Exponential Mean Square Stability of the BE Method for Linear Neutral Hybrid Stochastic Delay Differential Equations

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Abstract: In this paper, the backward Euler (BE) method is introduced and analyzed for linear neutral hybrid stochastic delay differential equations. The exponential mean square stability of the BE method is considered for linear neutral hybrid stochastic delay differential equations. It is proved that, under the one-sided Lipschitz condition and the linear growth condition, for some positive stepsizes which depend on the Markovian switching, the BE method is exponential mean square stable. A numerical example is provided to illustrate the theoretical results.

Key words: Linear neutral hybrid stochastic delay differential equations, exponential mean square stability, BE method, Markovian switching.

1. Introduction

The practical systems which experience abrupt changes in their structure and parameters caused by phenomena, such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances must be modeled by the hybrid systems driven by continuous-time Markov chains [1]. The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values. One of the important classes of hybrid systems is the stochastic delay differential equations (SDDEs) with Markovian switching. The natural extensions of this system are stochastic delay integro-differential equations (SDIDE) with Markovian switching and neutral stochastic delay differential equations (NSDDE) with Markovian switching. Although the theoretical study of stochastic differential delay equations with Markovian switching is considered in [2]-[4], the explicit solutions can hardly be obtained for the SDIDEs and NSDDEs with Markovian switching. Rathinasamy and Balachandran [5] analyzed the numerical MS-stability of second-order Runge-Kutta schemes for multi-dimensional stochastic differential systems with one multiplicative noise, the convergence and stability of the Euler method for a class of linear stochastic differential delay equations with Markovian switching is studied in [6]. The numerical method for SDIDEs with Markovian switching had been studied in [7], mean-square stability of Milstein method was obtained. For the neutral stochastic differential equations, Mao studied the exponential mean square stability of its analytical solution in [8], the delay-dependent exponential stability conditions for linear neutral stochastic differential systems was studied in [9]-[11]. Huang Chengming and Wang Wansheng studied the stability of semi-implicit Euler method and θ - method

for nonlinear neutral stochastic delay differential equations in [12], [13].

To best of our knowledge, no results on the numerical methods for linear NSDDEs with Markovian switching have been presented in the literatures. Thus, it is necessary to develop appropriate numerical methods and to study the properties of these approximate schemes for NSDDEs with Markovian switching.

In this paper we consider the linear d-dimensional neutral stochastic delay differential equation with Markovian switching of the form

$$\begin{cases} d[x(t) - N(x(t-\tau))] = \left[-A(r(t))x(t) + B(r(t))x(t-\tau)\right]dt \\ \qquad\qquad\qquad + \left[D(r(t))x(t) + E(r(t))x(t-\tau)\right]dw(t), t \geq 0, \\ x(t) = \varphi(t), t \in [-\tau, 0] \end{cases} \tag{1.1}$$

where $r(t)$ is a Markov chain taking values in $S = \{1, 2, \ldots, N\}$ and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ and $E(\cdot) \in \mathrm{R}$, N is the d-dimensional square matrices.

In Section 2, we will introduce some necessary notations and assumptions. In Section 3, the definition of exponential mean squares stability is defined, the BE method will be used to produce the numerical solutions and the main result will be shown and proved in this section. A numerical example is given in Section 4.

2. Notation and Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. If A is a vector or matrix, its transpose is denoted by A^{T}. Let $|\cdot|$ denote both the Euclidean norm in R^d and the trace (or Frobenius) norm in $\mathrm{R}^{d \times d}$ (denoted by trace $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$). $a \vee b$ represents $\max\{a, b\}$ and $a \wedge b$ denotes $\min\{a, b\}$. Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a complete probability space with a filtration $\{\mathrm{F}_t\}_{t \geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while F_0 contains all P-null sets. Let $w(t)$ be a d-dimensional Brownian motion defined on this probability space, and τ is a positive fixed delay.

Let $r(t)$, $t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathrm{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + 0(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii} + 0(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+. As for $r(t)$, the following lemma is satisfied.

Lemma 2.1 [6]. Given $h > 0$, then $\{r_n^h = r(nh), n = 0, 1, 2, \ldots\}$ is a discrete Markov chain with the one-step transition probability matrix $\mathrm{P}(h) = (\mathrm{P}_{ij}(h))_{N \times N} = e^{h\Gamma}$.

Since the γ_{ij} are independent of x, the paths of Markov chain r can be generated independent of x and, in fact, before computing x.

For the purpose of stability, assume that $f(0, 0, 0, i) = g(0, 0, 0, i) = 0$. This shows that (1.1) admits a trivial solution. We assume that f and g satisfy the local Lipschitz condition:

(H$_1$): Both f and g satisfy the local Lipschitz condition. That is, for each $k = 1,2,\ldots$, there is an $h_k > 0$ such that

$$\left| f(x,y,t,i) - f(\bar{x},\bar{y},t,i) \right| + \left| g(x,y,t,i) - g(\bar{x},\bar{y},t,i) \right| \le h_k \left(|x - \bar{x}| + |y - \bar{y}| \right) \tag{2.1}$$

for all $t \ge 0$ and $i \in S$ and those $x,y,\bar{x},\bar{y} \in \mathrm{R}^d$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le k$.

(H$_2$): For every $i \in S$, there are constants $a_i \in \mathrm{R}$ nonnegative constants μ_i, σ_i and β_i such that

$$2[x - N(y)]^{\mathrm{T}} f(x,y,t,i) \le -a_i |x|^2 + \mu_i |y|^2 \;, \tag{2.2}$$

$$\left| g(x,y,t,i) \right|^2 \le \sigma_i |x|^2 + \beta_i |y|^2 \;, \tag{2.3}$$

for all $x, y \in \mathrm{R}^d$ and $t \ge 0$.

We also assume that: exists a positive constant $0 < \lambda < \dfrac{1}{2}$, such that $|N| \le \lambda$.

Considering equation (1.1), we can match in the following form:

$$f(x(t), x(t-\tau), t, r(t)) = -A(r(t))x(t) + B(r(t))x(t-\tau)$$

and $g(x(t), x(t-\tau), t, r(t)) = D(r(t))x(t) + E(r(t))x(t-\tau)$.

If we take $a_i = (\lambda - 2)A(i) + |B(i)|$, $\mu_i = |A(i)|\lambda + |B(i)|$, $\sigma_i = |D(i)|$, $\beta_i = |E(i)|$, then the hypotheses (H$_1$) and (H$_2$) are satisfied naturally for Eq.(1.1). Then for any given initial data $\varphi \in C_{F_0}^b ([-\tau,0], \mathrm{R}^d)$. Eq. (1.1) has a unique continuous solution $x(t)$.

3. Math Stability Analysis of the BE Method

Applying the Backward Euler（BE）method to the linear neutral delay differential system with Markovian switching (1.1) leading to a numerical process of the following type:

$$x_{n+1} - Nx_{n+1-m} = x_n - Nx_{n-m} + \left(-A(r_{n+1}^h)x_{n+1} + B(r_{n+1}^h)x_{n+1-m} \right)h + \left(D(r_n^h)x_n + E(r_n^h)x_{n-m} \right)\Delta w_n, \tag{3.1}$$

where $h > 0$ is a stepsize which satisfies $\tau = mh$ for some positive integer m and $t_n = nh$, $r_n^h = r(t_n) \in S$, x_n is an approximation to $x(t_n) = x_n$, if $t_n \le 0$, we have $x_n = \varphi(t_n)$, the increments $\Delta w_k = w(t_{k+1}) - w(t_k)$, $k = 0,1,2,\ldots$, are independent $N(0,h)$ - distributed

Gaussian random variables. Further, we assume that x_n is F_{t_n} -measurable at the mesh points t_n.

Definition 3.1 A numerical method is said to be exponential mean square stable, if there exists a $h_0(A_i, B_i, D_i, E_i) > 0$ and positive constant γ, such that, for $h \in (0, h_0)$ with $h = \dfrac{\tau}{m}$, any application of the method to problem (1.1) generates a numerical approximation $\{x_k\}$, which satisfies

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathrm{E}|x_k|^2 \le -\frac{\gamma}{2} \tag{3.2}$$

Theorem 3.1 Assume that there are four nonnegative constants $a_i, \mu_i, \sigma_i, \beta_i$, such that conditions (2.2) –(2.4) hold, and for any $i \in S$, $3A_i + |D_i| + |E_i| < 0$, where $A_i = A(i), D_i = D(i)$ and $E_i = E(i)$. Then the BE method is exponential mean square stable.

Proof. Let $Z_k = x_k - N x_{k-m}$, then from conditions (2.2)-(2.4), we have

$$
\begin{aligned}
|Z_{k+1}|^2 &= \langle Z_{k+1}, Z_k + f(x_{k+1}, x_{k+1-m}, t_{k+1}, i)h + g(x_k, x_{k-m}, t_k, i)\Delta w_k \rangle \\
&= Z_{k+1}^{\mathrm{T}} f(x_{k+1}, x_{k+1-m}, t_{k+1}, i)h + \langle Z_{k+1}, Z_k + g(x_k, x_{k-m}, t_k, i)\Delta w_k \rangle \\
&\leq \frac{1}{2}(-a_i|x_{k+1}|^2 + \mu_i|x_{k+1-m}|^2)h + \frac{1}{2}[|Z_{k+1}|^2 + |Z_k|^2 + (\sigma_i|x_k|^2 + \beta_i|x_{k-m}|^2)h] + \frac{1}{2}m_k^h
\end{aligned}
$$

where $m_k^h = |g(x_k, x_{k-m}, kh, i)|^2 (|\Delta w_k|^2 - h) + 2Z_k^{\mathrm{T}} g(x_k, x_{k-m}, kh, i)\Delta w_k$.

Then we have

$$
|Z_{k+1}|^2 \leq |Z_k|^2 + (-a_i|x_{k+1}|^2 + \mu_i|x_{k+1-m}|^2)h + (\sigma_i|x_k|^2 + \beta_i|x_{k-m}|^2)h + m_k^h \tag{3.3}
$$

Note that $\mathrm{E}m_k^h = 0$ for all $k > 0$. Taking expectation on the both sides of inequality (3.3), we have

$$
\mathrm{E}|Z_{k+1}|^2 \leq \mathrm{E}|Z_k|^2 + (-a_i\mathrm{E}|x_{k+1}|^2 + \mu_i\mathrm{E}|x_{k+1-m}|^2)h + (\sigma_i\mathrm{E}|x_k|^2 + \beta_i|x_{k-m}|^2)h
$$

For any $C > 1$, we have

$$
\begin{aligned}
[C^{(k+1)h}\mathrm{E}|Z_{k+1}|^2 - C^{kh}\mathrm{E}|Z_k|^2] &\leq (C^{(k+1)h} - C^{kh})\mathrm{E}|Z_k|^2 + (-a_i\mathrm{E}|x_{k+1}|^2 + \mu_i\mathrm{E}|x_{k+1-m}|^2)hC^{(k+1)h} \\
&\quad + (\sigma_i\mathrm{E}|x_k|^2 + \beta_i\mathrm{E}|x_{k-m}|^2)hC^{(k+1)h}
\end{aligned} \tag{3.4}
$$

which implies

$$
\begin{aligned}
C^{kh}\mathrm{E}|Z_k|^2 &\leq \mathrm{E}|Z_0|^2 - a_ih\sum_{j=0}^{k-1} C^{(j+1)h}\mathrm{E}|x_{j+1}|^2 + \mu_ih\sum_{j=0}^{k-1} C^{(j+1)h}\mathrm{E}|x_{j+1-m}|^2 \\
&\quad + [\sigma_ihC^h + (1+\varepsilon_0)(C^h - 1)]\sum_{j=0}^{k-1} C^{jh}\mathrm{E}|x_j|^2 \\
&\quad + [\beta_ihC^h + \lambda^2(1+\varepsilon_0^{-1})(C^h - 1)]\sum_{j=0}^{k-1} C^{jh}\mathrm{E}|x_{j-m}|^2
\end{aligned}
$$

where we used the inequality $|Z_j|^2 = |x_j - N x_{j-m}|^2 \leq (1+\varepsilon_0)\left(|x_j|^2 + \frac{\lambda^2}{\varepsilon_0}|x_{j-m}|^2\right)$.

Note that

$$
\begin{aligned}
\sum_{j=0}^{k-1} C^{(j+1)h}\mathrm{E}|x_{j+1}|^2 &= \sum_{j=1}^{k} C^{jh}\mathrm{E}|x_j|^2 \\
&= \sum_{j=0}^{k-1} C^{jh}\mathrm{E}|x_j|^2 - \mathrm{E}|x_0|^2 + C^{kh}\mathrm{E}|x_k|^2
\end{aligned}
$$

$$\sum_{j=0}^{k-1} C^{(j+1)h} E|x_{j+1-m}|^2 = \sum_{j=-m+1}^{k-m} C^{(j+m)h} E|x_j|^2$$

$$= C^{mh} \sum_{j=-m+1}^{-1} C^{jh} E|x_j|^2 + C^{mh} \sum_{j=0}^{k-1} C^{jh} E|x_j|^2 - C^{mh} \sum_{j=k-m+1}^{k-1} C^{jh} E|x_j|^2$$

and

$$\sum_{i=0}^{k-1} C^{ih} E|x_{i-m}|^2 = \sum_{i=-m}^{k-m-1} C^{(i+m)h} E|x_i|^2$$

$$= C^{mh} \sum_{i=-m}^{-1} C^{ih} E|x_i|^2 + C^{mh} \sum_{i=0}^{k-1} C^{ih} E|x_i|^2 - C^{mh} \sum_{i=k-m}^{k-1} C^{ih} E|x_i|^2$$

We therefore have

$$C^{kh} E|Z_k|^2 \leq X + E|Z_0|^2 + a_i h E|x_0|^2 + \Delta(C) h \sum_{i=0}^{k-1} C^{ih} E|x_i|^2 \qquad (3.5)$$

where

$$X = [\mu_i + \beta_i C^h + \lambda^2 (1+\varepsilon_0^{-1})(C^h - 1)/h] C^\tau h \sum_{i=-m+1}^{-1} C^{ih} E|x_i|^2, \qquad (3.6)$$

$$\Delta(C) = -a_i + \mu_i C^\tau + \sigma_i C^h + (1+\varepsilon_0)\frac{C^h - 1}{h} + \left[\beta_i C^h + \lambda^2 (1+\varepsilon_0^{-1})\frac{C^h - 1}{h}\right] C^\tau .$$

It is easy to see that $\Delta\left(\frac{a_i}{\mu_i}^{\frac{1}{\tau}}\right) > 0$, $\Delta(1) = -a_i + \sigma_i + \mu_i + \beta_i \leq 3A_i + |D_i| + |E_i| < 0$ and $\Delta'(C) > 0$ for any $C \geq 1$,

hence, there exists unique constant $C_h^* \in (1, \frac{a_i}{\mu_i}^{\frac{1}{\tau}})$ such that $\Delta(C_h^*) = 0$.

Thus (3.5) yields

$$C_h^{*kh} E|Z_k|^2 \leq X + E|Z_0|^2 + a_i h E|x_0|^2 . \qquad (3.7)$$

In view of (3.6), for any bounded initial condition φ, X is nonnegative F_0-measurable random variable, thus the right-hand side of (3.7) converges to a finite random variable as $k \to +\infty$, that is,

$$\limsup_{k \to \infty} C_h^{*kh} E|Z_k|^2 \leq \limsup_{k \to \infty} \left(X + E|Z_0|^2 + a_i h E|x_0|^2\right) < \infty$$

Define $u = \log C$, then $\delta \in (0, \bar{\varepsilon}/2)$, and we also define $\lim_{h \to o} u_h^* = \bar{\varepsilon}$.

So, for any $\delta \in (0, \bar{\varepsilon}/2)$, there exists $h_0 = h_0(A_i, B_i, D_i, E_i)$ which is the solution of $\Delta(C_h^*) = 0$, such that for any $h \in (0, h_0)$, $u_h^* > \bar{\varepsilon} - 2\delta$.

Since we earlier obtained that $\limsup_{k \to \infty} C_h^{*kh} E|Z_k|^2 < \infty$, so we get, for any $h \in (0, h_0)$, that

$$\bar{\sigma} = \limsup_{k \to \infty} e^{(\varepsilon - 2\delta)kh} E|Z_k|^2 < \infty.$$

Consequently, for any $\gamma \in (0, e^{-\frac{1}{\tau} \log \lambda \wedge (\bar{\varepsilon} - 2\delta)})$, there exists an integer k_1, such that $e^{\gamma kh} E|Z_k|^2 \le \bar{\sigma} + \gamma, k \ge k_1$.

Applying the elementary inequality $(a + b)^p \le (1 + c)^{p-1}(a^p + c^{1-p} b^p)$, $a, b > 0, p > 1, c > 0$, with $p = 2$ and $c = \lambda/(1 - \lambda)$, as well as the condition (2.4), we obtain

$$e^{\gamma kh} E|x_k|^2 \le (1 - \lambda)^{-1} e^{\gamma kh} E|Z_k|^2 + \lambda e^{\gamma kh} E|x_{k-m}|^2, \tag{3.8}$$

then, for any integer $k_2 > k_1$,

$$\sup_{k_1 \le k \le k_2} e^{\gamma kh} E|x_k|^2 \le (1 - \lambda)^{-1}(\bar{\sigma} + \gamma) + \lambda \sup_{k_1 \le k \le k_2} e^{\gamma kh} E|x_{k-m}|^2 \le (1 - \lambda)^{-1}(\bar{\sigma} + \gamma) + \lambda e^{\gamma \tau} \sup_{k_1 \le k \le k_2} e^{\gamma (k-m)h} E|x_{k-m}|^2$$

$$\le (1 - \lambda)^{-1}(\bar{\sigma} + \gamma) + \lambda e^{\gamma \tau} \sup_{k_1 - m \le k \le k_1} e^{\gamma kh} E|x_k|^2 + \lambda e^{\gamma \tau} \sup_{k_1 \le k \le k_2} e^{\gamma kh} E|x_k|^2,$$

implying that

$$\sup_{k_1 \le k \le k_2} e^{\gamma kh} E|x_k|^2 \le (1 - \lambda e^{\gamma \tau})^{-1} \left[(1 - \lambda)^{-1}(\bar{\sigma} + \gamma) + \lambda e^{\gamma \tau} \sup_{k_1 - m \le k \le k_1} e^{\gamma kh} E|x_k|^2 \right]. \tag{3.9}$$

Letting $k_2 \to +\infty$ in (3.9), we obtain

$$\sup_{k_1 \le k \le +\infty} e^{\gamma kh} E|x_k|^2 \le (1 - \lambda e^{\gamma \tau})^{-1} \left[(1 - \lambda)^{-1}(\bar{\sigma} + \gamma) + \lambda e^{\gamma \tau} \sup_{k_1 - m \le k \le k_1} e^{\gamma kh} E|x_k|^2 \right] < \infty,$$

which gives $\lim_{k \to +\infty} e^{\gamma kh} E|x_k|^2 < \infty$.

Thus the estimate (3.8) yields

$$\limsup_{k \to +\infty} e^{\gamma kh} E|x_k|^2 \le (1 - \lambda)^{-1}(\bar{\sigma} + \gamma) + \lambda e^{\gamma \tau} \limsup_{k \to +\infty} e^{\gamma (k-m)h} E|x_{k-m}|^2,$$

so, foe any $\gamma \in (0, e^{-\frac{1}{\tau} \log \lambda \wedge (\bar{\varepsilon} - 2\delta)})$, we obtain $\limsup_{k \to +\infty} e^{\gamma kh} E|x_k|^2 \le (1 - \lambda e^{\gamma \tau})^{-1}(1 - \lambda)^{-1}(\bar{\sigma} + \gamma)$.

Consequently, $0 = \limsup_{k \to +\infty} \frac{\log e^{\gamma kh} E|x_k|^2}{kh} = \gamma + 2 \limsup_{k \to +\infty} \frac{\log E|x_k|}{kh}$, which gives (3.2).

4. Numerical Example

We shall discuss an example to illustrate our theory. Let $w(t)$ be a scalar Brownian motion. Let $r(t)$ be a right continuous Markov chain taking values in $S = \{1,2\}$ with the generator $\Gamma = \left(\gamma_{ij} \right)_{2 \times 2} = \begin{pmatrix} -1 & 1 \\ \gamma & -\gamma \end{pmatrix}$.

We assume $w(t)$ and $r(t)$ are independent.

Consider the following one-dimensional linear neutral stochastic delay differential equation with Markovian switching

$$\begin{cases} d[x(t) - \frac{1}{4}(x(t-1))] = \left[A(r(t))x(t) + B(r(t))x(t-1) \right] dt \\ \qquad\qquad\qquad + \left[D(r(t))x(t) + E(r(t))x(t-1) \right] dw(t), t \geq 0, \\ x(t) = t + 1, t \in [-1,0] \end{cases} \tag{4.1}$$

Let $A(1) = -2, B(1) = 1, D(1) = 0, E(1) = \frac{2}{5}$ and $A(2) = -9, B(2) = 4, D(2) = 0, E(2) = \frac{4}{5}$.

It is interesting to regard Eq. (4.1) as the result of the following two equations:

$$d[x(t) - \frac{1}{4}(x(t-1))] = \left[-2x(t) + x(t-1) \right] dt + \frac{2}{5} x(t-1) dw(t) \tag{4.2}$$

and

$$d[x(t) - \frac{1}{4}(x(t-1))] = \left[-9x(t) + 4(t-1) \right] dt + \frac{4}{5} x(t-1) dw(t) \tag{4.3}$$

switching from one to the other according to the movement of the Markov chain $r(t)$. It is known that Eqs. (4.2) and (4.3) are exponentially mean square stable, where $\lambda = \frac{1}{4}$, $a_1 = \frac{9}{2}$, $u_1 = 3$, $\sigma_1 = 0$, $\beta_1 = \frac{2}{5}$, $a_2 = \frac{79}{4}$, $u_2 = \frac{25}{4}$, $\sigma_2 = 0$, $\beta_2 = \frac{4}{5}$, $0 < \gamma < 4$, we take $\gamma = 1$, $h_0 = h_0(A_1, B_1, D_1, E_1) < \frac{1}{11}$, $h_0 = h_0(A_2, B_2, D_2, E_2) < \frac{1}{9}$.

Thus, when a method of form (3.1) with stepsize $h \in (0, \frac{1}{11})$ is used to solve the above system, the corresponding numerical solution is exponential mean-square stable by Theorem 3.1.

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