta|\neq 0, |\gamma|+|\delta|\neq 0,$ we denote by $\psi,\phi,\tilde{\psi}$ and $\tilde{\phi}$ the solutions of the following boundary value problems

$$\psi''(t)=0,\ \ 0<t<1;\ \ \psi(0)=\beta,\ \ \psi'(0)=\alpha,$$

$$\phi''(t)=0,\ \ 0<t<1;\ \ \phi(1)=\delta,\ \ \phi'(1)=-\gamma,$$

$$\tilde{\psi}''(t)=0,\ \ 0<t<1;\ \tilde{\psi}(0)=\tilde{\beta},\ \ \tilde{\psi}'(0)=\tilde{\alpha},$$

$$\tilde{\phi}''(t)=0,\ \ 0<t<1;\ \tilde{\phi}(1)=\tilde{\delta},\ \ \tilde{\phi}'(1)=-\tilde{\gamma},$$

respectively, that is the functions $\psi(t)=\alpha t+\beta$, $\phi(t)=-\gamma t+\gamma+\delta$, $\tilde{\psi}(t)=\tilde{\alpha}t+\tilde{\beta}$ and $\tilde{\phi}(t)=-\tilde{\gamma}t+\tilde{\gamma}+\tilde{\delta}$ for all $t\in[0,1]$. We denote by $\tau_1=\alpha\gamma+\alpha\delta+\beta\gamma$ and $\tau_2=\tilde{\alpha}\tilde{\gamma}+\tilde{\alpha}\tilde{\delta}+\tilde{\beta}\tilde{\gamma}$.

By using assumptions (J1)-(J3), we deduce that $\alpha>0$, $\gamma>0$, $\tilde{\alpha}>0$ and $\tilde{\gamma}>0$, and so $\alpha+\beta>0$, $\gamma+\delta>0$, $\alpha+\gamma>0$, $\tilde{\alpha}+\tilde{\beta}>0$, $\tilde{\gamma}+\tilde{\delta}>0$, $\tilde{\alpha}+\tilde{\gamma}>0$. Besides

$$\tau_1-\int_0^1\phi(s)\,dH_1(s)=(\gamma+\delta)\left(\alpha-\int_0^1 dH_1(s)\right)+\beta\gamma+\gamma\int_0^1 s\,dH_1(s)>0,$$

$$\tau_1-\int_0^1\psi(s)\,dH_2(s)=(\alpha+\beta)\left(\gamma-\int_0^1 dH_2(s)\right)+\alpha\delta+\alpha\int_0^1(1-s)\,dH_2(s)>0,$$

$$\tau_2-\int_0^1\tilde{\phi}(s)\,dK_1(s)=\left(\tilde{\gamma}+\tilde{\delta}\right)\left(\tilde{\alpha}-\int_0^1 dK_1(s)\right)+\tilde{\beta}\tilde{\gamma}+\tilde{\gamma}\int_0^1 s\,dK_1(s)>0,$$

$$\tau_2-\int_0^1\tilde{\psi}(s)\,dK_2(s)=\left(\tilde{\alpha}+\tilde{\beta}\right)\left(\tilde{\gamma}-\int_0^1 dK_2(s)\right)+\tilde{\alpha}\tilde{\delta}+\tilde{\alpha}\int_0^1(1-s)\,dK_2(s)>0.$$

In addition, we denote

$$\Delta_1=\left(\tau_1-\int_0^1\psi(s)\,dH_2(s)\right)\left(\tau_1-\int_0^1\phi(s)\,dH_1(s)\right)-\left(\int_0^1\psi(s)\,dH_1(s)\right)\left(\int_0^1\phi(s)\,dH_2(s)\right),$$

$$\Delta_2=\left(\tau_2-\int_0^1\tilde{\psi}(s)\,dK_2(s)\right)\left(\tau_2-\int_0^1\tilde{\phi}(s)\,dK_1(s)\right)-\left(\int_0^1\tilde{\psi}(s)dK_1(s)\right)\left(\int_0^1\tilde{\phi}(s)dK_2(s)\right).$$

After some computations, we obtain $\Delta_1=\tau_1\widetilde{\Delta}_1>0$, $\Delta_2=\tau_2\widetilde{\Delta}_2>0$, where

$$\widetilde{\Delta}_1=\left(\beta+\int_0^1 s\,dH_1(s)\right)\left(\gamma-\int_0^1 dH_2(s)\right)+\left(\alpha-\int_0^1 dH_1(s)\right)\left(\gamma+\delta-\int_0^1 s\,dH_2(s)\right),$$

$$\widetilde{\Delta}_2=\left(\tilde{\beta}+\int_0^1 s\,dK_1(s)\right)\left(\tilde{\gamma}-\int_0^1 dK_2(s)\right)+\left(\tilde{\alpha}-\int_0^1 dK_1(s)\right)\left(\tilde{\gamma}+\tilde{\delta}-\int_0^1 s\,dK_2(s)\right).$$

In this way, we conclude that assumptions (I2)-(I4) from [20] are satisfied. Hence, all the auxiliary results

Lemmas 2.1-2.7 from [20] for problem (1)-(2), and the corresponding auxiliary results for problem (3)-(4) are satisfied.

Therefore, under assumptions (J1)-(J3) and for $y, \tilde{y} \in C(0,1) \cap L^1(0,1)$, the solutions u and v for problems (1)-(2) and (3)-(4), respectively, are given by $u(t) = \int_0^1 G_1(t,s)y(s)\,ds$ and $v(t) = \int_0^1 G_2(t,s)\tilde{y}(s)ds$ for all $t \in [0,1]$, where G_1 and G_2 are given by

$$G_1(t,s) = g_1(t,s) + \frac{1}{\Delta_1}\Big[(\alpha t+\beta)\int_0^1 (-\gamma s+\gamma+\delta)\,dH_2(s) + (-\gamma t+\gamma+\delta)\Big(\alpha\gamma+\alpha\delta+\beta\gamma - \int_0^1 (\alpha s+\beta)\,dH_2(s)\Big)\Big]\int_0^1 g_1(\tau,s)\,dH_1(\tau) + \frac{1}{\Delta_1}\Big[(\alpha t+\beta)\Big(\alpha\gamma+\alpha\delta+\beta\gamma - \int_0^1 (-\gamma s+\gamma+\delta)\,dH_1(s)\Big) + (-\gamma t+\gamma+\delta)\int_0^1 (\alpha s+\beta)\,dH_1(s)\Big]\int_0^1 g_1(\tau,s)\,dH_2(\tau),$$

$$G_2(t,s) = g_2(t,s) + \frac{1}{\Delta_2}\Big[(\tilde{\alpha} t+\tilde{\beta})\int_0^1 (-\tilde{\gamma} s+\tilde{\gamma}+\tilde{\delta})\,dK_2(s) + (-\tilde{\gamma} t+\tilde{\gamma}+\tilde{\delta})\Big(\tilde{\alpha}\tilde{\gamma}+\tilde{\alpha}\tilde{\delta}+\tilde{\beta}\tilde{\gamma} - \int_0^1 (\tilde{\alpha} s+\tilde{\beta})\,dK_2(s)\Big)\Big]\int_0^1 g_2(\tau,s)\,dK_1(\tau) + \frac{1}{\Delta_2}\Big[(\tilde{\alpha} t+\tilde{\beta})\Big(\tilde{\alpha}\tilde{\gamma}+\tilde{\alpha}\tilde{\delta}+\tilde{\beta}\tilde{\gamma} - \int_0^1 (-\tilde{\gamma} s+\tilde{\gamma}+\tilde{\delta})\,dK_1(s)\Big) + (-\tilde{\gamma} t+\tilde{\gamma}+\tilde{\delta})\int_0^1 (\tilde{\alpha} s+\tilde{\beta})\,dK_1(s)\Big]\int_0^1 g_2(\tau,s)dK_2(\tau),$$

and

$$g_1(t,s) = \frac{1}{\alpha\gamma+\alpha\delta+\beta\gamma}\begin{cases}(-\gamma t+\gamma+\delta)(\alpha s+\beta), & 0 \le s \le t \le 1,\\ (-\gamma s+\gamma+\delta)(\alpha t+\beta), & 0 \le t \le s \le 1,\end{cases}$$

$$g_2(t,s) = \frac{1}{\tilde{\alpha}\tilde{\gamma}+\tilde{\alpha}\tilde{\delta}+\tilde{\beta}\tilde{\gamma}}\begin{cases}(-\tilde{\gamma} t+\tilde{\gamma}+\tilde{\delta})(\tilde{\alpha} s+\tilde{\beta}), & 0 \le s \le t \le 1,\\ (-\tilde{\gamma} s+\tilde{\gamma}+\tilde{\delta})(\tilde{\alpha} t+\tilde{\beta}), & 0 \le t \le s \le 1.\end{cases}$$

By Lemma 2.6 from [20], the Green's functions G_1 and G_2 above satisfy the inequalities $G_1(t,s) \le J_1(s)$, $In\ addition, by\ Lemma\ 2.7\ from$ [2], $we\ deduce\ that\ for\ G_2(t,s) \le J_2(s)$ for all $(t,s) \in [0,1]\times[0,1]$, and for every $\sigma \in \left(0,\frac{1}{2}\right)$, we have $min_{t\in[\sigma,1-\sigma]}G_1(t,s) \ge \nu_1 J_1(s)$ and $min_{t\in[\sigma,1-\sigma]}G_2(t,s) \ge \nu_2 J_2(s)$ for all $s \in [0,1]$, where

$$J_1(s) = g_1(s,s) + \frac{1}{\Delta_1}\Big[(\alpha+\beta)\int_0^1 (-\gamma s+\gamma+\delta)\,dH_2(s) + (\gamma+\delta)\Big(\alpha\gamma+\alpha\delta+\beta\gamma - \int_0^1 (\alpha s+\beta)\,dH_2(s)\Big)\Big]\int_0^1 g_1(\tau,s)\,dH_1(\tau) + \frac{1}{\Delta_1}\Big[(\alpha+\beta)\Big(\alpha\gamma+\alpha\delta+\beta\gamma - \int_0^1 (-\gamma s+\gamma+\delta)\,dH_1(s)\Big) + (\gamma+\delta)\int_0^1 (\alpha s+\beta)\,dH_1(s)\Big]\int_0^1 g_1(\tau,s)\,dH_2(\tau),$$

$$J_2(s) = g_2(s,s) + \frac{1}{\Delta_2}\left[(\tilde{\alpha}+\tilde{\beta})\int_0^1 (-\tilde{\gamma}s+\tilde{\gamma}+\tilde{\delta})\,dK_2(s) + (\tilde{\gamma}+\tilde{\delta})\left(\tilde{\alpha}\tilde{\gamma}+\tilde{\alpha}\tilde{\delta}+\tilde{\beta}\tilde{\gamma}\right.\right.$$
$$\left.\left.-\int_0^1 (\tilde{\alpha}s+\tilde{\beta})\,dK_2(s)\right)\right]\int_0^1 g_2(\tau,s)\,dK_1(\tau) + \frac{1}{\Delta_2}\left[(\tilde{\alpha}+\tilde{\beta})\left(\tilde{\alpha}\tilde{\gamma}+\tilde{\alpha}\tilde{\delta}+\tilde{\beta}\tilde{\gamma}\right.\right.$$
$$\left.\left.-\int_0^1 (-\tilde{\gamma}s+\tilde{\gamma}+\tilde{\delta})\,dK_1(s)\right) + (\tilde{\gamma}+\tilde{\delta})\int_0^1 (\tilde{\alpha}s+\tilde{\beta})\,dK_1(s)\right]\int_0^1 g_2(\tau,s)\,dK_2(\tau),$$

and $\nu_1 = \min\left\{\frac{\gamma\sigma+\delta}{\gamma+\delta}, \frac{\alpha\sigma+\beta}{\alpha+\beta}\right\}$, $\nu_2 = \min\left\{\frac{\tilde{\gamma}\sigma+\tilde{\delta}}{\tilde{\gamma}+\tilde{\delta}}, \frac{\tilde{\alpha}\sigma+\tilde{\beta}}{\tilde{\alpha}+\tilde{\beta}}\right\}$.

In addition, by Lemma 2.7 from [20], we deduce that for $\sigma \in (0,\frac{1}{2})$ and $y, \tilde{y} \in C(0,1) \cap L^1(0,1)$, with $y(t) \geq 0$, $\tilde{y}(t) \geq 0$ for all $t \in (0,1)$, the solutions u and v of problems (1)-(2) and (3)-(4), respectively, satisfy the inequalities $inf_{t\in[\sigma,1-\sigma]}u(t) \geq \nu_1 sup_{t'\in[0,1]}u(t')$ and $inf_{t\in[\sigma,1-\sigma]}v(t) \geq \nu_2 sup_{t'\in[0,1]}v(t')$.

3. Main Results

Our first theorem is the following existence result for problem $(S)-(BC)$.

Theorem 3.1 Assume that assumptions (J1)-(J5) hold. Then problem $(S)-(BC)$ has at least one positive solution for $a_0 > 0$ and $b_0 > 0$ sufficiently small.

Proof. By (J4) and (J5), we deduce that $\int_0^1 p(s)J_1(s)ds > 0$ and $\int_0^1 q(s)J_2(s)ds > 0$, that is the constant L from (J5) is positive.

We consider the problems

$$\begin{cases} h''(t) = 0, \ \ t \in (0,1), \\ \alpha h(0) - \beta h'(0) = \int_0^1 h(s)\,dH_1(s), \ \ \gamma h(1) + \delta h'(1) = \int_0^1 h(s)\,dH_2(s) + 1, \end{cases} \tag{5}$$

$$\begin{cases} k''(t) = 0, \ \ t \in (0,1), \\ \tilde{\alpha} k(0) - \tilde{\beta} k'(0) = \int_0^1 k(s)\,dK_1(s), \ \ \tilde{\gamma} k(1) + \tilde{\delta} k'(1) = \int_0^1 k(s)\,dK_2(s) + 1. \end{cases} \tag{6}$$

The above problems (5) and (6) have the solutions

$$h(t) = \frac{\tau_1}{\Delta_1}\left[t\left(\alpha - \int_0^1 dH_1(s)\right) + \beta + \int_0^1 s\,dH_1(s)\right], \qquad t \in [0,1], \tag{7}$$
$$k(t) = \frac{\tau_2}{\Delta_2}\left[t\left(\tilde{\alpha} - \int_0^1 dK_1(s)\right) + \tilde{\beta} + \int_0^1 s\,dK_1(s)\right], \qquad t \in [0,1],$$

respectively.

By assumptions (J1)-(J3) we obtain $h(t) > 0$ and $k(t) > 0$ for all $t \in (0,1]$.

We define the functions $x(t)$ and $y(t)$, $t \in [0,1]$ by

$$x(t) = u(t) - a_0 h(t), \qquad y(t) = v(t) - b_0 k(t), \qquad t \in [0,1],$$

where (u,v) is a solution of $(S)-(BC)$. Then $(S)-(BC)$ can be equivalently written as

$$\begin{cases} x''(t) + p(t)f\big(y(t) + b_0 k(t)\big) = 0, \ \ t \in (0,1), \\ y''(t) + q(t)g\big(x(t) + a_0 h(t)\big) = 0, \ \ t \in (0,1) \end{cases} \tag{8}$$

with the boundary conditions

$$\begin{cases}\alpha x(0)-\beta x'(0)=\int_0^1 x(s)\,dH_1(s), & \gamma x(1)+\delta x'(1)=\int_0^1 x(s)\,dH_2(s),\\ \tilde{\alpha} y(0)-\tilde{\beta} y'(0)=\int_0^1 y(s)\,dK_1(s), & \tilde{\gamma} y(1)+\tilde{\delta} y'(1)=\int_0^1 y(s)\,dK_2(s).\end{cases} \tag{9}$$

Using the Green's functions G_1, G_2 from Section 2, a pair (x, y) is a solution of problem (8)-(9) if and only if (x, y) is a solution for the nonlinear integral equations

$$\begin{cases} x(t)=\int_0^1 G_1(t,s)p(s)f\left(\int_0^1 G_2(s,\tau)q(\tau)(g(x(\tau)+a_0h(\tau))\,d\tau+b_0k(s)\right)ds,\\ y(t)=\int_0^1 G_2(t,s)q(s)g(x(s)+a_0h(s))\,ds, \quad 0\le t\le 1,\end{cases} \tag{10}$$

where $h(t)$, $k(t)$, $t\in[0,1]$ are given by (7).

We consider the Banach space $X=C([0,1])$ with the supremum norm $\|\cdot\|$ and define the set

$$E=\{x\in C([0,1]), \quad 0\le x(t)\le c_0, \quad \forall t\in[0,1]\}\subset X.$$

We also define the operator $S: E\to X$ by

$$(Sx)(t)=\int_0^1 G_1(t,s)p(s)f\left(\int_0^1 G_2(s,\tau)q(\tau)(g(x(\tau)+a_0h(\tau))\,d\tau+b_0k(s)\right)ds, \quad 0\le t\le 1, x\in E.$$

For sufficiently small $a_0>0$ and $b_0>0$, by (J5), we deduce

$$f(y(t)+b_0k(t))\le\frac{c_0}{L}, \qquad g(x(t)+a_0h(t))\le\frac{c_0}{L}, \quad \forall t\in[0,1], \qquad \forall x,y\in E.$$

Then, by using some remarks from Section 2, we obtain $(Sx)(t)\ge 0$ for all $t\in[0,1]$ and $x\in E$. In addition, for all $x\in E$, we have

$$\int_0^1 G_2(s,\tau)q(\tau)(g(x(\tau)+a_0h(\tau))\,d\tau\le\int_0^1 J_2(\tau)q(\tau)(g(x(\tau)+a_0h(\tau))\,d\tau$$
$$\le\frac{c_0}{L}\int_0^1 J_2(\tau)q(\tau)\,d\tau\le c_0, \forall s\in[0,1],$$

and

$$(Sx)(t)\le\int_0^1 J_1(s)p(s)f\left(\int_0^1 G_2(s,\tau)q(\tau)(g(x(\tau)+a_0h(\tau))\,d\tau+b_0k(s)\right)ds$$
$$\le\frac{c_0}{L}\int_0^1 J_1(s)p(s)\,ds\le c_0, \forall t\in[0,1].$$

Therefore $S(E)\subset E$.

Using standard arguments, we deduce that S is completely continuous (S is compact, that is for any bounded set $B \subset E, S(B) \subset E$ is relatively compact by Arzèla-Ascoli theorem, and S is continuous). By Theorem 1.1, we conclude that S has a fixed point $x \in E$. This element together with y given by (10) represents a solution for (8)-(9). This shows that our problem $(S)-(BC)$ has a positive solution (u,v) with $u = x + a_0 h,\ v = y + b_0 k$ for sufficiently small $a_0 > 0$ and $b_0 > 0$.

In what follows, we present sufficient conditions for the nonexistence of positive solutions of problem $(S)-(BC)$.

Theorem 3.2 Assume that assumptions (J1)-(J4) and (J6) hold. Then problem $(S)-(BC)$ has no positive solution for a_0 and b_0 sufficiently large.

Proof. We suppose that (u,v) is a positive solution of $(S)-(BC)$. Then (x,y) with $x = u - a_0 h,\ y = v - b_0 k$ is a solution for (8)-(9), where h and k are the solutions of problems (5) and (6), respectively, (given by (7)). By (J4) there exists $\sigma \in (0, \frac{1}{2})$ such that $t_1, t_2 \in (\sigma, 1-\sigma)$, and then $\int_\sigma^{1-\sigma} p(s) J_1(s)\, ds >$ $0, \quad \sigma 1-\sigma q s J 2 s\ ds>0.$ Now by using some remarks from Section 2, we have $xt\geq 0, \quad y(t)\geq 0$ for all $t\in 0,1,$ and $min_{t\in[\sigma,1-\sigma]} x(t) \geq \upsilon_1 \|x\|$ and $min_{t\in[\sigma,1-\sigma]} y(t) \geq \upsilon_2 \|y\|$.

Using now (7), we deduce that

$$\min_{t\in[\sigma,1-\sigma]} h(t) = h(\sigma) = \frac{h(\sigma)}{h(1)}\|h\|, \qquad \min_{t\in[\sigma,1-\sigma]} k(t) = k(\sigma) = \frac{k(\sigma)}{k(1)}\|k\|.$$

Therefore, we obtain

$$\min_{t\in[\sigma,1-\sigma]} (x(t) + a_0 h(t)) \geq \upsilon_1 \|x\| + a_0 \frac{h(\sigma)}{h(1)} \|h\| \geq r_1 (\|x\| + a_0 \|h\|) \geq r_1 \|x + a_0 h\|,$$

$$\min_{t\in[\sigma,1-\sigma]} (y(t) + b_0 k(t)) \geq \upsilon_2 \|y\| + b_0 \frac{k(\sigma)}{k(1)} \|k\| \geq r_2 (\|y\| + b_0 \|k\|) \geq r_2 \|y + b_0 k\|,$$

where

$$r_1 = \min\left\{\upsilon_1, \frac{h(\sigma)}{h(1)}\right\} = \min\left\{\upsilon_1, \frac{\sigma\left(\alpha - \int_0^1 dH_1(s)\right) + \beta + \int_0^1 s\, dH_1(s)}{\alpha - \int_0^1 dH_1(s) + \beta + \int_0^1 s\, dH_1(s)}\right\},$$

$$r_2 = \min\left\{\upsilon_2, \frac{k(\sigma)}{k(1)}\right\} = \min\left\{\upsilon_2, \frac{\sigma\left(\tilde{\alpha} - \int_0^1 dK_1(s)\right) + \tilde{\beta} + \int_0^1 s\, dK_1(s)}{\tilde{\alpha} - \int_0^1 dK_1(s) + \tilde{\beta} + \int_0^1 s\, dK_1(s)}\right\}.$$

We now consider $R = \left(\min\left\{\upsilon_2 r_1 \int_\sigma^{1-\sigma} q(s) J_2(s)\, ds,\ \upsilon_1 r_2 \int_\sigma^{1-\sigma} p(s) J_1(s)\, ds\ \right\}\right)^{-1} > 0$.

By (J6), for R defined above, we conclude that there exists $M > 0$ such that $f(u) > 2Ru,\ g(u) > 2Ru$ for all $u \geq M$. We consider $a_0 > 0$ and $b_0 > 0$ sufficiently large such that

$$\min_{t\in[\sigma,1-\sigma]} (x(t) + a_0 h(t)) \geq M, \min_{t\in[\sigma,1-\sigma]} (y(t) + b_0 k(t)) \geq M.$$

By (J4), (8), (9) and the above inequalities, we deduce that $\|x\| > 0$ and $\|y\| > 0$.

Now by the above considerations, we have

$$y(\sigma)=\int_0^1 G_2(\sigma,s)q(s)g\big(x(s)+a_0h(s)\big)\,ds \ge \upsilon_2\int_0^1 J_2(s)q(s)g\big(x(s)+a_0h(s)\big)\,ds$$
$$\ge \upsilon_2\int_\sigma^{1-\sigma} J_2(s)q(s)g\big(x(s)+a_0h(s)\big)\,ds \ge 2R\upsilon_2\int_\sigma^{1-\sigma} J_2(s)q(s)\big(x(s)+a_0h(s)\big)\,ds$$
$$\ge 2R\upsilon_2\int_\sigma^{1-\sigma} J_2(s)q(s)\min_{\tau\in[\sigma,1-\sigma]}\big(x(\tau)+a_0h(\tau)\big)\,ds$$
$$\ge 2R\upsilon_2 r_1\int_\sigma^{1-\sigma} J_2(s)q(s)\|x+a_0h\|\,ds \ge 2\|x+a_0h\| \ge 2\|x\|.$$

Therefore we obtain

$$\|x\| \le \frac{y(\sigma)}{2} \le \frac{\|y\|}{2}. \tag{11}$$

In a similar manner, we deduce

$$x(\sigma)=\int_0^1 G_1(\sigma,s)p(s)f\big(y(s)+b_0k(s)\big)\,ds \ge \upsilon_1\int_0^1 J_1(s)p(s)(f\big(y(s)+b_0k(s)\big)\,ds$$
$$\ge \upsilon_1\int_\sigma^{1-\sigma} J_1(s)p(s)f\big(y(s)+b_0k(s)\big)\,ds \ge 2R\upsilon_1\int_\sigma^{1-\sigma} J_1(s)p(s)\big(y(s)+b_0k(s)\big)\,ds$$
$$\ge 2R\upsilon_1\int_\sigma^{1-\sigma} J_1(s)p(s)\min_{\tau\in[\sigma,1-\sigma]}\big(y(\tau)+a_0k(\tau)\big)ds$$
$$\ge 2R\upsilon_1 r_2\int_\sigma^{1-\sigma} J_1(s)p(s)\|y+b_0k\|\,ds \ge 2\|y+b_0k\| \ge 2\|y\|.$$

So, we obtain

$$\|y\| \le \frac{x(\sigma)}{2} \le \frac{\|x\|}{2}. \tag{12}$$

By (11) and (12), we conclude that $\|x\| \le \frac{\|y\|}{2} \le \frac{\|x\|}{4}$, which is a contradiction, because $\|x\|>0$. Then, for a_0 and b_0 sufficiently large, our problem $(S)-(BC)$ has no positive solution.

Similar results as Theorems 3.1 and 3.2 can be obtained if instead of boundary conditions (BC) we have

$$(BC_1)\quad \begin{cases} \alpha u(0)-\beta u'(0)=\int_0^1 u(s)\,dH_1(s)+a_0, & \gamma u(1)+\delta u'(1)=\int_0^1 u(s)\,dH_2(s),\\ \tilde\alpha v(0)-\tilde\beta v'(0)=\int_0^1 v(s)\,dK_1(s)+b_0, & \tilde\gamma v(1)+\tilde\delta v'(1)=\int_0^1 v(s)\,dK_2(s),\end{cases}$$

or

$$(BC_2)\quad \begin{cases} \alpha u(0)-\beta u'(0)=\int_0^1 u(s)\,dH_1(s)+a_0, & \gamma u(1)+\delta u'(1)=\int_0^1 u(s)\,dH_2(s),\\ \tilde\alpha v(0)-\tilde\beta v'(0)=\int_0^1 v(s)\,dK_1(s), & \tilde\gamma v(1)+\tilde\delta v'(1)=\int_0^1 v(s)\,dK_2(s)+b_0,\end{cases}$$

or

$$(BC_3)\qquad \begin{cases} \alpha u(0)-\beta u'(0)=\int_0^1 u(s)\,dH_1(s), \quad \gamma u(1)+\delta u'(1)=\int_0^1 u(s)\,dH_2(s)+a_0, \\ \tilde{\alpha} v(0)-\tilde{\beta} v'(0)=\int_0^1 v(s)\,dK_1(s)+b_0, \quad \tilde{\gamma} v(1)+\tilde{\delta} v'(1)=\int_0^1 v(s)\,dK_2(s), \end{cases}$$

where a_0 and b_0 are positive constants.

For problem $(S)-(BC_1)$, instead of functions h and k from the proof of Theorem 3.1, the solutions of problems

$$\begin{cases} h_1''(t)=0, \quad t\in(0,1), \\ \alpha h_1(0)-\beta h_1'(0)=\int_0^1 h_1(s)\,dH_1(s)+1, \quad \gamma h_1(1)+\delta h_1'(1)=\int_0^1 h_1(s)\,dH_2(s), \end{cases} \tag{13}$$

$$\begin{cases} k_1''(t)=0, \quad t\in(0,1), \\ \tilde{\alpha} k_1(0)-\tilde{\beta} k_1'(0)=\int_0^1 k_1(s)\,dK_1(s)+1, \quad \tilde{\gamma} k_1(1)+\tilde{\delta} k_1'(1)=\int_0^1 k_1(s)\,dK_2(s), \end{cases} \tag{14}$$

are

$$h_1(t)=\frac{\tau_1}{\Delta_1}\left[-t\left(\gamma-\int_0^1 dH_2(s)\right)+\gamma+\delta-\int_0^1 s\,dH_2(s)\right], \qquad t\in[0,1],$$

$$k_1(t)=\frac{\tau_2}{\Delta_2}\left[-t\left(\tilde{\gamma}-\int_0^1 dK_2(s)\right)+\tilde{\gamma}+\tilde{\delta}-\int_0^1 s\,dK_2(s)\right], \qquad t\in[0,1],$$

respectively. By assumptions (J1)-(J3) we obtain $h_1(t)>0$ and $k_1(t)>0$ for all $t\in[0,1)$.

For problem $(S)-(BC_2)$, instead of functions h and k from the proof of Theorem 3.1, the solutions of problems (13) and (6) are the functions h_1 and k, respectively, which satisfy $h_1(t)>0$ for all $t\in[0,1)$ and $k(t)>0$ for all $t\in(0,1]$. For problem $(S)-(BC_3)$, instead of functions h and k from the proof of Theorem 3.1, the solutions of problems (5) and (14) are the functions h and k_1, respectively, which satisfy $h(t)>0$ for all $t\in(0,1]$ and $k_1(t)>0$ for all $t\in[0,1)$.

Therefore we also obtain the following results.

Theorem 3.3 Assume that assumptions (J1)-(J5) hold. Then problem $(S)-(BC_1)$ has at least one positive solution ($u(t)>0$ and $v(t)>0$ for all $t\in[0,1)$) for $a_0>0$ and $b_0>0$ sufficiently small.

Theorem 3.4 Assume that assumptions (J1)-(J4) and (J6) hold. Then problem $(S)-(BC_1)$ has no positive solution ($u(t)>0$ and $v(t)>0$ for all $t\in[0,1)$) for a_0 and b_0 sufficiently large.

Theorem 3.5 Assume that assumptions (J1)-(J5) hold. Then problem $(S)-(BC_2)$ has at least one positive solution ($u(t)>0$ for all $t\in[0,1)$, and $v(t)>0$ for all $t\in(0,1]$) for $a_0>0$ and $b_0>0$ sufficiently small.

Theorem 3.6 Assume that assumptions (J1)-(J4) and (J6) hold. Then problem $(S)-(BC_2)$ has no positive solution ($u(t)>0$ for all $t\in[0,1)$, and $v(t)>0$ for all $t\in(0,1]$) for a_0 and b_0 sufficiently large.

Theorem 3.7 Assume that assumptions (J1)-(J5) hold. Then problem $(S)-(BC_3)$ has at least one positive solution ($u(t) > 0$ for all $t \in (0,1]$, and $v(t) > 0$ for all $t \in [0,1)$) for $a_0 > 0$ and $b_0 > 0$ sufficiently small.

Theorem 3.8 Assume that assumptions (J1)-(J4) and (J6) hold. Then problem $(S) - (BC_3)$ has no positive solution ($u(t) > 0$ for all $t \in (0,1]$, and $v(t) > 0$ for all $t \in [0,1)$) for a_0 and b_0 sufficiently large.

4. An Example

We consider $p(t) = at$, $q(t) = bt$ for all $t \in [0,1]$ with $a, b > 0$, $\alpha = 3, \beta = 2, \gamma = 2, \delta = 1, \tilde{\alpha} = 5, \tilde{\beta} = 2$, $\tilde{\gamma} = 2, \tilde{\delta} = 1/2$, $H_1(t) = t$, $H_2(t) = t^2$, $K_1(t) = t^3$, $K_2(t) = \sqrt{t}$ for all $t \in [0,1]$. We also consider the functions $f, g: [0,\infty) \to [0,\infty)$, $f(x) = \frac{cx^3}{x+1}$, $g(x) = \frac{dx^4}{2x+3}$ for all $x \in [0,\infty)$ with $c, d > 0$. We have $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{g(x)}{x} = \infty$.

Therefore, we consider the nonlinear second-order differential system

$$(S_0) \qquad \begin{cases} u''(t) + at\dfrac{cv^3(t)}{v(t)+1} = 0, & t \in (0,1), \\ v''(t) + bt\dfrac{du^4(t)}{2u(t)+3} = 0, & t \in (0,1), \end{cases}$$

with the boundary conditions

$$(BC_0) \qquad \begin{cases} 3u(0) - 2u'(0) = \displaystyle\int_0^1 u(s)\,ds, \;\; 2u(1) + u'(1) = 2\int_0^1 su(s)\,ds + a_0, \\ 5v(0) - 2v'(0) = 3\displaystyle\int_0^1 s^2 v(s)\,ds, \;\; 2v(1) + \frac{1}{2}v'(1) = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{s}} v(s)\,ds + b_0. \end{cases}$$

We obtain $\alpha - \int_0^1 dH_1(\tau) = 2 > 0$, $\gamma - \int_0^1 dH_2(\tau) = 1 > 0$, $\tilde{\alpha} - \int_0^1 dK_1(\tau) = 4 > 0$, $\tilde{\gamma} - \int_0^1 dK_2(\tau) = 1>0$, ψt=3t+2, ϕt=−2t+3 for all t∈0,1, τ1=13, Δ1=436, Δ1=5596,

$\tilde{\psi}(t) = 5t + 2$, $\tilde{\phi}(t) = -2t + \frac{5}{2}$ for all $t \in [0,1]$, $\tau_2 = \frac{33}{2}$, $\widetilde{\Delta}_2 = \frac{137}{12}$, $\Delta_2 = \frac{1507}{6}$. So assumptions (J1)-(J4) and (J6) are satisfied.

In addition, we have

$$g_1(t,s) = \frac{1}{13}\begin{cases}(-2t+3)(3s+2), & 0 \le s \le t \le 1, \\ (-2s+3)(3t+2), & 0 \le t \le s \le 1,\end{cases}$$

$$g_2(t,s) = \frac{2}{33}\begin{cases}(-2t+5/2)(5s+2), & 0 \le s \le t \le 1, \\ (-2s+5/2)(5t+2), & 0 \le t \le s \le 1,\end{cases}$$

and the functions J_1 and J_2 are of the form

$$J_1(s) = g_1(s,s) + \frac{212}{559}\int_0^1 g_1(\tau,s)\,d\tau + \frac{786}{559}\int_0^1 \tau g_1(\tau,s)\,d\tau = \frac{424 + 464s - 364s^2 - 131s^3}{559},$$

$$\begin{aligned} J_2(s) &= g_2(s,s) + \frac{98}{137}\int_0^1 \tau^2 g_2(\tau,s)\,d\tau + \frac{983}{3014}\int_0^1 \frac{1}{\sqrt{\tau}} g_2(\tau,s)\,d\tau \\ &= \frac{4312 + 8588s - 3932s^{3/2} - 5480s^2 - 539s^4}{9042}. \end{aligned}$$

Then we deduce $L = \max\left\{a\int_0^1 sJ_1(s)\,ds,\ b\int_0^1 sJ_2(s)\,ds\right\}$, with $\int_0^1 sJ_1(s)\,ds \approx 0.4462731$ and $\int_0^1 sJ_2(s)\,ds \approx 0.2693436$. We choose $c_0 = 1$ and if we select c and d satisfying the conditions $c < \frac{2}{L}, d < \frac{5}{L}$, then we obtain $f(x) \le \frac{c}{2} < \frac{1}{L}$, $g(x) \le \frac{d}{5} < \frac{1}{L}$ for all $x \in [0,1]$. For example, if $a = 1, b = 1/2$, then for $c \le 4.48$ and $d \le 11.2$ the above conditions for f and g are satisfied. So, assumption (J5) is also satisfied. By Theorems 3.1 and 3.2 we conclude that problem $(S_0) - (BC_0)$ has at least one positive solution for sufficiently small $\mathrm{a}_0 > 0$ and $\mathrm{b}_0 > 0$, and no positive solution for suficiently large a_0 and b_0.

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