Nonlinear Super Integrable Couplings of Super Classical Boussinesq Hierarchy

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Manuscript submitted January 10, 2015; accepted June 5, 2015.
doi: 10.17706/ijapm.2015.5.3.177-184

Abstract: Enlargement of Lie super algebra $B(0, 1)$ was given firstly. Then nonlinear super integrable couplings of the super classical Boussinesq hierarchy based upon this enlarged matrix Lie super algebra were constructed secondly. And its super Hamiltonian structures were established by using super trace identity thirdly. As its reduction, special integrable couplings of classical Boussinesq hierarchy were obtained finally.

Key words: Lie super algebra, super integrable couplings,super classical Boussinesq hierarchy, super Hamiltonian structures.

1. Introduction

With the development of soliton theory, super integrable systems associated with Lie super algebra have aroused growing attentions by many mathematicians and physicists. It was known that super integrable systems contained the odd variables, which would provide more prolific fields for mathematical researchers and physical ones. Several super integrable systems including super AKNS hierarchy, super KdV hierarchyand super classical Boussinseq hierarchy, etc., have been studied [1]-[4]. There are some interesting results on the super integrable systems, such as Darboux transformation [5], super Hamiltonian structures [6], binary nonlinearization [7] and reciprocal transformation [8] and so on.

The research of integrable couplings of the well knownintegrable hierarchy has been received considerable attentions [9]-[12]. One approach to construct linear integrable couplings of the classical soliton equation are presented by using matrix Lie algebra constructing new loop Lie algebra [13]. Recently, Ma and Zhu [14], [15] presented a scheme for constructing nonlinear continuous and discrete integrable couplings using the block type matrix algebra. However, there is one interesting question for us is how to generate nonlinear super integrable couplings for the super integrable hierarchy.

In this paper, We take the Lie algebra $B(0, 1)$ as an example to illustrate the approach for extending Lie super algebras. Based on the enlarged Lie super algebra $gl(6, 2)$, we work out nonlinear super integrable Hamiltonian couplings of the super classical Boussinesq hierarchy. Finally, we will reduce the nonlinear super super classical Boussinesqintegrable Hamiltonian couplings to some special cases.

2. Enlargement of Lie Super Algebra

Consider the Lie super algebra $B(0, 1)$. Its basis is

$$E_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, E_2=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, E_3=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, E_4=\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix}, E_5=\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix}. \tag{1}$$

where E_1, E_2, E_3 are even element and E_4, E_5 are odd elements. Their non-zero (anti) commutation relations are

$$\begin{aligned}&[E_1,E_2]=2E_2,[E_1,E_3]=-2E_3,[E_1,E_4]=E_4,[E_1,E_5]=-E_5,[E_2,E_3]=E_1,[E_2,E_5]=E_4,\\&[E_3,E_4]=E_5,[E_4,E_4]=-2E_2,[E_4,E_5]=E_1,[E_5,E_5]=2E_3.\end{aligned} \tag{2}$$

Let us enlarge the Lie super algebra $B(0, 1)$ to the Lie super algebra $gl(6, 2)$ with a basis

$$\begin{aligned}&e_1=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0\end{pmatrix}, e_2=\begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, e_3=\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix}, e_4=\begin{pmatrix}0&0&1&0&0\\0&0&0&-1&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0\end{pmatrix},\\&e_5=\begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, e_6=\begin{pmatrix}0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix}, e_7=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&-1&0&1&0\end{pmatrix}, e_8=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\1&0&-1&0&0\end{pmatrix}.\end{aligned} \tag{3}$$

where $e_1, e_2, e_3, e_4, e_5, e_6$ are even, and e_7, e_8 are odd.

The generator of Lie super algebra $gl(6, 2)$, $e_i (1 \le i \le 8)$ satisfy the following (anti) commutation relations:

$$\begin{aligned}&[e_1,e_2]=2e_2,[e_1,e_3]=-2e_3,[e_1,e_5]=2e_5,[e_1,e_6]=-2e_6,[e_1,e_7]=e_7,[e_1,e_8]=-e_8,[e_2,e_3]=e_1,\\&[e_2,e_4]=-2e_5,[e_2,e_6]=e_4,[e_2,e_8]=e_7,[e_3,e_4]=2e_6,[e_3,e_5]=-e_4,[e_3,e_7]=e_8,[e_4,e_5]=2e_5,\\&[e_4,e_6]=-2e_6,[e_5,e_6]=e_4,[e_7,e_7]=2e_5-2e_2,[e_7,e_8]=e_1-e_4,[e_8,e_8]=2e_3-2e_6,\\&[e_1,e_4]=[e_2,e_5]=[e_2,e_7]=[e_3,e_6]=[e_3,e_8]=[e_4,e_7]=[e_4,e_8]=[e_5,e_7]=[e_5,e_8]=[e_6,e_7]=[e_6,e_8]=0.\end{aligned} \tag{4}$$

Define a loop super algebra corresponding to the Lie super algebra $gl(6, 2)$, denote by

$$gl(6,2)\otimes C[\lambda,\lambda^{-1}]=\{e_i\lambda^m \mid e_i\in gl(6,2), i=1,\cdots,8; m=0,\pm1,\pm2,\cdots\}. \tag{5}$$

The corresponding (anti)commutative relations are given as

$$[e_i\lambda^m, e_j\lambda^n]=[e_i,\ e_j]\lambda^{m+n}, \forall e_i,e_j\in gl(6,2). \tag{6}$$

3. Nonlinear Super Integrable couplings of the Super Classical Boussinesq Hierarchy

If let us start from an enlarged spectral problem associated with $gl(6, 2)$,

$$\phi_x=U(u,\lambda)\phi, U=-e_1(1)-\tfrac{1}{4}qe_1(0)+re_2(0)-e_3(0)+u_1e_4(0)+u_2e_5(0)+\alpha e_7(0)+\beta e_8(0)$$

$$
=\begin{pmatrix} -\frac{1}{2}\lambda+\frac{1}{2}q & -r & u_1 & u_2 & \alpha \\ 1 & \frac{1}{2}\lambda-\frac{1}{2}q & 0 & -u_1 & \beta \\ 0 & 0 & -\frac{1}{2}\lambda+\frac{1}{2}q+u_1 & -r+u_2 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\lambda-\frac{1}{2}q-u_1 & 0 \\ \beta & -\alpha & -\beta & \alpha & 0 \end{pmatrix}. \tag{7}
$$

where q,r,u_1,u_2 are even potentials, but α,β are odd ones.

In order to obtain super integrable couplings of super integrable hierarchy, we first solve the adjoint representation of (7),

$$
V_x=[U,V], \tag{8}
$$

with

$$
V=Ae_1(0)+Be_2(0)+Ce_3(0)+Ee_4(0)+Fe_5(0)+Ge_6(0)+\rho e_7(0)+\delta e_8(0)=\begin{pmatrix} A & B & E & F & \rho \\ C & -A & G & -E & \delta \\ 0 & 0 & A+E & B+F & 0 \\ 0 & 0 & C+G & -A-E & 0 \\ \delta & -\rho & -\delta & \rho & 0 \end{pmatrix}. \tag{9}
$$

where *A, B, C, E, F* and *G* are commuting fields, and ρ, δ are anti-commuting fields.

Substituting

$$
A=\sum_{m\geq 0}A_m\lambda^{-m},B=\sum_{m\geq 0}B_m\lambda^{-m},C=\sum_{m\geq 0}C_m\lambda^{-m},E=\sum_{m\geq 0}E_m\lambda^{-m},F=\sum_{m\geq 0}F_m\lambda^{-m},G=\sum_{m\geq 0}G_m\lambda^{-m},\rho=\sum_{m\geq 0}\rho_m\lambda^{-m},\delta=\sum_{m\geq 0}\delta_m\lambda^{-m}. \tag{10}
$$

Intoprevious equation gives the following recursive formulas

$$
\begin{cases} A_{m,x}=B_m+rC_m+\beta\rho_m+\alpha\delta_m, \\ B_{m,x}=-2rA_m-2B_{m+1}-\frac{1}{2}qB_m-2\alpha\rho_m, \\ C_{m,x}=-2A_m+2C_{m+1}+\frac{1}{2}qC_m+2\beta\delta_m, \\ E_{m,x}=u_2C_m+F_m+rG_m+u_2G_m-\beta\rho_m-\alpha\delta_m, \\ F_{m,x}=-2u_2A_m+2u_1B_m-2rE_m-2u_2E_m-2F_{m+1}-\frac{1}{2}qF_m+2u_1F_m+2\alpha\rho_m, \\ G_{m,x}=-2u_1C_m-2E_m+2G_{m+1}+\frac{1}{2}qG_m-2u_1G_m-2\beta\delta_m, \\ \rho_{m,x}=-\alpha A_m-\beta B_m-\rho_{m+1}-\frac{1}{4}q\rho_m+r\delta_m, \\ \delta_{m,x}=\beta A_m-\alpha C_m+\rho_m+\delta_{m+1}+\frac{1}{4}q\delta_m. \end{cases} \tag{11}
$$

From previous equations, we can successively deduce

$$
\begin{aligned}
&A_0=1,B_0=C_0=F_0=G_0=\rho_0=\delta_0=0,E_0=\varepsilon=\text{const.},A_1=0,B_1=-r,C_1=1,E_1=0,F_1=-u_2-\varepsilon r-\varepsilon u_2,G_1=\varepsilon,\\
&\rho_1=-\alpha,\delta_1=-\beta,A_2=\tfrac{1}{2}r-\alpha\beta,B_2=\tfrac{1}{2}r_x+\tfrac{1}{4}qr,C_2=-\tfrac{1}{4}q,E_2=\tfrac{1}{2}u_2+\tfrac{1}{2}\varepsilon u_2+\tfrac{1}{2}\varepsilon r+\alpha\beta,F_2=\tfrac{1}{2}u_{2x}+\tfrac{1}{2}\varepsilon u_{2x}+\tfrac{1}{2}\varepsilon r_x-ru_1\\
&+\tfrac{1}{4}\varepsilon qr+\tfrac{1}{4}qu_2+\tfrac{1}{4}\varepsilon qu_2-u_1u_2-\varepsilon u_1u_2-\varepsilon ru_1,G_2=u_1+\varepsilon u_1-\tfrac{1}{4}\varepsilon q,\rho_2=\alpha_x+\tfrac{1}{4}q\alpha,\delta_2=-\beta_x+\tfrac{1}{4}q\beta,A_3=-\tfrac{1}{4}r_x+\\
&-\tfrac{1}{4}qr+\alpha_x\beta-\alpha\beta_x+\tfrac{1}{2}q\alpha\beta,B_3=-\tfrac{1}{4}r_{xx}-\tfrac{1}{8}q_xr-\tfrac{1}{4}qr_x-\tfrac{1}{2}r^2-\tfrac{1}{16}q^2r+r\alpha\beta-\alpha\alpha_x,C_3=-\tfrac{1}{8}q_x+\tfrac{1}{2}r+\tfrac{1}{16}q^2-\alpha\beta+\beta\beta_x,\\
&E_3=-\tfrac{1}{4}qu_2-\tfrac{1}{4}\varepsilon qu_2-\tfrac{1}{4}\varepsilon qr+ru_1+\varepsilon ru_1-\tfrac{1}{4}\varepsilon r_x+u_1u_2+\varepsilon u_1u_2-\tfrac{1}{4}u_{2x}-\tfrac{1}{4}\varepsilon u_{2x}-\alpha_x\beta+\alpha\beta_x-\tfrac{1}{2}q\alpha\beta,F_3=(\varepsilon+1)\cdot\\
&(-ru_1^2-u_1^2u_2-\tfrac{1}{8}q_xu_2-\tfrac{1}{4}qu_{2x}+\tfrac{1}{2}u_{1x}u_2+u_1u_{2x}+r_xu_1-\tfrac{1}{2}ru_{1x}-\tfrac{1}{2}u_2^2-\tfrac{1}{4}u_{2xx}-\tfrac{1}{16}q^2u_2+\tfrac{1}{2}qru_1+\tfrac{1}{2}qu_1u_2-ru_2)-r\alpha\beta\\
&+\alpha\alpha_x-\tfrac{1}{16}\varepsilon q^2r-\tfrac{1}{4}\varepsilon qr_x-\tfrac{1}{8}\varepsilon q_xr-\tfrac{1}{2}\varepsilon r^2-\tfrac{1}{4}\varepsilon r_{xx},G_3=\tfrac{1}{2}u_{1x}+\tfrac{1}{2}\varepsilon u_{1x}-\tfrac{1}{8}\varepsilon q_x+\tfrac{1}{2}u_2+\tfrac{1}{2}\varepsilon u_2-2\varepsilon r+\alpha\beta-\tfrac{1}{2}qu_1-\tfrac{1}{2}\varepsilon qu_1+\tfrac{1}{16}\varepsilon q^2\\
&+u_1^2+\varepsilon u_1^2-\beta\beta_x,\rho_3=-\alpha_{xx}-\tfrac{1}{4}q_x\alpha-\tfrac{1}{2}q\alpha_x-\tfrac{1}{2}r\alpha-\tfrac{1}{2}r_x\beta-\tfrac{1}{16}q^2\alpha-4r\beta_x,\delta_3=-\beta_{xx}+\tfrac{1}{4}q_x\beta+\tfrac{1}{2}q\beta_x-\tfrac{1}{2}r\beta+\alpha_x-\tfrac{1}{16}q^2\beta.
\end{aligned}
$$

Equations (11) can be written as

$$\begin{pmatrix} -A_{m+1}-\frac{1}{2}E_{m+1} \\ 2C_{m+1}+G_{m+1} \\ 2A_{m+1}+2E_{m+1} \\ C_{m+1}+G_{m+1} \\ 2\delta_{m+1} \\ -2\rho_{m+1} \end{pmatrix} = L\begin{pmatrix} -A_m-\frac{1}{2}E_m \\ 2C_m+G_m \\ 2A_m+2E_m \\ C_m+G_m \\ 2\delta_m \\ -2\rho_m \end{pmatrix}, \tag{12}$$

where

$$L=\begin{pmatrix} -\frac{1}{2}\partial-\frac{1}{4}\partial^{-1}q\partial & -\frac{1}{4}r-\frac{1}{4}\partial^{-1}r\partial & -\frac{1}{4}\partial^{-1}u_1\partial & -\frac{1}{4}u_2-\frac{1}{4}\partial^{-1}u_2\partial & -\frac{1}{8}\alpha-\frac{1}{4}\partial^{-1}\alpha\partial & \frac{1}{8}\beta-\frac{1}{4}\partial^{-1}\beta\partial \\ -2 & \frac{1}{2}\partial-\frac{1}{4}q & 0 & u_1 & -\frac{1}{2}\beta & 0 \\ 0 & 0 & -\frac{1}{2}\partial-\frac{1}{4}\partial^{-1}q\partial+\partial^{-1}u_1\partial & r+u_2+\partial^{-1}r\partial+\partial^{-1}u_2\partial & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\partial-2u_1-\frac{1}{4}q & 0 & 0 \\ 4\beta & 2\alpha & \beta & -2\alpha & \partial-\frac{1}{4}q & -1 \\ -4\alpha-4\beta\partial & -2r\beta & -\beta\partial & 2r\beta & -r+\alpha\beta & -\partial-\frac{1}{4}q \end{pmatrix}. \tag{13}$$

Then, let us consider the spectral problem (7) with the following auxiliary problem

$$\phi_{t_n}=V^{(n)}\phi, \tag{14}$$

with

$$V^{(n)}=\sum_{j=0}^{n}\begin{pmatrix} A_j & B_j & E_j & F_j & \rho_j \\ C_j & -A_j & G_j & -E_j & \delta_j \\ 0 & 0 & A_j+E_j & B_j+F_j & 0 \\ 0 & 0 & C_j+G_j & -A_j-E_j & 0 \\ \delta_j & -\rho_j & -\delta_j & \rho_j & 0 \end{pmatrix}\lambda^{n-j}-C_{n+1}e_1(0)-G_{n+1}e_4(0), \tag{15}$$

From the compatible condition $\phi_{x,t_n}=\phi_{t_n,x}$, according to (7) and (14), we get the zero curvature equation

$$U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=0. \tag{16}$$

which gives a nonlinear Lax super integrable hierarchy

$$u_{t_n}=\begin{pmatrix} q \\ r \\ u_1 \\ u_2 \\ \alpha \\ \beta \end{pmatrix}_{t_n}=\begin{pmatrix} 4C_{n+1,x} \\ -2B_{n+1}-2rC_{n+1} \\ -G_{n+1,x} \\ -2u_2C_{n+1}-2F_{n+1}-2rG_{n+1}-u_2G_{n+1} \\ -\alpha C_{n+1}-\rho_{n+1} \\ \beta C_{n+1}+\delta_{n+1} \end{pmatrix}. \tag{17}$$

The super integrable hierarchy (17) is a nonlinear super integrablecouplings for the super classical Boussinesq hierarchy

$$\tilde{u}_{t_n} = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}_{t_n} = \begin{pmatrix} 4C_{n+1,x} \\ -2B_{n+1} - 2rC_{n+1} \\ -\alpha C_{n+1} - \rho_{n+1} \\ \beta C_{n+1} + \delta_{n+1} \end{pmatrix}. \tag{18}$$

4. Super Hamiltonian Structure

A direct calculation reads

$$\mathrm{Str}(U_\lambda, V) = -4A - 2E, \mathrm{Str}(U_q, V) = -A - \tfrac{1}{2}E, \mathrm{Str}(U_r, V) = 2C + G, \mathrm{Str}(U_{u_1}, V) = 2A + 2E, \mathrm{Str}(U_{u_2}, V) = C + G, \mathrm{Str}(U_\alpha, V) = 2\delta, \mathrm{Str}(U_\beta, V) = -2\rho. \tag{19}$$

Substituting above results into the super trace identity [6]

$$\frac{\delta}{\delta u}\int \mathrm{Str}\left(\frac{\delta U}{\delta \lambda}V\right)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{Str}\left(\frac{\delta U}{\delta u}V\right), \tag{20}$$

yields that

$$\frac{\delta}{\delta u}\int(-4A - 2E)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} -A + \frac{1}{2}E \\ 2C + G \\ 2A + 2E \\ C + G \\ 2\delta \\ -2\rho \end{pmatrix}. \tag{21}$$

Comparing the coefficients of λ^{-n-1} on both side of (21)

$$\frac{\delta}{\delta u}\int(-4A_{n+1} - 2E_{n+1})\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} -A_n - \frac{1}{2}E_n \\ 2C_n + G_n \\ 2A_n + 2E_n \\ C_n + G_n \\ 2\delta_n \\ -2\rho_n \end{pmatrix}, n \geq 0. \tag{22}$$

From the initial values in (11), we obtain $\gamma = 0$. Thus we have

$$\frac{\delta H_n}{\delta u} = \begin{pmatrix} -A_n - \frac{1}{2}E_n \\ 2C_n + G_n \\ 2A_n + 2E_n \\ C_n + G_n \\ 2\delta_n \\ -2\rho_n \end{pmatrix}, H_n = \int \frac{4A_{n+1} + 2E_{n+1}}{n+1}\mathrm{d}x, n \geq 0. \tag{23}$$

It then follows that the nonlinear super integrable couplings (17) possess the following super

Hamiltonian form

$$u_{t_n}=K_n(u)=J\frac{\delta H_n}{\delta u}. \tag{24}$$

where

$$J=\begin{pmatrix} 0 & 4\partial & 0 & -4\partial & 0 & 0 \\ 4\partial & 0 & \partial & 0 & \alpha & -\beta \\ 0 & \partial & 0 & -2\partial & 0 & 0 \\ -4\partial & 0 & -2\partial & 0 & -\alpha & \beta \\ 0 & -\alpha & 0 & \alpha & 0 & \frac{1}{2} \\ 0 & \beta & 0 & -\beta & \frac{1}{2} & 0 \end{pmatrix}. \tag{25}$$

is a super Hamiltonian operator and $H_n(n\geq 0)$ are Hamiltonian functions.

5. Reductions

Taking $\alpha=\beta=0$, the hierarchy (24) reduces to a nonlinear integrable couplings of the classical Boussinesqhirerarchy.

When $n=2$ in (24), we obtain the nonlinear super integrable couplings of the second order superclassical Boussinesqequatiaons

$$\begin{cases} q_{t_2}=-\frac{1}{2}q_{xx}+2r_x+\frac{1}{2}qq_x-4\alpha_x\beta-4\alpha\beta_x+4\beta\beta_{xx}, \\ r_{t_2}=\frac{1}{2}r_{xx}+\frac{1}{2}q_xr+\frac{1}{2}qr_x+2\alpha\alpha_x-2r\beta\beta_x, \\ u_{1,t_2}=(\varepsilon+1)(-\frac{1}{2}u_{1xx}+\frac{1}{2}q_xu_1+\frac{1}{2}qu_{1x}-2u_1u_{1x}-\frac{1}{2}u_{2x}-\frac{1}{8}q_{xx}-\frac{1}{2}r_x-\frac{1}{8}qq_x)-\frac{1}{8}q_{xx}+\frac{1}{2}r_x+\frac{1}{8}qq_x-\alpha_x\beta-\alpha\beta_x+\beta\beta_{xx}, \\ u_{2,t_2}=(\varepsilon+1)(\frac{1}{2}q_xu_2-2ru_{1x}-2u_{1x}u_2-2u_1u_{2x}+\frac{1}{2}qu_{2x}-2r_xu_1+\frac{1}{2}u_{2xx}+\frac{1}{2}qr_x+\frac{1}{2}q_xr+\frac{1}{2}r_{xx})-\frac{1}{2}r_{xx}-\frac{1}{2}q_xr-\frac{1}{2}qr_x-2\alpha\alpha_x+2r\beta\beta_x, \\ \alpha_{t_2}=\alpha_{xx}+\frac{3}{8}q_x\alpha+\frac{1}{2}q\alpha_x+\frac{1}{2}r_x\beta+r\beta_x-\alpha\beta\beta_x, \\ \beta_{t_2}=-\beta_{xx}+\alpha_x+\frac{1}{8}q_x\beta+\frac{1}{2}q\beta_x. \end{cases} \tag{26}$$

Especially, taking $\alpha=\beta=0$ in (26), we can obtain the nonlinear integrable couplings of the second order classical Boussinesq equation

$$\begin{cases} q_{t_2}=-\frac{1}{2}q_{xx}+2r_x+\frac{1}{2}qq_x, \\ r_{t_2}=\frac{1}{2}r_{xx}+\frac{1}{2}q_xr+\frac{1}{2}qr_x, \\ u_{1,t_2}=(\varepsilon+1)(-\frac{1}{2}u_{1xx}+\frac{1}{2}q_xu_1+\frac{1}{2}qu_{1x}-2u_1u_{1x}-\frac{1}{2}u_{2x}-\frac{1}{8}q_{xx}-\frac{1}{2}r_x-\frac{1}{8}qq_x)-\frac{1}{8}q_{xx}+\frac{1}{2}r_x+\frac{1}{8}qq_x, \\ u_{2,t_2}=(\varepsilon+1)(\frac{1}{2}q_xu_2-2ru_{1x}-2u_{1x}u_2-2u_1u_{2x}+\frac{1}{2}qu_{2x}-2r_xu_1+\frac{1}{2}u_{2xx}+\frac{1}{2}qr_x+\frac{1}{2}q_xr+\frac{1}{2}r_{xx})-\frac{1}{2}r_{xx}-\frac{1}{2}q_xr-\frac{1}{2}qr_x. \end{cases} \tag{27}$$

If setting $\varepsilon=-1, u_1=\frac{1}{4}q, u_2=-r$ in (26), we obtain the second order super classical Boussinesq equation

$$\begin{cases} q_{t_2}=-\frac{1}{2}q_{xx}+2r_x+\frac{1}{2}qq_x-4\alpha_x\beta-4\alpha\beta_x+4\beta\beta_{xx}, \\ r_{t_2}=\frac{1}{2}r_{xx}+\frac{1}{2}q_xr+\frac{1}{2}qr_x+2\alpha\alpha_x-2r\beta\beta_x, \\ \alpha_{t_2}=\alpha_{xx}+\frac{3}{8}q_x\alpha+\frac{1}{2}q\alpha_x+\frac{1}{2}r_x\beta+r\beta_x-2\alpha\beta\beta_x, \\ \beta_{t_2}=-\beta_{xx}+\alpha_x+\frac{1}{8}q_x\beta+\frac{1}{2}q\beta_x. \end{cases} \tag{28}$$

6. Conclusions

In this paper, we introduced an approach for constructing nonlinear integrable couplings of super integrable hierarchy. Zhang [16] once employed two kinds of explicit Lie algebra F and G to obtain the nonlinear integrable couplings of the GJ hierarchy and Yang hierarchy, respectively. It is easy to see that Lie algebra F given in [16] is isomorphic to the Lie algebra span $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ in $gl(6, 2)$. So we can obtain nonlinear integrable couplings of super GJ and Yang hierarchy easily. The method in this paper can be applied to other super integrable systems for constructing their super integrable couplings.

Acknowledgment

This work was supported by the Natural Science Foundation of Henan Province (No.132300410202), the Science and Technology Key Research Foundation of the Education Department of Henan Province (No. 14A110010), the Youth Backbone Teacher Foundation of Shangqiu Normal University(No. 2013GGJS02).

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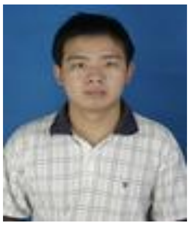

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