Asymptotic Behaviors of an Opinion Dynamical System without or with a Leader

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Abstract: Consensus problems of an opinion dynamical system without a leader and with a leader under CS and MT influence functions are studied and a simple proof to guarantee that an opinion dynamical system can reach a consensus is also obtained in this paper. Moreover, results show that the system with a leader reaches a consensus more slowly than that without a leader by numerical simulations.

Key words: Dini derivative, Lyapunov function, opinion consensus.

1. Introduction

The consensus problems as one of the forms of self-organized systems have been widely observed from Physics [1], Chemistry [2], Biology [3]-[5], Computer science [6], [7], Human society, Psychology and Education [8]-[11] (in the emergence of language and spectator violence), Management science [12] and so on. Opinion consensus systems as one kind of consensus problems have attracted more and more researchers to study. There has been a lot of literature which focused on a natural question that how to form an opinion consensus by the interaction between individuals in the organization? Baum and Katz [13] considered the convergence rates in the law of large numbers and Couzin *et al*. [14] studied the effective leadership and decision making in animal groups on the move, but they did not model the consensus phenomenon. A natural problem is that is a system more quickly to reach an opinion consensus with a leader than that without a leader? What attracts us most is under which circumstances the given simple rule can lead agents' opinions to a steady state, that is, whether agents' opinions can reach a consensus by the proposed algorithm. Such problem carries critical applications for decision-making in social systems or flocking behaviors of animal population.

In this paper, inspired by [5], [8], [9], [10], [15], we study the celebrated opinion consensus problem which reflects opinion compromise of a certain event by different agents and give a simple proof to guarantee that an opinion dynamical system can reach a consensus. Moreover, we find that the system with a leader reaches a consensus more slowly than that without a leader by simulations.

For the purpose of the paper, we first give the mathematical formulation of the opinion dynamical system. Let $\mathcal{N} = \{1, 2, \dots, N\}$ denote a group of N agents and $x_i \in \mathbb{R}$ denote the i-th personal opinion(or position). Then the classical opinion consensus problem can be written as follows in continuous time ($t \in \mathbb{R}_+$)(see, e.g., [5], [6]),

$$\frac{dx_i}{dt} = \alpha \sum_{j=1, j \neq i}^{N} a_{ij}(x_j - x_i), \qquad i, j \in \mathcal{N} \tag{0.1}$$

with the initial conditions

$$x_i(0) = x_{i0} \in \mathbb{R}, \quad i = 1, 2, \ldots, N, \tag{0.2}$$

where α is a positive constant of self-adapting from the pairwise influence function and a_{ij} is the so-called pairwise influence function, which measures the influence of agent *j* on agent *i*. Cucker and Smale [9], [10] gave a symmetric form and an asymmetric form is taken by Motsch and Tadmor in [4]. We use the follow to denote the different influence functions, respectively,

$$a_{ij} := a_{ij}^{CS} = \frac{I(|x_j - x_i|)}{N} \quad \text{and} \quad a_{ij} := a_{ij}^{MT} = \frac{I(|x_j - x_i|)}{\sum_{k=1}^{N} I(|x_k - x_i|)}, \tag{0.3}$$

here, $I(|x_j - x_i|)$ $(j \neq i)$ originally quantifies the interaction of agent *j* on the alignment of agent *i* and $I(r)$ is decreasing on $[0, +\infty)$ such that $I(0) = 1$ and $\lim_{r \to \infty} I(r) = 0$.

The classical model (1.1) shows that agents are assumed to be equal in terms of their influence on each other, but in reality, there always exists leader-follower relationship in aggregation (see, e.g., [2], [14]). In a corporation, for example, the boss plays a vital role in efficient teamwork for his decision-making exerts much more influence on his employees than that of employees exerted on themselves. Once the leader makes up his mind, a consensus opinion will always be achieved eventually. Therefore, we also consider an opinion dynamical system with a leader and assume that the *N*-th agent is the leader, then we can depict leader-follower relationship with a mathematical model as follows

$$\begin{cases} \dfrac{dx_N}{dt} &= 0, \\ \dfrac{dx_i}{dt} &= \alpha \sum_{j=1, j \neq i}^{N-1} a_{ij}(x_j(t) - x_i(t)) + \beta a_{iN}(x_N(t) - x_i(t)), \end{cases} \tag{0.4}$$

where $i = 1, 2, \ldots, N-1$ and $\beta > 0$ expresses the impacts on every agent exerted by the leader N.

In this paper, "consensus" refers to general phenomena where each agent's opinion reaches a same value finally. We will study opinion dynamics of the system (1.1) and (1.4). For the purpose, firstly, we will introduce the concept of reaching consensus.

Definition 1.1. The system (1.1) or (1.4) is said to reach a consensus, if for any initial value there finally exists only one constant (opinion) c such that

$$\lim_{t \to \infty} x_i(t) = c, \qquad for \quad \forall i \in \mathcal{N}, \tag{0.5}$$

the constant c is called the consensus value.

The rest of this paper is organized as follows. We give a simple proof to obtain that both the system (1.1) without a leader and the system (1.4) with a leader can reach a consensus in Section 2 and Section 3 respectively. The opinion consensus results are verified by numerical simulations and the sociological significance is discussed in the last section.

2. The Opinion Dynamics of the System without a Leader

In this section, we study the opinion consensus problem of the system (1.1). Assume that $\sum_{j\neq i} a_{ij} \leq 1$ after rescaling α whether $a_{ij} = a_{ij}^{CS}$ or a_{ij}^{MT} and let $a_{ii} = 1 - \sum_{j\neq i} a_{ij}$ and then we have

$$\sum_{j=1}^{N} a_{ij} = 1 \tag{2.1}$$

Assume that $\{x_i(t)\}_{i=1}^{N}$ be the solutions to the system (1.1) with the initial value (1.2) and

$$d_X(t) = \max_{i,j} | x_j(t) - x_i(t) |.$$

Then by Definition 1.1, the system (1.1) reaches a consensus if and only if $\lim_{t\to\infty} d_X(t) = 0$.

Theorem 2.1. The opinion dynamical system (1.1) can reach a consensus whether $a_{ij} = a_{ij}^{CS}$ or a_{ij}^{MT}.

Proof. It just needs to prove $\lim_{t\to\infty} d_X(t) = 0$. For a given time t, there must exist integers m and n such that $d_X(t) = | x_m(t) - x_n(t) |$. Noting that $\sum_{q=1}^{N} a_{mq} = \sum_{l=1}^{N} a_{nl} = 1$, we have

$$\begin{aligned}
D^+ d_X^2 &= 2(x_m - x_n)(\dot{x}_m - \dot{x}_n) \\
&= 2\alpha(x_m - x_n)[\sum_{q=1}^{N} a_{mq}(x_q - x_m) - \sum_{l=1}^{N} a_{nl}(x_l - x_n)] \\
&= 2\alpha(x_m - x_n)[\sum_{q=1}^{N} a_{mq} x_q - \sum_{l=1}^{N} a_{nl} x_l - (x_m - x_n)] \\
&= 2\alpha(x_m - x_n)\sum_{q=1}^{N}\sum_{l=1}^{N} a_{mq} a_{nl}(x_q - x_l) - 2\alpha(x_m - x_n)^2 \\
&= 2\alpha(x_m - x_n)\sum_{q\neq l}\sum_{l=1}^{N} a_{mq} a_{nl}(x_q - x_l) + 2\alpha(x_m - x_n)\sum_{l=1}^{N} a_{ml} a_{nl}(x_m - x_n) \\
&\quad -2\alpha(x_m - x_n)\sum_{l=1}^{N} a_{ml} a_{nl}(x_m - x_n) - 2\alpha d_X^2 \\
&\leq 2\alpha\sum_{q=1}^{N}\sum_{l=1}^{N} a_{mq} a_{nl} d_X^2 - 2\alpha\sum_{l=1}^{N} a_{ml} a_{nl} d_X^2 - 2\alpha d_X^2 = -2\alpha\sum_{l=1}^{N} a_{ml} a_{nl} d_X^2.
\end{aligned}$$

Here D^+ denote the upper Dini derivative (see, e.g., [16]) and $D^+ d_X^2 \leq 0$ gives to a constant $M_1 > 0$

such that $0 \le d_X \le M_1$. From (1.3) and the decreasing of I, we have for every l,

$$a_{ml} \ge \frac{I(d_X)}{N} \ge \frac{I(M_1)}{N} > 0, \quad a_{nl} \ge \frac{I(d_X)}{N} \ge \frac{I(M_1)}{N} > 0$$

at any given time t. Therefore, $D^+ d_X^2 \le -2\alpha \frac{I^2(M_1)}{N} d_X^2$ which combines the fact $D^+ d_X^2 = 2d_X D^+ d_X$ give to

$$D^+ d_X \le -\alpha \frac{I^2(M_1)}{N} d_X,$$

which implies that $\lim_{t\to\infty} d_X(t) = 0$. The proof is complete.

Remark 2.2. In fact, if a_{ij} is symmetry then the total momentum in the model (1.1) is conserved for the center of mass coordinate $x_c(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ which satisfies $\dot{x}_c(t) = 0$. For this case, we can prove that the solutions of (1.1) with the initial value (1.2) satisfy $\lim_{t\to\infty} x_i(t) = x_c(0)$ by using a Lyapunov function as follows

$$V(t) = V(x_1(t), x_2(t), \ldots, x_n(t)) = \frac{1}{2}\sum_{i=1}^{N} x_i^2(t)$$

This means the system (1.1) can reach a consensus and $x_c(0)$ is the consensus value from Definiton 1.1.

3. The Opinion Dynamics of the System with a Leader

In what follows, we extend the results to opinion dynamics of the system with a leader. Recall the model (1.4), which contains N agents, including a leader agent and $N-1$ followers. The agent N is viewed as the leader as it is consistently not influenced by other agents, namely, $x_N(t) = x_N(0)$ for any $t \in \mathbb{R}_+$. And a consensus will be achieved via a process of followers adjusting their individual state according to their relative opinions with others' in order to reach a convergence as usual. Similarly, assume that $\{x_i(t)\}_{i=1}^{N}$ be the solutions of the system (1.4) with the initial value (1.2) and let $d_N(t) = \max_i |x_i(t) - x_N|$, then we can get the following statement.

Theorem 3.1. The opinion dynamical system (1.4) can reach a consensus if a_{ij} satisfies (1.3). Moreover, if $\{x_i(t)\}_{i=1}^{N}$ is the solution of the system (1.4) with (1.2), then $x_N(0)$ is the unique consensus value of the system (1.4).

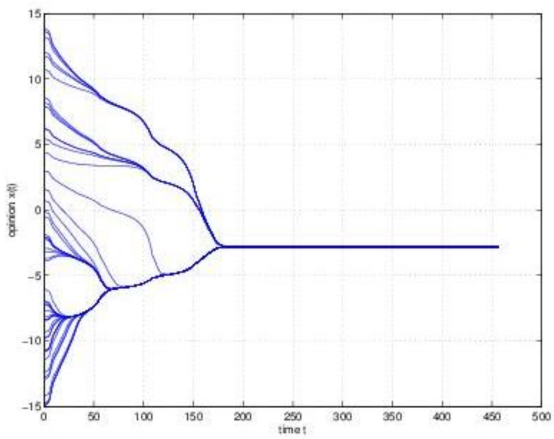
Proof. For a given time t, there exists an integer r such that $d_N(t) = |x_r(t) - x_N|$. Noting that (2.1) and using the same argument as in Section 2 show that

$$
\begin{aligned}
D^+ d_N^2 &= 2(x_r - x_N)(\dot{x}_r - \dot{x}_N) \\
&= 2(x_r - x_N)[\alpha \sum_{j=1}^{N-1} a_{rj}(x_j - x_r) + \beta a_{rN}(x_N - x_r)] \\
&= 2\alpha(x_r - x_N)\sum_{j=1}^{N} a_{rj}(x_j - x_r) + (2\alpha - 2\beta) a_{rN} d_N^2 \\
&= 2\alpha(x_r - x_N)[\sum_{j=1}^{N} a_{rj}(x_j - x_r) - x_N + x_N] + (2\alpha - 2\beta) a_{rN} d_N^2 \\
&= 2\alpha(x_r - x_N)[\sum_{j=1}^{N} a_{rj}(x_j - x_N) - (x_r - x_N)] + (2\alpha - 2\beta) a_{rN} d_N^2 \\
&= 2\alpha \sum_{j=1}^{N} a_{rj}(x_j - x_N)(x_r - x_N) - 2\alpha d_N^2 + (2\alpha - 2\beta) a_{rN} d_N^2 \\
&= 2\alpha \sum_{j=1}^{N-1} a_{rj}(x_j - x_N)(x_r - x_N) + 2\alpha a_{rN} d_N^2 - 2\alpha a_{rN} d_N^2 - 2\alpha d_N^2 + (2\alpha - 2\beta) a_{rN} d_N^2 \\
&\le 2\alpha d_N^2 - 2\alpha a_{rN} d_N^2 - 2\alpha d_N^2 + (2\alpha - 2\beta) a_{rN} d_N^2 \\
&= -2\beta a_{rN} d_N^2 .
\end{aligned}
$$

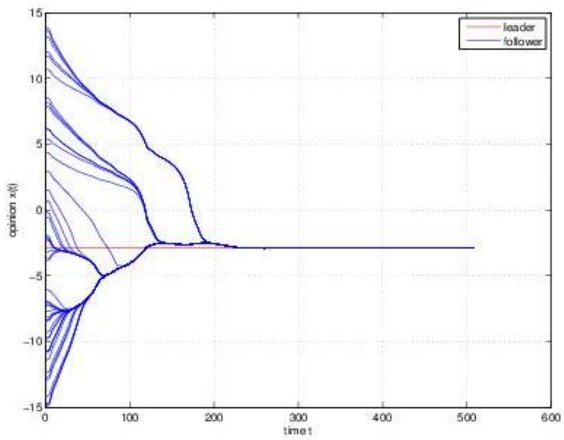
Therefore, $\lim_{t\to\infty} d_N(t) = 0$ which implies that $\lim_{t\to\infty} x_i(t) = x_N(0)$ for $i = 1, 2, \ldots, N-1$ and the consensus value will be $x_N(0)$.

Remark 3.2. From Theorem 3.1, all followers' opinions will ultimately tend to the leader's as long as the view of the leader is firm ($x_N(t) \equiv x_N(0)$).

4. Numerical Simulations



(a) Without a leader



(b) With a leader

Fig. 1. Numerical solutions of (1.1) and (1.4). Here $\alpha = 1$, $\beta = 5$, $N = 50$ and $a_{ij} = a_{ij}^{CS}$.

In this section, we carry out numerical simulations to illustrate our theoretical results. We assume $\alpha = 1$ and there are 50 agents in the two systems, i.e., $N = 50$. Also we choose the impact function

$I(r)=\frac{1}{1+r^2}$ in (1.3) from [5], [9]. Meanwhile, numerical simulations of the consensus of the system (1.1) without a leader and (1.4) with a leader will be shown according to different a_{ij} forms. Under stochastic initial value conditions which are randomly chosen from (-15,15) and a_{ij}^{CS} or a_{ij}^{MT}, we can find that both the system (1.1) without a leader and (1.4) with a leader can reach an opinion consensus (see Fig. 1 and Fig. 3 respectively).

It is interesting that the system without a leader than the system with a leader can more quickly reach an opinion consensus whether $a_{ij}=a_{ij}^{CS}$ or $a_{ij}=a_{ij}^{MT}$ (see Fig. 1 and Fig. 3). This phenomenon can be understood as the leader sticks to his own opinion ($x_N(t)\equiv x_N(0)$) and followers must obey the leader's order, which makes it more difficult to reach an opinion consensus. However, if the impact of the leader is not strong (β is smaller), then reaching an opinion consensus needs more time. This can be observed from (b) in Fig. 1 and Fig. 2 with $a_{ij}=a_{ij}^{CS}$ or from (b) in Fig. 3 and Fig. 4 with $a_{ij}=a_{ij}^{MT}$.

Two sets of initial conditions are overall same, randomly chosen from (-15,15). In (a) without a leader, the opinion consensus can be reached at $t=170$, while in (b) with a leader, the system (1.4) reaches the opinion consensus at $t=230$ which is more slowly than that of the case without a leader.

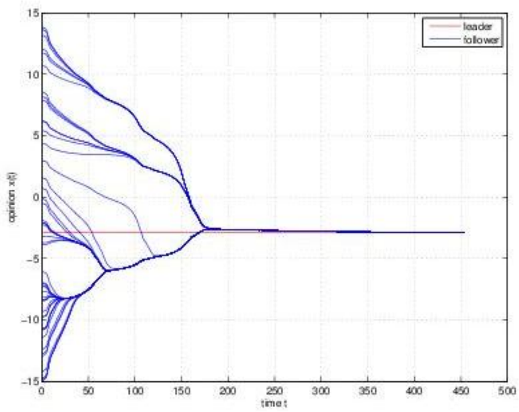
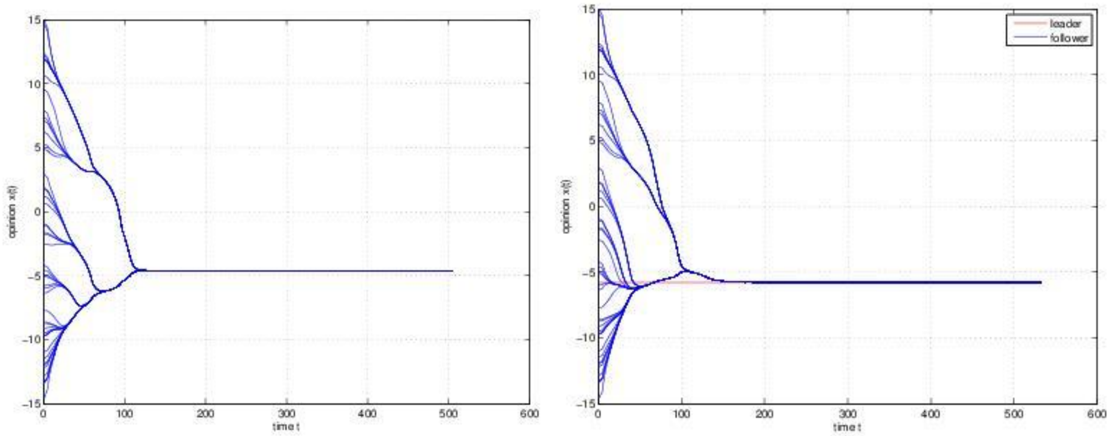


Fig. 2. Numerical solutions of (1.4) with $\alpha=1$, $\beta=\frac{1}{2}$, $N=50$, $a_{ij}=a_{ij}^{CS}$ and initial value conditions randomly chosen from (-15,15). The opinion consensus can be reached at $t>400$.

Opinion consensus phenomenon exists widely in our society and it is very interesting and worth to reveal how to form a consensus. Here, we give a simple proof to obtain that both the system (1.1) without a leader and the system (1.4) with a leader can reach a consensus. In our model (1.4), the external signal does not work to the leader, in other words, the latter is just like an arrant dictator. A consensus will not be easy to achieve unless it can exert a profound impact on others. The studied models are very close to real life, the simulation would be valuable for our understanding of emergent behaviors in social and biological systems.

Understanding the evolution of opinion dynamical system models is a significant issue. In this paper, we postulated relative models with no leader and one leader and demonstrated the process of agents' opinions in either model reaching an agreement respectively. On the other hand, we expect to hunt for some new adaptive ways of influencing others for a leader or to take the hierarchy structure of leader systems into consideration, for it reflects a wide range of realistic questions. For instance, a layered management system is vital for every corporation, so researches on it are exceedingly meaningful.



(a) Without a leader (b) With a leader

Fig. 3 Numerical solutions of (1.1) and (1.4). Here $\alpha = 1, \beta = 5, N = 50$ and $a_{ij} = a_{ij}^{MT}$.

Two sets of initial conditions are overall same, randomly chosen from (-15,15). In (a) without a leader, the opinion consensus can be reached at $t = 120$, while in (b) with a leader, the system (1.4) reaches the opinion consensus at $t = 170$ which is more slowly than that of the case without a leader.

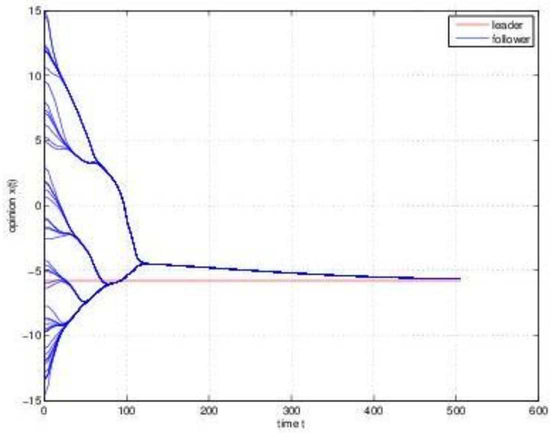


Fig. 4. Numerical solutions of (1.4) with $\alpha = 1, \beta = \frac{1}{2}, N = 50, a_{ij} = a_{ij}^{MT}$ and initial value conditions randomly chosen from (-15,15). The opinion consensus can be reached at $t > 500$.

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