Reality of the Division by Zero $z/0 = 0$

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Abstract: In this paper, we will give some clear evidences of the reality of the division by zero $z/0 = 0$ with a fundamental algebraic theorem, and physical and geometrical examples; that is, 1) a field structure containing the division by zero, 2) by the gradient of the y axis on the (x, y) plane, 3) by the reflection $1/\overline{z}$ of z with respect to the unit circle with center at the origin on the complex z plane, and 4) by considering rotation of a right circle cone having some very interesting phenomenon from some practical and physical problem.

Key words: Division by zero, 1/0 = 0, field, Y-field, computer science, reflection with respect to circle, point at infinity, gradient, circle, curvature, EM radius, right circular cone.

1. Introduction

By a natural extension of the fractions

$$\frac{b}{a} \tag{1.1}$$

for any complex numbers a and b, we found the surprising result, for any complex number b

$$\frac{b}{0} = 0, \tag{1.2}$$

incidentally in [1] by the Tikhonov regularization for the Hadamard product in versions for matrices and we discussed their properties and gave several physical interpretations on the general fractions in [2] for the case of real numbers. The result is a very special case for general fractional functions in [3].

The division by zero has a long and mysterious story over the world (see, for example, google site with the division by zero) with its physical viewpoints since the document of zero in India on AD 628. However, Sin-Ei, Takahasi ([4]) (see also [2]) established a simple and decisive interpretation (1.2) by analyzing the extensions of fractions and by showing the complete characterization for the property (1.2):

Proposition 1 ([4], [5]). Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ such that

$$F(b, a)F(c, d) = F(bc, ad)$$

for all

$$a, b, c, d \in C$$

And

$$F(b, a) = \frac{b}{a}, \quad a, b \in C, a \neq 0.$$

Then, we obtain, for any $b \in C$

$$F(b, 0) = 0.$$

We thus should consider, for any complex number b, as (1.2); that is, for the mapping

$$w = \frac{1}{z}, \tag{1.3}$$

the image of $z = 0$ is $w = 0$. This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([6]). Therefore, the division by zero will give great impacts to complex analysis and to our idea for the world.

However, the division by zero (1.2) is now clear, indeed, for the introduction of (1.2), we have several independent approaches as in:

A. by the generalization of the fractions by the Tikhonov regularization or by the Moore-Penrose generalized inverse ([1], [2]),
B. by the intuitive meaning of the fractions (division) by H. Michiwaki ([7]),
C. by the unique extension of the fractions by S. Takahasi, as in Proposition 1([4], [5]),
D. by the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C}$, that is, $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ gives a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$,

and

E. by considering the values of functions with the mean values of functions.

Furthermore, we were able to find several meanings in the elementary geometry and physical meanings of the division by zero.

In this paper, in order to show the importance of the division by zero, we will give:

1) a field structure containing the division by zero ([8]),
2) by the gradient of the y axis on the (x, y) plane ([9], [10]),
3) by the reflection $1/\overline{z}$ of z with respect to the unit circle with center at the origin on the complex z plane ([11]), and
4) by considering rotation of a right circle cone having some very interesting phenomenon from some practical and physical problem ([12]).

See also [13] for the relationship between fields and the division by zero, and the importance of the division by zero for computer science. However, they state that in the conclusion:

The theory of meadows depends upon the formal idea of a total inverse operator. We do not claim that division by zero is possible in numerical calculations involving the rationals or reals. But we do claim that zero totalized division is logically, algebraically and computationally useful: for some applications, allowing zero totalized division in formal calculations, based on equations and rewriting, is appropriate because it is conceptually and technically simpler than the conventional concept of partial division.

It seems that the relationship of the division by zero and field structures are abstract in their paper.

On the division by zero, see the survey style announcements 179,185,237,246,247,250 and 252 of the Institute of Reproducing Kernels ([7]-[12], [14]). The division by zero is not only mathematical problems, but also it will give great impacts to human beings and the idea on the universe. The Institute of Reproducing Kernels is presenting various opinions in Announcements (many in Japanese) on the universe. It is clear that the items A, B, C and D are serious results in our mathematics.

At this moment, the following theorem may be looked as the fundamental theorem of the division by zero:

Theorem ([15]). Any analytic function takes a definite value at an isolated singular point **with a natural meaning.**

The essential meaning of this theorem may be understood in the sense of the mean values of analytic functions.

2. Introduction of the Y-Field

We consider

$$\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$$

and the direct decomposition

$$\mathbf{C}^2 = (\mathbf{C} \setminus \{0\})^2 + (\{0\} \times (\mathbf{C} \setminus \{0\})) + ((\mathbf{C} \setminus \{0\}) \times \{0\}) + \{0\}^2.$$

Then, we note

Theorem 1. For the set $\mathbf{C}^2$, we introduce the relation $\sim$: for any

$$(a, b), (c, d) \in (\mathbf{C} \setminus \{0\})^2,$$

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

and, for any $(a, b), (c, d) \notin (\mathbf{C} \setminus \{0\})^2$, in the above direct decomposition

$$(a, b) \sim (c, d).$$

Then, the relation $\sim$ satisfies the equivalent relation.

Definition 1. For the quotient set by the relation $\sim$ of the set $\mathbf{C}^2$, we write it by A and for the class containing (a, b), we shall write it by $\frac{a}{b}$.

Then, we obtain the main result:

Theorem 2. For any members $\frac{a}{b}, \frac{c}{d} \in A$, we introduce the product

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

and the sum +

$$\frac{a}{b}+\frac{c}{d}=\begin{cases}\frac{c}{d}, & if \quad \frac{a}{b}=\frac{0}{1}\\ \frac{a}{b}, & if \quad \frac{c}{d}=\frac{0}{1}\\ \frac{ad+bc}{bd}, & if \quad \frac{a}{b},\frac{c}{d}\neq\frac{0}{1},\end{cases}$$

then, the product and the sum are well-defined and A becomes a field **Y**.

Indeed, we can see easily the followings: 1) Under the operation + ,**Y** becomes an abelian group and $\frac{0}{1}=0Y$ is the unit element.

Under the operation, **Y** $\setminus$ {0Y }becomes an abelian group and $\frac{1}{1}=1\mathrm{Y}$ is the unit element.

In **Y** , operations + and satisfy distributive law.

The results are very simple, however, we will have some confusions for the long history of the division by zero. The statement that for any (a, b), (c, d) $\in(\mathbf{C}\setminus\{0\})^2$,

$$(a,b)\sim(c,d)\Leftrightarrow ad=bc,$$

that is for non zero members, we introduce the usual fractions $\frac{a}{b},\frac{c}{d}$ and the identity $\frac{a}{b}=\frac{c}{d}$. Other members like $\frac{0}{1},\frac{1}{0},\frac{0}{0}$ are identified, and therefore, from $\frac{0}{1}=0$, they are all zero. Hence, here, the division by zero $\frac{1}{0}$ is given by the definition as zero. As stated in the introduction, we can give the definition of the division by zero $\frac{1}{0}$, in several ways. Here, following the structure of fields, the division by zero is introduced as zero. The division by zero $\frac{1}{0}$ is not a mysterious one, but to look for its various meanings will be interesting research topics.

Remark. We note that

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

is not well-defined.

Theorem 3. The two fields **Y** and **C** are homomorphic.

Indeed, consider the mapping *f* from **Y** to **C**:

$$f: x=\frac{a}{b}|\rightarrow\begin{cases}ab^{-1}\left(\frac{a}{b}\neq 0\mathrm{Y}\right)\\ 0 \qquad \left(\frac{a}{b}=0\mathrm{Y}\right).\end{cases}$$

Then, we can see easily the followings: 1) $f(x+y)=f(x)+f(y)$, 2) $f(x\cdot y)=f(x)f(y)$, 3) $f(1\mathrm{Y})=1$, and 4) *f* is a one to one and onto mapping from **Y** to **C**.

We define an inversion operation $\varphi\mathrm{Y}$ on **Y** as

$$\varphi\mathrm{Y}\left(\frac{a}{b}\right)=\frac{b}{a}.$$

For the inverse element of $x=a/b\neq 0\mathrm{Y}$, we will denote it by x^{-1}.

Definition 2. We demote a binary operation / on **Y** as follows: For any x, y $\in$ **Y**

$$x/y = x \cdot \varphi Y(y) = \begin{cases} xy^{-1}(y \neq 0Y), \\ 0 \quad\quad (y = 0Y). \end{cases}$$

We will call the field **Y** with the operation φY*0-divisible field*. **C** becomesa 0-divisible field.

Theorem 3. By the homeophismf :**Y** → **C**, the inversion $\varphi = f \circ \varphi Y \circ f^{-1}$ from **C** onto **C** is induced.

Indeed, in **C**, the operation $\varphi = f \circ \varphi Y \circ f^{1}$ is induced by the homomorphic f from the 0-divisible field **C**. Then, for any $z \in$ **C**,

$$\varphi(z) = \begin{cases} z^{-1}(z \neq 0), \\ 0 \quad\quad (z = 0). \end{cases}$$

We, however, would like to state that the division by zero z/0 = 0 is essentially, just the definition, and we can derive all the properties of the division by zero, essentially, from the definition. Furthermore, by the idea of this session, we can introduce the fundamental concept of the divisions (fractions) in any field.

We should use the 0-divisible field **Y** for the complex numbers field **C** as complex numbers, by this simple modification. Then, we can see the new world that the point at infinity is represented by zero and the zero division $z/0$ is possible. Note that, however, the point ∞ at infinity makes sense still as the limit

$$\lim_{z \to 0} \frac{1}{z} = \infty.$$

The ∞ is not a definite number and is considered as the limit.

3. The Gradient of y-Axis Is Zero

Consider the lines $y = ax$ with gradients a through the origin (0, 0) on the (x, y) plane. Consider the two limits that a (>0) tends to +∞ and a (<0) tends to −∞, respectively. As their limits, we see that the limiting lines are y axis, in a sense. Note that the gradient of the y axis is zero, and is not infinity. This example shows as in the graph of the function $y = f(x) = 1/x$ at $x = 0$ as $f(0) = 0$, that was introduced by the division by zero 1/0 = 0 mathematically ([1], [2], [5], [10]).

For this result, Professor H. Begehr kindly referred to the gradient of the y axis in the above: If the gradient of the imaginary axis is 0 this would mean $\tan(\pi/2) = 0$, right? Of course this would be a consequence of 1/0 = 0! We had sent the e-mail, soon as follows:

For the gradient of y axis, we can define it as zero, in a very natural way and in the intuitive sense; of course, we can give its definition precisely.

However, as you stated, we can derive it formally by the division by zero

1/0 = 0; this deduction will be very interested in itself, because, the formal result 1/0 = 0 coincidents with the natural sense.

Surprisingly enough, this would mean $\tan(\pi/2) = 0$, right? This is right in our sense; we gave the definition of the values for analytic functions at an isolated singular point as in the theorem in the introduction.

As the fundamental results, we would like to state that

1) The gradient of the y axis is zero,

and

2) $\tan\frac{\pi}{2} = 0$.

4. Reflection Points

For simplicity, we will consider the unit circle $|z| = 1$ on the complex $z = x + iy$ plane. Then, we have the reflection formula

$$z^* = \frac{1}{\overline{z}} \tag{4.1}$$

for any point z, as well-known ([6]). For the reflection point $1/\overline{z}$, there is no problem for the points $z \neq 0, \infty$. As the classical result, the reflection of zero is the point at infinity and conversely, for the point at infinity we have the zero point. The reflection is a one to one and onto mapping between the inside and the outside of the unit circle.

Are these correspondences, however, suitable? Does there exist the point at ∞, really? Is the point at infinity corresponding to the zero point? Is the point at ∞ reasonable from the practical point of view? Indeed, where can we find the point at infinity? Of course, we know pleasantly the point at infinity on the Riemann sphere, however, on the complex z-plane it seems that we can not find the corresponding point. When we approach to the origin on a radial line, it seems that the correspondence reflection points approach to *the point at infinity* with the direction (on the radial line).

On the concept of the division by zero, there is no point at infinity ∞ as the numbers. For any point z such that $|z| > 1$, there exists the unique point z^* by (4.1). Meanwhile, for any point z such that $|z| < 1$ except $z = 0$, there exits the unique point z^* by (4.1). Here, note that for $z = 0$, by the division by zero, $z^* = 0$. Furthermore, we can see that

$$\lim_{z \to 0} z^* = \infty, \tag{4.2}$$

however, for $z = 0$ itself, by the division by zero, we have $z^* = 0$. This will mean a strong discontinuity of the functions $w = \frac{1}{z}$ and (4.1) at the origin $z = 0$; that is a typical property of the division by zero. This strong discontinuity may be looked in the above reflection property, physically.

Should we exclude the point at infinity, from the numbers? We were able to look the strong discontinuity of the division by zero in the reflection with respect to circles, physically (geometrical optics). The division by zero gives a one to one and onto mapping of the reflection (4.1) from the whole complex plane onto the whole complex plane.

The infinity ∞ may be considered as in (4.2) as the usual sense of limits, however, the infinity ∞ is not a definite number.

5. Circles and Curvature — An Interpretation of the Division by Zero $r/0 = 0$

We consider a solid body called right circular cone whose bottom is a disc with radius r_2. We cut the body with a disc of radius $r_1(0 < r_1 < r_2)$ that is parallel to the bottom disc. We denote the distance by d between the both discs and R the distance between the top point of the cone and the bottom circle on the surface of the cone. Then, R is calculated by Eko Michiwachi

(8 years old daughter of Mr. H. Michiwaki) as follows:

$$R = \frac{r_2}{r_2 - r_1}\sqrt{d^2 - (r_2 - r_1)^2},$$

that is called *EM radius*, because by the rotation of the cone on the plane, the bottom circle writes the circle of radius R. We denote by $K = K(R) = 1/R$ the curvature of the circle with radius R. We fix the distance d. Now note that:

$$r_1 \to r_2 \Rightarrow R \to \infty.$$

This will be natural in the sense that when $r_1 = r_2$, the circle with radius R becomes a line.

However, the division by zero will mean that when $r_1 = r_2$, the above EM radius formula makes sense and $R = 0$. What does it mean? Here, note that, however, then the curvature $K = K(0) = 0$ by the division by zero; that is, the circle with radius R becomes a line, similarly. The curvature of a point (circle of radius zero) is zero.

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References

[1] Saitoh, S., (2014). Generalized inversions of Hadamard and tensorproducts for matrices. *Advances in Linear Algebra & Matrix Theory, 4(2),* 87-95.

[2] Kuroda, M., Michiwaki, H., Saitoh, S., & Yamane, M. (2014). New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$. *Int. J. Appl. Math, 27(2)*, 191-198.

[3] Castro, L. P., & Saitoh, S. (2013). Fractional functions and their representations. *Complex Anal. Oper. Theory,* 7*(4),* 1049-1063.

[4] Takahasi, S.-E. (2014). On the identities $100/0 = 0$ and $0/0 = 0$.

[5] Takahasi, S.-E., Tsukada, M., & Kobayashi, Y. (2015). Classification of continuous fractional binary operations on the real and complex fields. *Tokyo Journal of Mathematics*, *8(2)*.

[6] Ahlfors, L. V. (1966). *Complex Analysis*. McGraw-Hill Book Company.

[7] Division by zero is clear as z/0=0 and it is fundamental in mathematics. *Announcement 179*.

[8] What are numbers? — The Yamada field containing the division by zero z/0 = 0. *Announcement 250*.

[9] An interpretation of the division by zero 1/0 = 0 by the gradients of lines. *Announcement 246.*

[10] The gradient of y-axis is zero and $\tan(\pi/2) = 0$ by the division by zero $1/0 = 0$. *Announcement 247*.

[11] A reality of the division by zero $z/0 = 0$ by geometrical optics. *Announcement 237*.

[12] Circles and curvature — An interpretation by Mr. Hiroshi Michiwaki of the division by zero $r/0 = 0$. *Announcement 252*.

[13] Bergstra, J. A., Hirshfeld Y., & Tucker, J. V. (2009). *Meadows and the Equational Specification of Division*.

[14] The importance of the division by zero $z/0 = 0$. *Announcement 185.*

[15] Michiwaki, H., Saitoh, S., & Takagi, M. A new concept for the point at infinity and the division by zero z/0=0.

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