Some Formulae of n-Norms and Their Identicalness in a Hilbert Space

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Abstract: In this paper, we discuss the concept of n-normed spaces and generalize a formulae of n-norm. Further we prove the equality of seven formulae of n-norms on a Hilbert space and eight formulae of n-norms on a separable Hilbert space. An alternative formula of n-norm on the dual of an n-normed space is introduced. Also, we show its equality with two alternative formulae.

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1. Introduction

Let X be a real vector space with dim $X \geqslant n$, where n is a positive integer. A real valued function $\|.,...,.\|: X^n \to \mathbb{R}$ is called an n-norm on X if the following conditions hold:

1) $\|x_1,...,x_n\| = 0$ iff $x_1,...,x_n$ are linearly dependent.
2) $\|x_1,...,x_n\|$ is invariant under permutations of $x_1,...,x_n$.
3) $\|\alpha x_1, x_2,...,x_n\| = |\alpha| \|x_1,...,x_n\|$ for any $\alpha \in \mathbb{R}$.
4) $\|x_0 + x_1, x_2,...,x_n\| \leq \|x_0,...,x_n\| + \|x_1,...,x_n\|$ for all $x_0, x_1,...,x_n \in X$.

The pair $(X, \|.,...,.\|)$ is called an n-normed space. An n-norm is always non-negative. The combination of conditions (3) and (4) above gives the non-negativity of an n-norm. If X is an n-normed space with dual X', the following formula (as formulated by "Gähler [1]").

$$\|x_1,...,x_n\|^G = \sup_{f_j \in X', \|f_j\| \leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on X.

If X is equipped with an inner product $\langle .,. \rangle$, we can define the standard n-norm on X by $\|x_1,...,x_n\|^S = \sqrt{\det\left[\langle x_i, x_j \rangle\right]}$.

Note that the value of $\|x_1,...,x_n\|^S$ represents the volume of n-dimensional parallelepiped spanned by $x_1,...,x_n$. Let X be a Hilbert space with dual X'. Then Gähler's formula on X becomes $\|x_1,...,x_n\|^G = \sup_{y_j \in X, \|y_j\| \leq 1} \det\left[\langle x_i, y_j \rangle\right]$.

Also the function

$$\|x_1,...,x_n\|^D = \underset{y_j\in X,\|y_1,...,y_n\|^S\leq 1}{Sup}\begin{vmatrix}\langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}$$

defines an n-norm on a Hilbert space X. Then $\|.,...,.\|^G$ and $\|.,...,.\|^D$ are identical on a Hilbert space X [2].

If X is a separable Hilbert space and $\{e_1,e_2,...\}$ is a complete orthonormal set in X, we can define an n-norm on X by

$$\|x_1,...,x_n\|_2 = \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left|\det\left[\alpha_{ij_k}\right]\right|^2\right]^{\frac{1}{2}}$$

where $\alpha_{ij}=\langle x_i,e_j\rangle$ [2], [3].

Further, the function $\|x_1,...,x_n\|^E = \underset{y_j\in X,\|y_1,...,y_n\|^S=1}{Sup}\begin{vmatrix}\langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}$ defines an n-norm on a Hilbert space and the function

$$\|x_1,...,x_n\|^r = \underset{f_j\in X',\|f_j\|=1}{Sup}\begin{vmatrix}f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}$$

defines an n-norm on a normed space X with dual X' [4].

If X is a Hilbert space, $\|.,...,.\|^r$ becomes $\|x_1,...,x_n\|^r = \underset{y_j\in X,\|y_j\|=1}{Sup}\det\left[\langle x_i,y_j\rangle\right]$. Then $\|.,...,.\|^D, \|.,...,.\|^E, \|.,...,.\|^G, \|.,...,.\|^r$ and $\|.,...,.\|^S$ are identical on a Hilbert space and they are identical with $\|.,...,.\|_2$ on a separable Hilbert space.

Also, $\|f_1,...,f_n\|' = \underset{x_i\in X,\|x_1,...,x_n\|\leq 1}{Sup}\begin{vmatrix}f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}$ and $\|f_1,...,f_n\|'_1 = \underset{x_i\in X,\|x_1,...,x_n\|=1}{Sup}\begin{vmatrix}f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}$ are identical n-norms on X',the dual of an n-normed space X [4].

The theory of 2-normed spaces and n-normed spaces were initially developed by Gähler [1], [5]-[7] in the 1960's. Recent works and related works can be found in [2], [3], [8]-[10]. The most recent work can be seen in [4]. Our interest here is to study alternative formulae of n-norms especially in a Hilbert space. The alternative formulae are identical with the n-norms mentioned above. In the last part we study the equality of three n-norms defined on the dual space of an n-normed space.

2. Generalization of an n-Norm

Let X be a real vector space with dim $X\geq n$ equipped with an inner product $\langle .,.\rangle$. Then the function

$$\|x_1,...,x_n\|^R = Abs\left(\begin{vmatrix}\langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}\right)$$

defines an n-norm on X for fixed linearly independent n elements $y_1,...,y_n\in X$ [4].

The following proposition is the generalization of the above proposition.

Proposition 2.1. Let X be a normed space of $\dim X \geq n$ with dual X'. Then the function

$$\|x_1,...,x_n\| = Abs\left(\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$$

defines an n-norm on X for fixed linearly independent n funtionals $f_1, f_2, ..., f_n \in X'$.

Proof: (i) It is easy to show that $x_1,...,x_n$ are linearly dependent iff $\|x_1,...,x_n\| = 0$.

(ii) The absolute value of a determinant remains invariant under the interchange of rows (or columns). $\Rightarrow$ $\|x_1,...,x_n\|$ is invariant under the permutations of $x_1,...,x_n$.

(iii) $\forall \alpha \in \mathbb{R}$,

$$\|\alpha x_1,...,x_n\| = Abs\left(\begin{vmatrix} f_1(\alpha x_1) & \cdots & f_n(\alpha x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$$

$$= Abs\left(\begin{vmatrix} \alpha f_1(x_1) & \cdots & \alpha f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right) \text{ (}\because f_i\text{'s are linear)}$$

$$= |\alpha| Abs\left(\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$$

$$= |\alpha| \|x_1,...,x_n\|$$

(iv) For $x_0, x_1, ..., x_n \in X$,

$$\begin{vmatrix} f_1(x_0+x_1) & \cdots & f_n(x_0+x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} = \begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} + \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

$$\Rightarrow Abs\left(\begin{vmatrix} f_1(x_0+x_1) & \cdots & f_n(x_0+x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right) \leq Abs\left(\begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right) + Abs\left(\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$$

$$\Rightarrow \|x_0 + x_1,...,x_n\| \leq \|x_0,...,x_n\| + \|x_1,...,x_n\|.$$

This completes the proof.

Remark: If X is a Hilbert space with dual X', the above n-norm $\|.,...,.\|$ becomes $\|.,...,.\|^R$. It follows from:

By *Riesz-representation theorem,* for each fixed bounded linear functional $f_j \in X'$, there exists unique $y_j \in X$ such that $f_j(x_i) = \langle x_i, y_j \rangle$ & $\|f_j\| = \|y_j\|$. Then, $\|x_1,...,x_n\| = Abs\left(\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}\right)$

$$= Abs\left(\begin{vmatrix} \langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}\right)$$ for linearly independent n elements $y_1,...,y_n$ $=\|x_1,...,x_n\|^R$.

3. Identicalness of Alternative n-Norms

Proposition 3.1. The function $\|x_1,...,x_n\|^F = \sup_{y_j\in X, \|y_1,...,y_n\|^S\neq 0} \frac{\begin{vmatrix} \langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S}$ defines an n-norm on a Hilbert space X.

Proof: (i) It is easy to show that $x_1,...,x_n$ are linearly dependent iff $\|x_1,...,x_n\|^F = 0$.

(ii) By the properties of determinant and definition of supremum, $\|x_1,...,x_n\|^F$ remains invariant under the permutations of $x_1,...,x_n$.

(iii) $\forall \alpha \in \mathbb{R}$,

$$\|\alpha x_1,...,x_n\|^F = \sup_{y_j\in X, \|y_1,...,y_n\|^S\neq 0} \frac{\begin{vmatrix} \langle \alpha x_1, y_1\rangle & \cdots & \langle \alpha x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S}$$

$$= \sup_{y_j\in X, \|y_1,...,y_n\|^S\neq 0} \frac{\begin{vmatrix} \alpha\langle x_1, y_1\rangle & \cdots & \alpha\langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S}$$

$$= \sup_{y_j\in X, \|y_1,...,y_n\|^S\neq 0} \frac{\alpha\begin{vmatrix} \langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S}$$

$$= |\alpha| \sup_{y_j\in X, \|y_1,...,y_n\|^S\neq 0} \frac{\begin{vmatrix} \langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S}$$

$$= |\alpha| \|x_1,...,x_n\|^F$$

(iv) For $x_0, x_1,...,x_n \in X$,

$$\frac{\begin{vmatrix} \langle x_0 + x_1, y_1\rangle & \cdots & \langle x_0 + x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S} = \frac{\begin{vmatrix} \langle x_0, y_1\rangle + \langle x_1, y_1\rangle & \cdots & \langle x_0, y_n\rangle + \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S}$$

$$= \frac{\begin{vmatrix} \langle x_0, y_1\rangle & \cdots & \langle x_0, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S} + \frac{\begin{vmatrix} \langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S}$$

$$\leq \underset{y_j\in X,\|y_1,...,y_n\|^S\neq 0}{Sup}\frac{\begin{vmatrix}\langle x_0,y_1\rangle & \cdots & \langle x_0,y_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}}{\|y_1,...,y_n\|^S}+\underset{y_j\in X,\|y_1,...,y_n\|^S\neq 0}{Sup}\frac{\begin{vmatrix}\langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}}{\|y_1,...,y_n\|^S}$$

$$\Rightarrow \frac{\begin{vmatrix}\langle x_0+x_1,y_1\rangle & \cdots & \langle x_0+x_1,y_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}}{\|y_1,...,y_n\|^S}\leq \|x_0,x_2,...,x_n\|^F+\|x_1,x_2,...,x_n\|^F \text{ for all } y_1,...,y_n\in X \text{ with } \|y_1,...,y_n\|^S\neq 0.$$

$$\Rightarrow \|x_0+x_1,x_2,...,x_n\|^F\leq\|x_0,x_2,...,x_n\|^F+\|x_1,x_2,...,x_n\|^F.$$

This completes the proof.

Proposition 3.2. On a Hilbert space X with $\dim X\geq n$, the two formulae $\|.,...,.\|^E$ and $\|.,...,.\|^F$ are identical.

Proof:

$$\|x_1,...,x_n\|^E=\underset{y_j\in X,\|y_1,...,y_n\|^S=1}{Sup}\begin{vmatrix}\langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}$$

And

$$\|x_1,...,x_n\|^F=\underset{y_j\in X,\|y_1,...,y_n\|^S\neq 0}{Sup}\frac{\begin{vmatrix}\langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}}{\|y_1,...,y_n\|^S}.$$

Clearly,

$$\|x_1,...,x_n\|^E\leq\|x_1,...,x_n\|^F.$$

Conversely, we choose $z_j=\frac{y_j}{\sqrt[n]{\|y_1,...,y_n\|^S}}=\frac{y_j}{a},a=\sqrt[n]{\|y_1,...,y_n\|^S}\neq 0$ for $j=1,2,...,n.$

Then,

$$\frac{\begin{vmatrix}\langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}}{\|y_1,...,y_n\|^S}=\frac{\begin{vmatrix}\langle x_1,az_1\rangle & \cdots & \langle x_1,az_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,az_1\rangle & \cdots & \langle x_n,az_n\rangle\end{vmatrix}}{\|az_1,...,az_n\|^S}$$

$$=\frac{a^n\begin{vmatrix}\langle x_1,z_1\rangle & \cdots & \langle x_1,z_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,z_1\rangle & \cdots & \langle x_n,z_n\rangle\end{vmatrix}}{a^n\|z_1,...,z_n\|^S}$$

$$=\begin{vmatrix}\langle x_1,z_1\rangle & \cdots & \langle x_1,z_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,z_1\rangle & \cdots & \langle x_n,z_n\rangle\end{vmatrix}\left(\because\|z_1,...,z_n\|^S=1\right)$$

$$\leq\underset{z_j\in X,\|z_1,...,z_n\|^S=1}{Sup}\begin{vmatrix}\langle x_1,z_1\rangle & \cdots & \langle x_1,z_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n,z_1\rangle & \cdots & \langle x_n,z_n\rangle\end{vmatrix}$$

$=\|x_1,...,x_n\|^E$ for all $y_j \in x$ with $\|y_1,...,y_n\|^S \neq 0.$

$$\Rightarrow \underset{y_j \in X, \|y_1,...,y_n\|^S \neq 0}{Sup} \frac{\begin{vmatrix} \langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S} \leq \|x_1,...,x_n\|^E .$$

$$\Rightarrow \|x_1,...,x_n\|^F \leq \|x_1,...,x_n\|^E .$$

This completes the proof.

Corollary 3.1. $\|.,...,.\|^G, \|.,...,.\|^S, \|.,...,.\|^D, \|.,...,.\|^E$ and $\|.,...,.\|^F$ are identical.

Proposition 3.3. On a separable Hilbert space X, $\|.,...,.\|^F$ and $\|.,...,.\|_2$ are identical.

Proof: Let $\{e_1,e_2,...\}$ be a complete orthonormal set in X .Then, $\|.,...,.\|_2$ may be derived directly from standard n-norm $\|.,...,.\|^S$ [6] $\Rightarrow \|.,...,.\|^S$ and $\|.,...,.\|_2$ are identical. Also, $\|.,...,.\|^S$ and $\|.,...,.\|^E$ are identical. So, $\|.,...,.\|^E$ and $\|.,...,.\|_2$ are identical. But, $\|.,...,.\|^E$ and $\|.,...,.\|^F$ are identical [proposition 3.2]. Therefore, $\|.,...,.\|^F$ and $\|.,...,.\|_2$ are identical.

Corollary 3.2. On a separable Hilbert space X, $\|.,...,.\|^G, \|.,...,.\|^S, \|.,...,.\|^D, \|.,...,.\|^E, \|.,...,.\|_2$ and $\|.,...,.\|^F$ are identical.

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