An Exponential Spline Approach to the Generalized Sine-Gordon Equation

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Abstract: The nonlinear sine-Gordon equation is used to model many nonlinear phenomena. Numerical simulation of the solution to the one-dimensional generalized sine-Gordon equation is considered here. Two implicit three time-level difference schemes are developed, by using the exponential spline function approximation. We consider both Dirichlet and Neumann boundary conditions. The resulting spline difference schemes are analyzed for local truncation error, stability and convergence. It has been shown that by suitably choosing the parameters, we can obtain two schemes of $O(k^2+k^2h^2+h^2)$ and $O(k^2+k^2h^2+h^4)$. In the end, some numerical examples are provided to demonstrate the effectiveness of the proposed schemes.

Key words: Exponential spline, finite difference, generalizedsine-gordon equation, dirichlet and neumann boundary conditions, stability analysis, convergence.

1. Introduction

This paper is devoted to the numerical computation of the one-dimensional time-dependent nonlinear sine-Gordon equation. The initial-value problem of the one-dimensional generalized sine-Gordon equation is given by the following equation:

$$u_{tt}+\rho u_t=u_{xx}-f(x,t,u),\quad x\in(a,b),\quad t>t_0, \tag{1}$$

subjected to the initial conditions

$$u(x,t_0)=\phi(x),\quad u_t(x,t_0)=\psi(x),\quad x\in[a,b], \tag{2}$$

and with following Dirichletboundary conditions:

$$u(a,t)=g_0(t),\quad u(b,t)=g_1(t),\quad t\geq t_0, \tag{3}$$

where the functions $\phi(x)$ and $\psi(x)$ are wave modes or kinks and velocity, respectively. The parameter ρ is the so-called dissipative term, which is assumed to be a real number with $\rho\geq 0$ [1], [2]. We shall assume that $\phi(x)$, $\psi(x)$, $g_0(t)$ and $g_1(t)$ are continuously differentiable up to order 2.

Klein-Gordon and sine-Gordon equations have applications in various research areas, such as differential geometry and relativistic field theory [3]-[9].

The numerical solutions to the nonlinear sine-Gordon equation have received considerable attention in the literature (see [10-16]). Mohammadi [17]-[19] developed different spline schemes to find the numerical solution of different type of partial differential equations.

In this paper we have developed a three time-level implicit method by using the exponential spline function for solution of the nonlinear partial differential equation (1). The method involves some parameters, and its order can be increased from $O(k^2+k^2h^2+h^2)$ to $O(k^2+k^2h^2+h^4)$ by an appropriate choice of the parameters.

2. Proposition of the Method

The domain $[a,b]\times[0,T]$ is divided to a $n\times m$ mesh with the special step size $h=(b-a)/n$ in x direction and time step size $k=T/m$, respectively. Denote

$$\Omega_h=\{x_l=a+lh, l=0,1,\dots,n\}, \qquad \Omega_k=\{t_j=t_0+jk, j=0,1,\dots,m\}. \tag{4}$$

in which n and m are integers. The notation u_l^j be a grid function on $\Omega_h\times\Omega_k$, which is used for the difference approximation of $u(lh,jk)$. A function $S(x,t_j)$ of class $C^2[a,b]$ which interpolates $u(x,t_j)$ at the mesh points (x_l,t_j), depends on a parameter τ, reduces to cubic spline $S(x,t_j)$, in $[a,b]$ as $\tau\to 0$, is termed an exponential spline function. For each segment $[x_l,x_{l+1}]$, $l=0,1,\dots,n-1$ the function $S(x,t_j)$, can be defined in the following form

$$S(x,t_j)=a_l^*(t_j)+b_l^*(t_j)(x-x_l)+c_l^*(t_j)e^{i\tau(x-x_l)}+d_l^*(t_j)e^{-i\tau(x-x_l)}, \tag{5}$$

where $a_l^*(t_j)$, $b_l^*(t_j)$, $c_l^*(t_j)$ and $d_l^*(t_j)$ are unknown coefficients, τ is a free parameter and $i=\sqrt{-1}$. We first develop the explicit expressions for the four coefficients in (5) in terms of u_l^j, u_{l+1}^j, M_l^j and M_{l+1}^j. We can determine the four unknown coefficients in (5) as

$$a_l^*(t_j)=u_l^j+\frac{M_l^j}{\tau^2}, b_l^*(t_j)=\frac{M_{l+1}^j-M_l^j+\tau^2(u_{l+1}^j-u_l^j)}{h\tau^2}, c_l^*(t_j)=\frac{M_l^j-e^{i\tau h}M_{l+1}^j}{\tau^2(e^{2i\tau h}-1)}, d_l^*(t_j)=\frac{e^{i\tau h}(M_{l+1}^j-e^{i\tau h}M_l^j)}{\tau^2(e^{2i\tau h}-1)}.$$

The continuity of the first derivative of $S(x,t_j)$ at (x_l,t_j) yields the following consistency relation:

$$u_{l+1}^j-2u_l^j+u_{l-1}^j=h^2(\lambda_1 M_{l-1}^j+2\lambda_2 M_l^j+\lambda_1 M_{l+1}^j), \tag{6}$$

where

$$\lambda_1=(\theta\csc\theta-1)/\theta^2, \quad \lambda_2=(1-\theta\cot\theta)/\theta^2, \quad \theta=\tau h, \quad M_l^j=S''(x_l,t_j) \quad l=0,1,\dots,n \quad and \quad j=0,1,\dots,m.$$

Similar to (6) for the $(j+1)$ th and $(j-1)$ th time levels we have

$$u_{l+1}^{j\pm1}-2u_{l}^{j\pm1}+u_{l-1}^{j\pm1}=h^{2}\left(\lambda_{1}M_{l-1}^{j\pm1}+2\lambda_{2}M_{l}^{j\pm1}+\lambda_{1}M_{l+1}^{j\pm1}\right). \tag{7}$$

We develop an approximation for (1) in which the time derivative is replaced by following finite difference approximation

$$\overline{u_{tt}}_{l}^{j}=(u_{l}^{j+1}-2u_{l}^{j}+u_{l}^{j-1})/k^{2}=u_{ttl}^{j}+O(k^{2}), \qquad \overline{u_{t}}_{l}^{j}=(u_{l}^{j+1}-u_{l}^{j-1})/2k=u_{tl}^{j}+O(k^{2}), \tag{8}$$

and the space derivative by the non-polynomial spline function approximation

$$u_{xxl}^{j}=S''(x_{l},t_{j})=M_{l}^{j}+O(h^{2}). \tag{9}$$

To develop a new approximation for (1) the sine-Gordon equation (1) is then replaced by

$$\left[\left(u_{l}^{j+1}-2u_{l}^{j}+u_{l}^{j-1}\right)/k^{2}\right]+\left[\rho\left(u_{l}^{j+1}-u_{l}^{j-1}\right)/(2k)\right]+f_{l}^{j}=\sigma M_{l}^{j-1}+(1-2\sigma)M_{l}^{j}+\sigma M_{l}^{j+1}, \tag{10}$$

where $0\le\sigma\le1$, $f_{l}^{j}=f(x_{l},t_{j},u_{l}^{j})$ and $l=0,1,\ldots,n$, $j\ge0$.

The addition of (6) multiplied by $(1-2\sigma)$ to (7) multiplied by σ gives

$$\begin{aligned}&\lambda_{1}\left(\sigma M_{l-1}^{j-1}+(1-2\sigma)M_{l-1}^{j}+\sigma M_{l-1}^{j+1}\right)+2\lambda_{2}\left(\sigma M_{l}^{j-1}+(1-2\sigma)M_{l}^{j}+\sigma M_{l}^{j+1}\right)+\\&\lambda_{1}\left(\sigma M_{l+1}^{j-1}+(1-2\sigma)M_{l+1}^{j}+\sigma M_{l+1}^{j+1}\right)=\delta_{x}^{2}\left(\sigma u_{l}^{j-1}+(1-2\sigma)u_{l}^{j}+\sigma u_{l}^{j+1}\right)/h^{2},\end{aligned} \tag{11}$$

where δ_{x} is the central difference operator with respect to x so that $\delta_{x}^{2}u_{l}^{j}=u_{l-1}^{j}-2u_{l}^{j}+u_{l+1}^{j}$.

After the elimination of the M 's in (11) by means of (10) we have

$$\begin{aligned}&\left(\lambda_{1}\left(1-(k\rho/2)\right)-\sigma\mu^{2}\right)u_{l-1}^{j-1}+2\left(\lambda_{2}\left(1-(k\rho/2)\right)+\sigma\mu^{2}\right)u_{l}^{j-1}+\left(\lambda_{1}\left(1-(k\rho/2)\right)-\sigma\mu^{2}\right)u_{l+1}^{j-1}-\\&\quad\left(2\lambda_{1}+(1-2\sigma)\mu^{2}\right)u_{l-1}^{j}-2\left(2\lambda_{2}+(1-2\sigma)\mu^{2}\right)u_{l}^{j}-\left(2\lambda_{1}+(1-2\sigma)\mu^{2}\right)u_{l+1}^{j}+\\&\left(\lambda_{1}\left(1+(k\rho/2)\right)-\sigma\mu^{2}\right)u_{l-1}^{j+1}+2\left(\lambda_{2}\left(1+(k\rho/2)\right)-\sigma\mu^{2}\right)u_{l}^{j+1}+\left(\lambda_{1}\left(1+(k\rho/2)\right)-\sigma\mu^{2}\right)u_{l+1}^{j-1}+\\&\qquad k^{2}\left(\lambda_{1}f_{l-1}^{j}+2\lambda_{2}f_{l}^{j}+\lambda_{1}f_{l+1}^{j}\right)=0, \qquad l=1,2,\ldots,n-1,\end{aligned} \tag{12}$$

where $\mu=k/h$ is the mesh ratio, λ_{1} and λ_{2} are parameters defined in (6).

So by expanding (12) in Taylor series in terms of $u(x_{l},t_{j})$ and it's derivatives, and replacing the derivatives involving t by the relation

$$\partial u^{l+j}/\partial x^{l}\partial t^{j}=-\partial u^{l+2j}/\partial x^{l+2j}, \tag{13}$$

we obtain the local truncation error. The principal part of the local truncation error of the method is

$$\begin{aligned}T_{l}^{j}=&\Big\{\left[2(\lambda_{1}+\lambda_{2})-1\right]u_{xx}+h^{2}\left(-1/12+\lambda_{1}\right)\partial^{4}/\partial x^{4}-\left[h^{4}(1-30\lambda_{1})/360-k^{2}\left(\rho\left(\lambda_{1}+\lambda_{2}\right)-3\sigma\right)/3\right]\times\\&\partial^{6}/\partial x^{6}-(1/20160)\left[h^{6}\left(1-56\lambda_{1}\right)+3360k^{2}(\lambda_{1}+\lambda_{2})+1680k^{2}h^{2}\left(2\lambda_{1}\rho-\sigma\right)\right]\partial^{8}/\partial x^{8}+\ldots\Big\}u_{l}^{j},\end{aligned} \tag{14}$$

which tends to zero as h and k tend to zero simultaneously, so method is consistent with (1).

By choosing suitable values of parameters λ_1 and λ_2 in (12) we obtain the following classes of methods:

- If we choose $\lambda_1+\lambda_2=1/2$ and $\lambda_1\neq 1/12$ in (12) we obtain various schemes of $O(k^2+k^2h^2+h^2)$.
- If we choose $\lambda_1=1/12$ and $\lambda_2=5/12$, in (12) we obtain a new scheme of $O(k^2+k^2h^2+h^4)$.

3. Stability and Convergence Analysis

In this section, we will discuss the stability and convergence of the scheme (12) for numerical solution of Dirichlet boundary problem (1)-(3), with homogenous conditions $g_0(t)=g_1(t)=0$. First we define the following difference operators

$$\delta_{2t}u_l^j=u_l^{j+1}-u_l^{j-1},\ \ \delta_t^+u_l^j=u_l^{j+1}-u_l^j,\ \ \delta_t^-u_l^j=u_l^j-u_l^{j-1},\ \ \delta_{2t}u_l^j=(\delta_t^++\delta_t^-)u_l^j,\ \ \delta_t^2u_l^j=(\delta_t^+-\delta_t^-)u_l^j,$$
$$\delta_x^+u_l^j=u_{l+1}^j-u_l^j,\ \ \delta_x^-u_l^j=u_l^j-u_{l-1}^j,\ \ \delta_x^2u_l^j=u_{l+1}^j-2u_l^j+u_{l-1}^j,\ \ (u_t)_l^j=\delta_t^+u_l^j/k,\ \ (u_x)_l^j=\delta_x^+u_l^j/h.$$

If $v=(v_0,v_1,\ldots,v_n)$ and $w=(w_0,w_1,\ldots,w_n)$ are two grid functions on Ω_h, define the following inner product and the discrete L^2-norm

$$\langle v,w\rangle=h\sum_{l=0}^{n}v_lw_l,\quad \|v\|=\sqrt{\langle v,v\rangle}=(h\sum_{l=0}^{n}v_l^2)^{\frac{1}{2}}.$$

Denote $W^{i,per}(\Omega)$ the i-periodic Sobolev spaces [20], [21].

Now, as in papers [2], [15], we investigate the stability and convergence analysis of the main scheme (12) in the following Theorems.

Theorem 1. Let us assume that the function $U:\Omega\to R$ be the exact solution of (1)-(3) which

$$U\in W^{4,\infty}(0,+\infty;L^\infty(A,B))\cap L^\infty(0,+\infty;W^{4,\infty}(A,B)).$$

Let u be the solution of the difference scheme (12) and there exist constants $\chi_l,\ l=1,2,3$ such that

$$|f(x,t,u)|\le\chi_1(1+|x|+|t|+|u|),\quad |\partial f(x,t,u)/\partial x|\le\chi_2\ \text{ and }\ |\partial f(x,t,u)/\partial u|\le\chi_3.$$

Then, when

$$\mu^2<\begin{cases}(1-4\lambda_1)/(1-4\sigma), & if\quad \rho>5k/(1-4\lambda_1),\\ \left[(1-4\lambda_1)(1+k\rho/4)-5/4\right]/(1-4\sigma), & if\quad \rho\le 5k/(1-4\lambda_1),\end{cases}$$

for sufficiently small k there exists a constant $\chi>0$, independent of h and k such that

$$\|u^{m+1}\|+\|u_t^m\|+\|u_x^{m+1}\|\le\chi.$$

Proof. We can rewrite (12) in the following form

$$\left(\delta_t^2+(k\rho/2)\delta_{2t}\right)\left(1+\lambda_1\delta_x^2\right)u^j-\mu^2\delta_x^2\left(1+\sigma\delta_t^2\right)u^j+k^2\left(1+\lambda_1\delta_x^2\right)f^j=0. \quad (15)$$

Multiplying (15) by $\left(h\delta_{2t}u_l^j\right)$ and summing up for l from 0 to n, we have

$$\left\langle \delta_t^2\left(1+\lambda_1\delta_x^2\right)u^j,\delta_{2t}u^j\right\rangle+(k\rho/2)\left\langle \delta_{2t}\left(1+\lambda_1\delta_x^2\right)u^j,\delta_{2t}u^j\right\rangle-\mu^2\left\langle \delta_x^2\left(1+\sigma\delta_t^2\right)u^j,\delta_{2t}u^j\right\rangle+ k^2\left\langle\left(1+\lambda_1\delta_x^2\right)f^j,\delta_{2t}u^j\right\rangle=0. \quad (16)$$

By using the preliminary definitions and Lemmas in [2], we have

$$\left\|\delta_t^+u^j\right\|^2-\left\|\delta_t^-u^j\right\|^2-\lambda_1\left(\left\|\delta_x^+\delta_t^+u^j\right\|^2-\left\|\delta_x^+\delta_t^-u^j\right\|^2\right)+(k\rho/2)\left(\left\|\delta_{2t}u^j\right\|^2-\lambda_1\left\|\delta_x^+\delta_{2t}u^j\right\|^2\right)+ \sigma\mu^2\left(\left\|\delta_x^+u^{j+1}\right\|^2-\left\|\delta_x^+u^{j-1}\right\|^2\right)+(1-2\sigma)\left(\left\langle\delta_x^+u^{j+1},\delta_x^+u^j\right\rangle-\left\langle\delta_x^+u^j,\delta_x^+u^{j-1}\right\rangle\right)\le k^2\left[(1/2)\left(\left\|\delta_{2t}u^j\right\|^2+\left\|\delta_x^+\delta_{2t}u^j\right\|^2\right)+4\chi_1^2\left(1+\left\|u^j\right\|^2\right)+4\lambda_1^2\left(\chi_2^2h^2(b-a)+\chi_3^2\left\|\delta_x^+u^j\right\|^2\right)\right]. \quad (17)$$

By summing up (17) for j from 1 to m and using $\delta_t^-u^j=\delta_t^+u^{j-1}$, we have

$$\left\|\delta_t^+u^m\right\|^2-\lambda_1\left\|\delta_x^+\delta_t^+u^m\right\|^2+\left((k\rho/2)-(k^2/2)\right)\sum_{j=1}^m\left\|\delta_{2t}u^j\right\|^2-\left((k\rho\lambda_1/2)+(k^2/2)\right)\sum_{j=1}^m\left\|\delta_x^+\delta_{2t}u^j\right\|^2+ \sigma\mu^2\left(\left\|\delta_x^+u^{m+1}\right\|^2+\left\|\delta_x^+u^m\right\|^2\right)+(1-2\sigma)\mu^2\left\langle\delta_x^+u^{m+1},\delta_x^+u^m\right\rangle\le\left\|\delta_t^-u^1\right\|^2-\lambda_1\left\|\delta_x^+\delta_t^+u^0\right\|^2+\sigma\mu^2\left(\left\|\delta_x^+u^0\right\|+\left\|\delta_x^+u^1\right\|\right)+ +(1-2\sigma)\mu^2\left\langle\delta_x^+u^1,\delta_x^+u^0\right\rangle+k^2\left[4m\chi_1^2\left(1+\left\|u^j\right\|^2\right)+4\chi_1^2\sum_{j=1}^m\left\|u^j\right\|^2+4\lambda_1^2\left(m\chi_2^2h^2(b-a)+\chi_3^2\sum_{j=1}^m\left\|\delta_x^+u^j\right\|^2\right)\right]. \quad (18)$$

By using the identity defined in [2], we have

$$\left\|\delta_t^+u^m\right\|^2-\left(\lambda_1+\left((1-2\sigma)\mu^2/2\right)\right)\left\|\delta_x^+\delta_t^+u^m\right\|^2+\left((k\rho/2)-(k^2/2)\right)\sum_{j=1}^m\left\|\delta_{2t}u^j\right\|^2-\left((k\rho\lambda_1/2)+ (k^2/2)\right)\sum_{j=1}^m\left\|\delta_x^+\delta_{2t}u^j\right\|^2+(\mu^2/2)\left(\left\|\delta_x^+u^{m+1}\right\|^2+\left\|\delta_x^+u^m\right\|^2\right)\le\left\|\delta_t^-u^1\right\|^2+\sigma\left\|\delta_x^+u^0\right\|+\left((1-2\sigma)/2\right)\times \left(\left\|\delta_x^+u^1\right\|^2+\left\|\delta_x^+u^0\right\|^2-\left\|\delta_x^+\delta_t^+u^0\right\|^2\right)+k^2\left[4m\chi_1^2+4\chi_1^2\sum_{j=1}^m\left\|u^j\right\|^2+4\lambda_1^2\left(m\chi_2^2h^2(b-a)+\chi_3^2\sum_{j=1}^m\left\|\delta_x^+u^j\right\|^2\right)\right]. \quad (19)$$

We know that $\delta_t^-u^1=\delta_t^+u^0$, by help of Lemma 2 in [2] the fourth term on the left hand side of (18) can be written as

$$\left[(1-4\lambda_1)-(1-4\sigma)\mu^2-2\varepsilon\mu^2\right]\left\|\delta_t^+u^m\right\|^2+\left((k\rho(1-4\lambda_1)/2)-(5k^2/2)\right)\sum_{j=1}^m\left\|\delta_{2t}u^m\right\|^2+\varepsilon\mu^2\left(\left\|\delta_x^+u^{m+1}\right\|^2+ \left\|\delta_x^+u^m\right\|^2\right)\le\left\|\delta_t^+u^0\right\|+\sigma\left\|\delta_x^+u^0\right\|+\left((1-2\sigma)/2\right)\left(\left\|\delta_x^+u^1\right\|^2+\left\|\delta_x^+u^0\right\|^2\right)+k^2\left\{\left\|\delta_t^+u^0/k\right\|^2+(\sigma/\mu^2)\left\|\delta_x^+u^0/h\right\|^2+ \left((1-2\sigma)/\mu^2\right)\left(\left\|\delta_x^+u^1/h\right\|^2+\left\|\delta_x^+u^0/h\right\|^2\right)+\left[4m\chi_1^2+4\chi_1^2\sum_{j=1}^m\left\|u^j\right\|^2+4\lambda_1^2\left(m\chi_2^2h^2(b-a)+\chi_3^2\sum_{j=1}^m\left\|\delta_x^+u^j\right\|^2\right)\right]\right\}. \quad (20)$$

where ε is a sufficiently small coefficient. In the case $\left(\rho(1-4\lambda_1)-5k\right)/2>0$, i.e., $\rho>5k/(1-4\lambda_1)$, as ε can be sufficiently small, we need that $\mu^2<(1-4\lambda_1)/(1-4\sigma)$. In other case, as

$$\left\|\delta_{2t}u^m\right\|^2 = \left\|\delta_t^+ u^j + \delta_t^- u^j\right\|^2 \le \left\|\delta_t^+ u^j\right\|^2 + \left\|\delta_t^- u^j\right\|^2,$$

holds, we require that μ satisfy the following condition (for $j = m$)

$$(1-4\lambda_1)-(1-4\sigma)\mu^2+\left[k\rho(1-4\lambda_1)-5k^2\right]/2>0.$$

Then we have $\mu^2 < \left[(1-4\lambda_1)(1+k\rho/4)-5k^2/4\right]/(1-4\sigma)$. Therefore we proved the stability condition. Now we can easily prove that $\left\|\delta_t^+ u^0/k\right\|$, $\left\|\delta_x^+ u^0/h\right\|$ and $\left\|\delta_x^+ u^1/h\right\|$ are all bounded, consequently, all the terms on the right hand side of (20) have upper bound.

Therefore we confirm that all the terms on the right hand side of (20) are bounded. From the above notations we conclude that all the coefficients on the left hand side of (20) are positive when $\rho > 5k/(1-4\lambda_1)$. Hence from Lemma 2 in [2] in both cases we obtain that $\left\|\delta_t^+ u^m\right\| + \mu\left\|\delta_x^+ u^{m+1}\right\| \le k\chi$, i.e, $\left\|u_t^{\,m}\right\| + \mu\left\|u_x^{\,m+1}\right\| \le \chi$, under the stability condition. The relation $u^{m+1} = u^0 + \sum_{j=1}^{m}\delta_t^+ u^j$, gives that $\left\|u^{m+1}\right\| \le \left\|u^0\right\| + mk\chi$. The proof is completed.

Theorem 2. Suppose that the solution $u(x,t)$ of (1)-(3) is sufficiently regular and the assumptions of Theorem (1) are valid. For k sufficiently small, the solution of the spline difference scheme (12) converges to the solution of (1)-(3) in the discrete L^2-norm and we have

$$\left\|(U-u)^{m+1}\right\| + \left\|(U-u)_t^m\right\| + \left\|(U-u)_x^{m+1}\right\| \le \begin{cases} \chi(k^2+k^2h^2+h^2), & if \quad 2(\lambda_1+\lambda_2)=1, \quad \lambda_1 \ne 1/12 \\ \chi(k^2+k^2h^2+h^4), & if \quad \lambda_1 = 1/12, \quad \lambda_2 = 5/12. \end{cases}$$

4. Numerical Illustrations

We applied the presented schemes to the following nonlinear generalized Sine-Gordon equation. We solve following example by our $O(k^2+k^2h^2+h^2)$ method with $(\lambda_1,\lambda_2)=(1/6,1/3)$ and our $O(k^2+k^2h^2+h^4)$ method with $(\lambda_1,\lambda_2)=(1/12,5/12)$. We tabulate the computed errors in solution in our Tables and compare our results with the results in [2].

Example. Consider the nonlinear Sine-Gordon equation:

$$u_{tt}+\rho u_t = u_{xx} - 2\sin u - \pi^2 e^{-t}\cos\pi x + 2\sin\left(e^{-t}(1-\cos\pi x)\right), \quad x\in(0,2), \quad t>0,$$

subjected to the initial conditions

$$u(x,0)=1-\cos\pi x, \qquad u_t(x,0)=-1+\cos\pi x, \qquad x\in[0,2],$$

and with Dirichlet boundary conditions

$$u(0,t)=u(2,t)=0, \qquad t\ge 0.$$

The exact solution for this problem is $u(x,t)=e^{-t}(1-\cos\pi x)$.

We solve this problem with $h = 0.1, 0.05$, $k = 0.01, 0.005$, $\sigma = \rho/6$ and different values of the parameters λ_1 and λ_2. Observed L_2 and RMS errors in the computed solution are tabulated in Tables 1-2 for different times. The results in Tables show that our $O(k^2 + k^2h^2 + h^4)$ method is more accurate than the methods in [2].

Table 1. Observed L_2 and RMS Errors in Our $O(k^2 + k^2h^2 + h^2)$ Method

t	$h = 0.1, \quad k = 0.01$		$h = 0.05, \quad k = 0.01$		$h = 0.05, \quad k = 0.005$	
	L_2	RMS	L_2	RMS	L_2	RMS
1.0	2.13(−2)	4.66(−3)	7.61(−3)	1.18(−3)	7.54(−3)	1.17(−3)
2.0	8.71(−3)	1.90(−3)	3.13(−3)	4.89(−4)	3.13(−3)	4.89(−4)
3.0	5.79(−3)	1.26(−3)	2.05(−3)	3.20(−4)	2.03(−3)	3.17(−4)
4.0	2.63(−3)	5.74(−4)	9.41(−4)	1.47(−4)	9.30(−4)	1.45(−4)
5.0	1.55(−3)	3.40(−4)	5.53(−4)	8.64(−5)	5.52(−4)	8.63(−5)

Table 2. Observed L_2 and RMS Errors in Our $O(k^2 + k^2h^2 + h^4)$ Method

t	$h = 0.1, \quad k = 0.01$		$h = 0.05, \quad k = 0.01$		$h = 0.05, \quad k = 0.005$	
	L_2	RMS	L_2	RMS	L_2	RMS
1.0	7.12(−5)	1.55(−5)	3.99(−5)	6.24(−6)	2.98(−6)	4.65(−7)
2.0	2.88(−5)	6.29(−6)	1.74(−5)	2.71(−6)	1.50(−6)	2.35(−7)
3.0	1.90(−5)	4.14(−6)	1.09(−5)	1.71(−6)	8.72(−7)	1.36(−7)
4.0	8.77(−6)	1.91(−6)	5.04(−6)	7.87(−7)	3.98(−7)	6.21(−8)
5.0	5.10(−6)	1.11(−6)	2.99(−6)	4.67(−7)	2.44(−7)	3.81(−8)

5. Conclusion

In this article, we constructed a three time-level spline-difference scheme for the one-dimensional generalized sine-Gordon equation. We proved the stability and convergence of the developed schemes by energy method for our scheme. We prove that our presented schemes are unconditionally stable. To examine the accuracy and efficiency of the proposed algorithm, we give one numerical examples. These computational results show that our proposed algorithm is effective and accurate in comparison with [2].

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