Numerical Solution of Nonlinear Weakly Singular Fredholm-Volterra Integral Equations of the Second Kind by Using Sinc-Collocation Methods

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Abstract: In this paper, the efficient methods based on the Sinc approximation with the single exponential (SE) and double exponential (DE) transformations are presented to solve nonlinear Fredholm-Volterra integral equations with weakly singular kernel. Sinc approximation has considerable advantages. Some of them are the exponential convergence of an approximate solution and simply implementation, even in the presence of singularities. Properties of the SE-Sinc and DE-Sinc methods are utilized to reduce the problem to a set of algebraic equations. Finally, we give some numerical results that confirm efficiency and accuracy of the numerical schemes.

Key words: Nonlinear Fredholm-Volterra integral equation, Sinc approximation, weakly singular kernel.

1. Introduction

Many problems of engineering, contact problems in the theory of elasticity [1]-[3], mathematical physics and chemical reactions, such as heat conduction and crystal growth lead to singular integral equations. In this paper, we consider the weakly singular Fredholm-Volterra integral equations (F-VIEs) of the second kind of the form

$$x(t) = g(t) + \int_a^b \frac{k_1(t,s,x(s))}{|t-s|^{1-\alpha}} ds + \int_a^t \frac{k_2(t,s,x(s))}{(t-s)^{1-\alpha}} ds, \qquad a \le t \le b, \tag{1}$$

where $0 < \alpha < 1$. k_1, k_2 and g are given functions, and x is an unknown function to be determined. There is an increasing demand for studying singular integral equations and these problems of course cannot be solved explicitly. Therefore, it is important to find their approximate solutions by using some numerical methods. The product Nyström method has been devised to find numerical solution of singular F-VIEs in [4]. In [5], Fayazzadeh and Lotfi proposed collocation method to solve weakly singular F-VIEs.

A great deal of interest has been focused on applications of the Sinc methods. These methods have considerable advantages over classical methods that use polynomials as bases. For example, in the presence of singularities, these methods give an exponential convergence and accuracy than polynomial methods. Therefore, this paper describes procedures for solving nonlinear weakly singular F-VIE based on sinc approximation. The Sinc methods have been studied by many authors, e.g., Saadatmandi and Razzaghi [6], Okayama et al. [7], [8], Rashidinia [9], [10] and Maleknejad [10], [11].

The paper is organized as follows. In Section 2, we review some basic facts about the sinc approximation. Section 3 is devoted to solve (1) by using Sinc-collocation methods. As a result, a set of algebraic equations is achieved and the solution of the considered problem is introduced. In Section 4, we report our numerical findings and demonstrate validity of the proposed schemes by considering numerical examples.

2. Sinc Function and Basic Definition

In this section, we will review Sinc function properties. These are discussed thoroughly in [12], [13]. Originally, Sinc approximation for a function f is expressed as

$$f(x) \approx \sum_{j=-N}^{N} f(jh)S(j,h)(x), \quad x \in \mathbb{R}. \tag{2}$$

To construct approximation on the interval $L=(a,b)$, which are used in (1), we consider the $tanh$ transformation (and its inverse)

$$\emptyset^{SE}(x) = \frac{b-a}{2}\tanh\left(\frac{x}{2}\right) + \frac{b+a}{2},$$

$$\{\emptyset^{SE}\}^{-1}(t) = \log\left(\frac{t-a}{b-t}\right).$$

Interpolation formula for $f(t)$ over (a,b) is

$$f(t) \approx \sum_{j=-N}^{N} f\left(\emptyset^{SE}(jh)\right) S(j,h)(\{\emptyset^{SE}\}^{-1}(t)).$$

Sinc approximation can be applied to definite integration based on the function approximation described above; it is called the Sinc quadrature. Where

$$\int_a^b f(s)ds \approx h \sum_{j=-m}^{n} f\left(\emptyset^{SE}(jh)\right) \{\emptyset^{SE}\}'(jh).$$

The following theorems give us an error bound for the SE-Sinc approximation and quadrature.

Theorem 1 ([12]): Let $f \in L_\alpha(\phi^{SE}(D_d))$ for d with $0 < d < \pi$. Let also N be a positive integer, and h be given by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}} \tag{3}$$

Then there exists a constant C independent of N, such that

$$\left| f(t) - \sum_{j=-N}^{N} f(\emptyset^{SE}(jh)) S(j,h) \left(\{\emptyset^{SE}\}^{-1}(t)\right) \right| \leq C\sqrt{N}\exp(-\sqrt{\pi d \alpha N}). \tag{4}$$

Theorem 2 ([12]): Let $(fQ) \in L_\alpha(\phi^{SE}(D_d))$ for d with $0 < d < \pi$. Suppose that N is a positive integer and h is selected by (3). Then there exists a constant C independent of N, such that

$$\left|\int_a^b f(s)ds - h\sum_{j=-m}^{n} f\big(\emptyset^{SE}(jh)\big)\{\emptyset^{SE}\}'(jh)\right| \le \mathrm{C}\exp\big(-\sqrt{\pi d\alpha N}\big). \tag{5}$$

Also, double exponential transformation can be used

$$\emptyset^{DE}(x) = \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh(x)\right) + \frac{b+a}{2},$$

$$\{\emptyset^{DE}\}^{-1}(t) = \log\left(\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right) + \sqrt{1+\left(\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right)\right)^2}\right).$$

The following theorems describe the accuracy of DE-Sinc method.

Theorem 3 ([14]): Let $f \in L_\alpha(\phi^{DE}(D_d))$ for d with $0 < d < \frac{\pi}{2}$. Let also N be a positive integer, and h be given by the formula

$$h = \frac{\log(2dN/\alpha)}{N} \tag{6}$$

Then there exists a constant C which is independent of N, such that

$$\left|f(t) - \sum_{j=-N}^{N} f\big(\emptyset^{DE}(jh)\big)S(j,h)\left(\{\emptyset^{DE}\}^{-1}(t)\right)\right| \le C\exp\left(\frac{-\pi dN}{\log\left(\frac{2dN}{\alpha}\right)}\right). \tag{7}$$

Such a function is required to be zero at the endpoints, $t = a$ and $t = b$, which seems to be an impractical assumption. In order to handle more general cases, we introduce the translated function

$$\Gamma[f](t) = f(t) - \left[\left(\frac{b-t}{b-a}\right)f(a) + \left(\frac{t-a}{b-a}\right)f(b)\right]. \tag{8}$$

Theorem 4 ([8]): Assume that $(fQ) \in L_\alpha(\emptyset^{DE}(D_d))$ for d with $0 < d < \frac{\pi}{2}$. Suppose that N is a positive integer and h is selected by (6). Then there exists a constant C independent of N, such that

$$\left|\int_a^b f(s)ds - h\sum_{j=-m}^{n} f\big(\emptyset^{DE}(jh)\big)\{\emptyset^{DE}\}'(jh)\right| \le \mathrm{C}\exp\left(\frac{-2\pi dN}{\log\left(\frac{2dN}{\alpha}\right)}\right). \tag{9}$$

3. Sinc-Collocation Methods

In this Section, illustrate how the Sinc methods may be used to replace (**1**) by a system of nonlinear algebraic equations. Equation (**1**) can be written in the operator form $\boldsymbol{x} = \boldsymbol{\kappa}[\boldsymbol{x}] + \boldsymbol{g}$ where

$$\kappa[x](t) = \int_a^b \frac{k_1(t,s,x(s))}{|t-s|^{1-\alpha}}ds + \int_a^t \frac{k_2(t,s,x(s))}{(t-s)^{1-\alpha}}ds.$$

3.1. SE-Sinc Scheme

A Sinc approximation x_N^{SE} to the solution $x \in M_\alpha(\phi^{SE}(D_d))$ of above equation is described in this part.

The function can be accurately approximated as

$$\Gamma[x](t) \approx \sum_{j=-N}^{N} \Gamma[x]\big(\phi^{SE}(jh)\big)S(j,h)\big(\{\phi^{SE}\}^{-1}(t)\big).$$

So, the approximate solution x is considered that has the form

$$P_N^{SE}[x](t) = x(a)\tfrac{b-t}{b-a} + \textstyle\sum_{j=-N}^{N} \Gamma[x](\phi^{SE}(jh))S(j,h)(\{\phi^{SE}\}^{-1}(t)) + x(b)\tfrac{t-a}{b-a}, \tag{10}$$

where h is given by (3). Now, we can obtain the convergence theorem corresponding to Theorem 1.

Theorem 5: Let $f \in M_\alpha(\phi^{SE}(D_d))$ for d with $0 < d < \pi$. Let also N be a positive integer, and h be given by (3). Then there exists a constant C^{SE} independent of N, such that

$$\max_{a\le t\le b}\big|f(t) - P_N^{SE}[f]\big| \le C^{SE}\sqrt{N}\exp\big(-\sqrt{\pi d\alpha N}\big). \tag{11}$$

There are $(2N+3)$ unknown coefficients on the right-hand side of (10) that should be determined. So, the approximate solution x_N^{SE} has a form like

$$x_N^{SE} = c_{-N-1}\tfrac{b-t}{b-a} + \textstyle\sum_{j=-N}^{N} c_N S(j,h)(\{\phi^{SE}\}^{-1}(t)) + c_{N+1}\tfrac{t-a}{b-a}, \tag{12}$$

In order to determine unknowns, we apply the collocation method. Summarily, by setting $t = t_j^{SE}$ where

$$t_j^{SE} = \begin{cases} a & j = -N-1, \\ \phi^{SE}(jh) & j = -N,\dots,N, \\ b & j = N+1, \end{cases}$$

in (12) and then by substituting x_N^{SE} into (1) and applying the collocation method to it, we obtain the following system of nonlinear equations with $(2N+3)$ unknowns $c_j, j = -N-1,\dots,N+1$:

$$x_N^{SE}\left(t_i^{SE}\right) - \int_a^b k_1\left(t_i^{SE}, s, x_N^{SE}(s)\right)\left|t_i^{SE} - s\right|^{\alpha-1} ds - \int_a^t k_2\left(t_i^{SE}, s, x_N^{SE}(s)\right)\left(t_i^{SE} - s\right)^{\alpha-1} ds = g\big(t_i^{SE}\big). \tag{13}$$

Since the SE-Sinc quadrature does not allow any singularity in the target interval, we split the first integral into two parts at $s = t_j^{SE}$. In order to approximate (13) we employ (5) and then

$$g\big(t_i^{SE}\big) = x_N^{SE}\big(t_i^{SE}\big) - \kappa_N^{SE}[x_N^{SE}]\big(t_i^{SE}\big), \tag{14}$$

where κ_N^{SE} is defined by

$$\begin{aligned}\kappa_N^{SE}[x](t) = h\sum_{j=-M}^{N}\left(t - \phi_{a,t}^{SE}(jh)\right)^{\alpha-1} k_1\left(t, \phi_{a,t}^{SE}(jh), x\left(\phi_{a,t}^{SE}(jh)\right)\right)\left\{\phi_{a,t}^{SE}\right\}'(jh) \\ + h\sum_{j=-N}^{M}\left(\phi_{t,b}^{SE}(jh) - t\right)^{\alpha-1} k_1\left(t, \phi_{t,b}^{SE}(jh), x\left(\phi_{t,b}^{SE}(jh)\right)\right)\left\{\phi_{t,b}^{SE}\right\}'(jh) \\ + h\sum_{j=-M}^{N}\left(t - \phi_{a,t}^{SE}(jh)\right)^{\alpha-1} k_2\left(t, \phi_{a,t}^{SE}(jh), x\left(\phi_{a,t}^{SE}(jh)\right)\right)\left\{\phi_{a,t}^{SE}\right\}'(jh)\end{aligned}$$

It should be pointed out that M is set by $M = [\alpha N]$. The above nonlinear system consists of $2N + 3$ equations with $2N + 3$ unknown $\{c_j\}_{-N-1}^{N+1}$. This system can be rewritten as $\mathcal{F}^{SE}(x_N^{SE}) = 0$, where $\mathcal{F}^{SE}: \mathbb{R}^{2N+3} \longrightarrow \mathbb{R}^{2N+3}$ with $\mathcal{F}^{SE}(x_N^{SE}) = x_N^{SE} - \kappa_N^{SE}[x_N^{SE}] - g$. By solving this nonlinear system by Newton's method, we can obtain the approximate solution x_N^{SE}.

3.2. DE-Sinc Scheme

In this case, we assume that the solution of (1) belongs to $M_\alpha(\phi^{DE}(D_d))$. Similar to the SE-Sinc method, by Theorem 3, the following theorem can be inferred.

Theorem 6: Let $f \in M_\alpha(\phi^{DE}(D_d))$ for d with $0 < d < \frac{\pi}{2}$. Let also N be a positive integer, and h be given by (6). Then there exists a constant C^{DE} which is independent of N, such that

$$\max_{a \le t \le b}\left|f(t) - P_N^{DE}[f]\right| \le C^{DE} \exp\left(\frac{-\pi d N}{\log\left(\frac{2dN}{\alpha}\right)}\right). \tag{15}$$

To apply the collocation method, set $t = t_j^{DE}$, $j = -N-1, \dots, N+1$ are Sinc grid points

$$t_j^{DE} = \begin{cases} a & j = -N-1, \\ \phi^{DE}(jh) & j = -N, \dots, N, \\ b & j = N+1, \end{cases}$$

It should be noted that in DE case M is set by $M = N + [\frac{log(\alpha)}{h}]$.

4. Numerical Examples

In order to illustrate the performance of the Sinc methods in solving weakly singular F-VIE and justify the accuracy and efficiency of the presented methods, we consider the following examples.

Example 1: Consider the following weakly singular F-VIE

$$x(t) = \sqrt{t} - \frac{\pi t^{3/2}}{2\sqrt{t}} - \frac{2}{3}\left(\sqrt{1-t} + 2t\sqrt{1-t} + 2t^{\frac{3}{2}}\right) + \int_0^1 \frac{(s(s))^2}{|t-s|^{\frac{1}{2}}} ds + \int_0^t \frac{x(s)}{(t-s)^{1/2}} ds, \quad t \epsilon [0,1]. \tag{16}$$

where the exact solution is $x(t) = \sqrt{t}$. Fig. 1 show maximum absolute errors corresponding to SE and DE-Sins methods. It shows that DE-Sinc method is more accurate than SE-Sinc method.

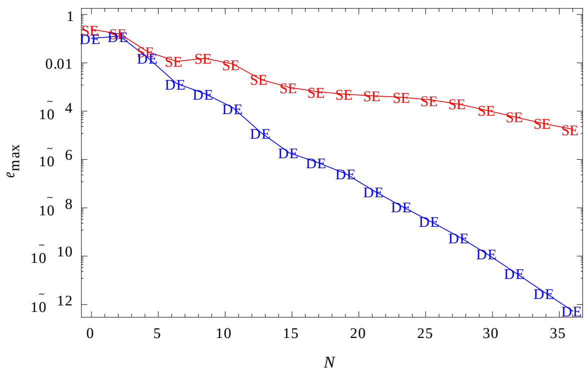

Fig. 1. The SE and DE-Sinc results for example 1.

Example 2: Consider the problem

$$x(t) = g(t) + \int_0^1 \frac{(s(s))^2}{|t-s|^{\frac{1}{4}}} ds + \int_0^t \frac{\sin(x(s))}{(t-s)^{1/4}} ds, \quad t\epsilon[0,1]. \tag{17}$$

where $g(t)$ is obtained so that $x(t) = \sqrt{t}$ is the solution. We list the absolute errors for several selected values of N for SE and DE-Sinc methods in Tables 1 and 2. Tables show that the convergence rate of the DE-Sinc method is much faster than the SE-Sinc scheme. Moreover, Fig. 2 shows maximum absolute errors for each method. This figure shows that DE-Sinc method is more accurate than SE-Sinc method.

Table 1. Absolute Errors of the SE-Sinc Method for Example 2

t	N=5	N=15	N=25	N=35
0.1	1.66E-3	4.86E-6	4.73E-7	1.91E-8
0.3	6.94E-4	1.07E-5	9.22E-7	6.81E-8
0.5	5.16E-4	1.07E-5	6.08E-7	5.79E-8
0.7	2.49E-3	1.91E-5	4.74E-7	3.25E-8
0.9	7.15E-4	1.70E-5	7.19E-7	3.77E-8

Table 2. Absolute Errors of the DE-Sinc Method for Example 2

t	N=5	N=15	N=25	N=35
0.1	5.81E-3	2.69E-7	7.20E-11	2.79E-14
0.3	7.83E-3	4.38E-9	8.10E-11	1.09E-14
0.5	3.58E-3	4.72E-8	8.22E-12	2.78E-15
0.7	1.40E-2	7.24E-8	7.33E-11	1.41E-14
0.9	1.15E-3	2.18E-7	6.55E-11	2.61E-14

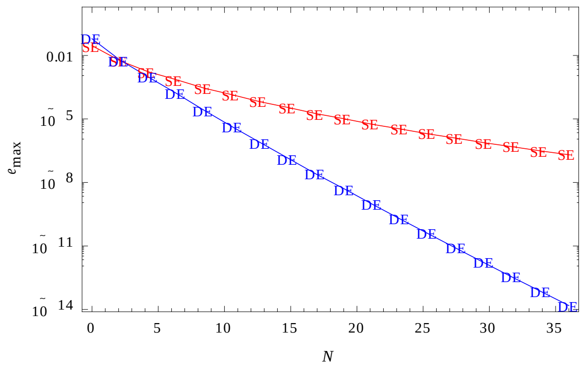

Fig. 2. The SE and DE-Sinc results for example 2.

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