Simplified Method of Evaluating Integrals of Powers of Sine Using Reduction Formula as Mathematical Algorithm

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Abstract: This paper presents a simpler and shorter method of evaluating integrals of powers of sine. The reduction formula for sine is repeatedly applied to the integral of the nth power of sine until generalized formulas are derived. Since the derivation process involves recursive relations, the coefficients and exponents of the derived formulas showed certain patterns and sequences which were used as the basis for developing an easier algorithm.

Key words: Integration, mathematical algorithm, powers of sine, reduction formula, trigonometric identities.

1. Introduction

Evaluating integrals of powers of trigonometric functions is always part of the study of Integral Calculus. Integrals of powers of sine are usually evaluated using trigonometric identities and the solution depends on whether the power is odd or even. For odd powers, the integrand is transformed by factoring out one sine and the remaining even powered sine is converted into cosine using the identity $\sin^2 x = 1 - \cos^2 x$. The integral is then evaluated using power formula with the factored sine used as the differential of cosine. For even powers, the double angle identity $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ is used to reduce the power of sine into an expression where direct integration formulas can already be applied [1]-[3].

Another method used to evaluate powers of sine is by using reduction formula. A reduction formula transforms the integral into an integral of the same or similar expression with a lower integer exponent [4]. It is repeatedly applied until the power of the last term is reduced to two or one, and the final integral can be evaluated. Using integration by parts, the reduction formula for sine is [5].

$$\int \sin^n ax dx = -\frac{1}{a(n)} \sin^{n-1} ax \cos ax + \frac{n-1}{n} \int \sin^{n-2} ax dx$$

The methods discussed above are normally tedious and time consuming depending on the given power of sine. As shown in the study of Dampil [6], deriving generalized formulas can simplify solutions, hence, the objective of this paper is to come up with a shorter and simpler method of integrating powers of sine. Generalized formulas are derived by successive application of the reduction formula to the integral of the nth power of sine. Because of the recursive nature of the reduction formula, an algorithm is developed based on the sequences and patterns of the coefficients and exponents of the terms of the derived formulas.

2. Derivation of Formulas

Given: $\int \sin^n ax dx$, where n is any integer

Using the reduction formula,

$$\int \sin^n ax dx = -\frac{1}{a(n)}\sin^{n-1} ax \cos ax + \frac{n-1}{n}\int \sin^{n-2} ax dx$$

Applying the reduction formula to the last term

$$\int \sin^n ax dx = -\frac{1}{a(n)}\sin^{n-1} ax \cos ax + \frac{n-1}{n}\left[-\frac{1}{a(n-2)}\sin^{n-3} ax \cos ax + \frac{n-3}{n-2}\int \sin^{n-4} ax dx\right]$$

Applying the reduction formula again,

$$\int \sin^n ax dx = -\frac{1}{a(n)}\sin^{n-1} ax \cos ax - \frac{n-1}{a(n)(n-2)}\sin^{n-3} ax \cos ax -$$
$$\frac{(n-1)(n-3)}{n(n-2)}\left[-\frac{1}{a(n-4)}\sin^{n-5} ax \cos ax + \frac{n-3}{n-4}\int \sin^{n-6} ax dx\right]$$

Simplifying,

$$\int \sin^n ax dx = -\frac{1}{a(n)}\sin^{n-1} ax \cos ax - \frac{n-1}{a(n)(n-2)}\sin^{n-3} ax \cos ax -$$
$$\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)}\sin^{n-5} ax \cos ax + \frac{(n-1)(n-3)}{n(n-2)(n-4)}\int \sin^{n-6} ax dx$$

The same trend continues until the last term becomes

$\int \sin ax dx$ if n is odd, or

$\int \sin^2 ax dx$ if n is even

2.1. Odd Powers

$$\int \sin^n ax dx = -\frac{1}{a(n)}\sin^{n-1} ax \cos ax - \frac{n-1}{a(n)(n-2)}\sin^{n-3} ax \cos ax -$$
$$\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)}\sin^{n-5} ax \cos ax - .... + \frac{(n-1)(n-3)(n-5)...(2)}{n(n-2)(n-4)(n-6)...(3)}\int \sin ax dx$$

Integrating the last term,

$$\int \sin^n ax dx = -\frac{1}{a(n)}\sin^{n-1} ax \cos ax - \frac{n-1}{a(n)(n-2)}\sin^{n-3} ax \cos ax - \frac{n-1}{a(n)(n-2)}\sin^{n-3} ax \cos ax -$$

$$\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)}\sin^{n-5}ax\cos ax-....-\frac{(n-1)(n-3)(n-5)....(2)}{a(n)(n-2)(n-4)(n-6)...(3)}\cos ax+C$$

Factoring out the common factor gives the formula,

$$\int\sin^{n}axdx=-\frac{\cos ax}{a}\left[\frac{1}{n}\sin^{n-1}ax+\frac{n-1}{n(n-2)}\sin^{n-3}ax+\frac{(n-1)(n-3)}{n(n-2)(n-4)}\sin^{n-5}ax+....+\frac{(n-1)(n-3)(n-5)...(2)}{n(n-2)(n-4)(n-6)...(3)}\right]+C$$

It can also be written as

$$\int\sin^{n}axdx=-\frac{\cos ax}{a}\left[C_0\sin^{n-1}ax+\sum_{j=1}^{\frac{n-1}{2}}C_j\sin^{n-2j-1}ax\right]+C$$

where: $C_0=\frac{1}{n}$, and $C_j=C_{j-1}\left[\frac{n-2j+1}{n-2j}\right]$

2.2. Even Powers

$$\int\sin^{n}axdx=-\frac{1}{a(n)}\sin^{n-1}ax\cos ax-\frac{n-1}{a(n)(n-2)}\sin^{n-3}ax\cos ax-\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)}\sin^{n-5}ax\cos ax-....+\frac{(n-1)(n-3)(n-5)...(3)}{n(n-2)(n-4)(n-6)...(4)}\int\sin^{2}axdx$$

Applying the reduction formula to the last term,

$$\int\sin^{n}axdx=-\frac{1}{a(n)}\sin^{n-1}ax\cos ax-\frac{n-1}{a(n)(n-2)}\sin^{n-3}ax\cos ax-\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)}\sin^{n-5}ax\cos ax-....+\frac{(n-1)(n-3)(n-5)...(3)}{n(n-2)(n-4)(n-6)...(4)}\left[-\frac{1}{a(2)}\sin ax\cos ax+\frac{1}{2}\sin^{0}axdx\right]$$

Simplifying,

$$\int\sin^{n}axdx=-\frac{1}{a(n)}\sin^{n-1}ax\cos ax-\frac{n-1}{a(n)(n-2)}\sin^{n-3}ax\cos ax-\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)}\sin^{n-5}ax\cos ax-...-\frac{(n-1)(n-3)....(3)}{a(n)(n-2)(n-4)...(2)}\sin ax\cos ax+\frac{(n-1)(n-3)...(3)}{a(n)(n-2)(n-4)...(2)}x+C$$

Factoring out the common factor gives,

$$\int\sin^{n}axdx=-\frac{\cos ax}{a}\left[\frac{1}{n}\sin^{n-1}ax+\frac{n-1}{n(n-2)}\sin^{n-3}ax+\frac{(n-1)(n-3)}{n(n-2)(n-4)}\sin^{n-5}ax+...+\right.$$

$$\frac{(n-1)(n-3)(n-5)\ldots(3)}{n(n-2)(n-4)(n-6)\ldots(2)}\sin ax\Bigg] + \frac{(n-1)(n-3)(n-5)\ldots(3)}{n(n-2)(n-4)(n-6)\ldots(2)}x + C$$

The formula may also be written as,

$$\int \sin^n ax dx = -\frac{\cos ax}{a}\left[C_0 \sin^{n-1} ax + \sum_{j=1}^{\frac{n-2}{2}} C_j \sin^{n-2j-1} ax\right] + C_{\frac{n-2}{2}} x + C$$

where: $C_0 = \frac{1}{n}$, and $C_j = C_{j-1}\left[\frac{n-2j+1}{n-2j}\right]$

3. Development of the Algorithm for the New Method

A simpler and easier procedure can be developed from the observed trends of the coefficients and exponents of the derived formulas. These are summarized as follows:

3.1. Odd Powers

- Write $-\frac{\cos ax}{a}$. This will be followed by a series of sine terms. For example, $\int \sin^5 2x dx$

$$-\frac{\cos 2x}{2}$$

- The first term of the series has a coefficient of $\frac{1}{n}$ and the exponent of sine is n-1. This coefficient and exponent will be used in determining the coefficient and exponent of the next term.

$$-\frac{\cos 2x}{2}\left[\frac{1}{5}\sin^4 2x + \right.$$

- For the next term, the coefficient has a numerator equal to the product of the exponent and the numerator of the preceding term. The denominator is the product of the denominator and exponent minus one of the preceding term. The exponent of sine is the exponent of the preceding term minus two.

$$-\frac{\cos 2x}{2}\left[\frac{1}{5}\sin^4 2x + \frac{(1)(4)}{(5)(3)}\sin^2 2x\right.$$

- Follow the same procedure until the exponent of sine becomes zero which terminates the series.

$$-\frac{\cos 2x}{2}\left[\frac{1}{5}\sin^4 2x + \frac{4}{15}\sin^2 2x + \frac{(4)(2)}{(15)(!)}\sin^0 2x\right]$$

- Add a constant of integration.

$$\int \sin^5 2x dx = -\frac{\cos 2x}{2}\left[\frac{1}{5}\sin^4 2x + \frac{4}{15}\sin^2 2x + \frac{5}{15}\right] + C$$

3.2. Even Powers

- Write $-\frac{\cos ax}{a}$. This will be followed by a series of sine terms. For example, $\int \sin^6 3x dx$

$$-\frac{\cos 3x}{3}$$

- The first term of the series has a coefficient of $\frac{1}{n}$ and the exponent of sine is n-1. This coefficient and exponent will be used in determining the coefficient and exponent of the next term.

$$-\frac{\cos 3x}{3}\left[\frac{1}{6}\sin^5 3x+\right.$$

- For the next term, the coefficient has a numerator equal to the product of the exponent and the numerator of the preceding term. The denominator is the product of the denominator and exponent minus one of the preceding term. The exponent of sine is the exponent of the preceding term minus two.

$$-\frac{\cos 3x}{3}\left[\frac{1}{6}\sin^5 3x+\frac{(1)(5)}{(6)(4)}\sin^3 3x+\right.$$

- Follow the same procedure until the exponent of sine becomes one which terminates the series.

$$-\frac{\cos 3x}{3}\left[\frac{1}{6}\sin^5 3x+\frac{5}{24}\sin^3 3x+\frac{(5)(3)}{(24)(2)}\sin^1 3x\right]$$

- The next term is the product of x and the coefficient of the last term in the sine series.

$$-\frac{\cos 3x}{3}\left[\frac{1}{6}\sin^5 3x+\frac{5}{24}\sin^3 3x+\frac{15}{48}\sin 3x\right]+\frac{15}{48}x$$

- Add a constant of integration.

$$\int \sin^6 3x dx=-\frac{\cos 3x}{3}\left[\frac{1}{6}\sin^5 3x+\frac{5}{24}\sin^3 3x+\frac{15}{48}\sin 3x\right]+\frac{15}{48}x+C$$

4. Comparison between the Old and the New Method

Evaluate $\int \sin^7 4x dx$

Using the Old Method

$$\int \sin^7 5x dx=-\frac{1}{5(7)}\sin^6 5x\cos 5x+\frac{6}{7}\int \sin^5 5x dx$$

$$= -\frac{1}{35}\sin^6 5x\cos 5x + \frac{6}{7}\left[-\frac{1}{5(5)}\sin^4 5x\cos 5x + \frac{4}{5}\int \sin^3 5x\right]$$

$$= -\frac{1}{35}\sin^6 5x\cos 5x - \frac{6}{175}\sin^4 5x\cos 5x + \frac{24}{35}\left[-\frac{1}{5(3)}\sin^2 5x\cos 5x + \frac{2}{3}\int \sin 5x dx\right]$$

$$= -\frac{1}{35}\sin^6 5x\cos 5x - \frac{6}{175}\sin^4 5x\cos 5x - \frac{24}{525}\sin^2 5x\cos 5x - \frac{48}{525}\cos 5x + C$$

$$= -\frac{\cos 5x}{5}\left[\frac{1}{7}\sin^6 5x + \frac{6}{35}\sin^4 5x + \frac{24}{105}\sin^2 5x + \frac{48}{105}\right] + C$$

Using the New Method

$$\int \sin^7 5x dx = -\frac{\cos 5x}{5}\left[\frac{1}{7}\sin^6 5x + \frac{(1)(6)}{(7)(5)}\sin^4 5x + \frac{(6)(4)}{(35)(3)}\sin^2 5x + \frac{(24)(2)}{(105)(1)}\sin^0 5x\right] + C$$

$$\int \sin^7 5x dx = -\frac{\cos 5x}{5}\left[\frac{1}{7}\sin^6 5x + \frac{6}{35}\sin^4 5x + \frac{24}{105}\sin^2 5x + \frac{48}{105}\right] + C$$

Evaluate $\int \sin^4 2x$

Using the Old Method

$$\int \sin^4 2x dx = -\frac{1}{2(4)}\sin^3 2x\cos 2x + \frac{3}{4}\int \sin^2 2x dx$$

$$\int \sin^4 2x dx = -\frac{1}{8}\sin^3 2x\cos 2x + \frac{3}{4}\left[-\frac{1}{2(2)}\sin 2x\cos 2x + \frac{1}{2}\int dx\right]$$

$$\int \sin^4 2x dx = -\frac{1}{8}\sin^3 2x\cos 2x - \frac{3}{16}\sin 2x\cos 2x + \frac{3}{8}x + C$$

$$\int \sin^4 2x dx = -\frac{\cos 2x}{2}\left[\frac{1}{4}\sin^3 2x + \frac{3}{8}\sin 2x\right] + \frac{3}{8}x + C$$

Using the New Method

$$\int \sin^4 2x dx = -\frac{\cos 2x}{2}\left[\frac{1}{4}\sin^3 2x + \frac{(1)(3)}{(4)(2)}\sin 2x\right] + \frac{3}{8}x + C$$

$$\int \sin^4 2x dx = -\frac{\cos 2x}{2}\left[\frac{1}{4}\sin^3 2x + \frac{3}{8}\sin 2x\right] + \frac{3}{8}x + C$$

5. Conclusion

The new method is simpler and easier to use since the tiresome repetitions of applying the reduction formula, or expansions of identities using the conventional methods are eliminated. Integrals can be evaluated directly since the procedure simply involves coefficients and exponents. It is very helpful since integrals of powers of sine are always encountered in higher mathematics courses like Differential Equations and Advanced Engineering Mathematics, and even in physics and mechanics. It can also be used in many engineering applications specifically in electricity and magnetism, waves, heat and mass transfer, and reaction kinetics. It is also suggested that the method be extended to integrals of powers of other trigonometric functions.

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