Application Analysis of the Curved Elements to a Two-Dimensional Laplace Problem Using the Boundary Element Method

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Manuscript submitted January 25, 2015; accepted June 4, 2015.
doi: 10.17706/ijapm.2015.5.3.167-176

Abstract: The boundary element method (BEM) has had an advantage of discretizing the boundary over the finite element method and the finite difference method for so long. The curved and straight elements are crucial for the discretization process. In this study, the BEM has been of fundamental importance in as far as solving a two –dimensional Laplace problem is concerned.

The BEM, finite element method (FEM), and the finite difference method (FDM) and their applications were reviewed, with the BEM taking advantage of the other two in relation to curved and straight elements. The Laplace equation with its boundary integral formulation was done and the BEM applied. The Dirac-Delta and the Green's functions were behind the use of the BEM. Finally, three model problems were tested for analysis of the BEM and curved elements in relation to solving the Laplace problem. The MATLAB programs and subprograms were used among others to solve the Laplace's equation in the exterior of the Ellipse, interior of a unit circle, calculating the element data for a Dirichlet type problem. The same programs were used in computing the capacitance between the concentric cylinders and eventually in the analysis.

Findings showed the fundamental advantageous stage of the curved to straight elements, the suitability of the former to the latter when finding the potential and capacitance uniquely using the BEM. However the method is not applicable to partial differential equations (PDEs) whose fundamental solutions cannot be expressed in the explicit form.

Key words: Boundary, curved, capacitance, potential.

1. Introduction

There are several mathematical problems that arise in our day-to-day lives. For instance, the boundary problems are a common phenomenon in mathematics [1]. An example of the boundary value problem is:

Let E be an open domain in R^N with boundary S,

$$-\nabla^2 u(x) = 0, x \in E,\ u(x) = g(x)\ \ on\ S \tag{1}$$

where $g(x)$ is the Dirichlet data. Such a problem can be solved by several analytical and numerical methods which include the BEM, FEM, and FDM. The FEM and FDM are approximate and therefore their accuracy can be improved by refining the mesh or by using elements whose shapes fit more closely to the boundary [1],

[2]. Noted however, is that the FEM with their variation principle are powerful as far as obtaining numerical solutions to the linear or non-linear partial differential equations (PDEs) [3]. The FEM is vital in the understanding and the use of the BEM [4]. It should be noted that the BEM is more recent than the other two and has several advantages over the others that include among others: Discretizing at infinity [5], [6], and its application to solving acoustic and general engineering science problems.

The manuscript mainly entailed the use of the BEM on the curved elements. This requires preliminary studies of the Dirac-delta distributions, the Green's functions, and the divergence theorem. Similarly, the Laplace equation and the BEM were briefly reviewed.

1.1. The Laplace Equation

Apart from using this equation in electro-statistics, it is also used in incompressible fluid flow and for the heat conduction in the steady state [7]. For a Laplace equation, the solutions within a given region say R, can be found given the specification of the potential or field around the boundary. Solutions of Laplace's equation are called harmonic functions [8]. The Laplace's boundary integral form required us to transform the PDE only unlike the FEM. A function $\boldsymbol{u}$ was considered to be satisfying equation (1) to give the following integral formulations:

$$\iint_D [G(x;\xi)\nabla^2 u(x(S)) - u(x)\nabla^2 G(x;\xi)]dA = \iint_D [-u(x)\nabla^2 G(x;\xi)]dA \tag{2}$$

$$\int_S \left[G(s;x)\frac{\partial u}{\partial n}(s) - u(s)\frac{\partial G}{\partial n}(s) \right] ds = \begin{cases} u(x), \xi \in D \\ 0.5u(x), \xi \in S \\ 0, \xi \in E \end{cases} \tag{3}$$

where G is the Green's function, D is the interior domain, S is the boundary and E is the exterior region. Equation (3) is simplified from equation (2) by applying Green's function properties, and by cosmetic changes. This accomplishes the first stage of the BEM [1].

1.2. The Boundary Element Method (BEM)

1.2.1. Introduction

The BEM as applied to the Laplace's equation is the most recent than the finite element and finite difference methods [2] and [3]. In the last three decades, the method has been developed into a robust technique for especially modeling elasticity and acoustics [8] and [9]. The term "element" meant the geometry and type of approximation to a given variable. The term "node" was used to define the elementgeometry or the variables involved in the problem.

In using the BEM on the interior Laplace problem, two relevant things were noted about elements;

- A different number of nodes were used for the variable approximations. For instance the constant and linear elements used the same number of nodes whereas the Gauss-Legendre elements use twice that number [1], [2].
- Since the element is made up of the geometry and type of approximation, this approximation may be linear, quadratic or otherwise. But some of these approximations like the linear and quadratic use global basis functions which are continuous over the boundary. Other approximations give a jump discontinuity which may not affect the accuracy of method because of integration. Hence such discontinuities in the basis function are acceptable for the method [2].

However, a difficulty may arise because of the normal between adjacent elements not being continuous.

This difficulty is the change of the derivative $\frac{\partial u}{\partial n} = v$ at these nodes. It is advantageous however to use the interior nodes only as with the constant or Guass-Legendre points especially when the geometry is not smooth. Hence the use of spline approximation may be inevitable for the required continuity in the geometry [2].

We sometimes use straight line segments to represent the geometry, and the mid-points of such elements to represent the nodes. We let the nodes to be n in number, such that the global approximations are:

$$u(s) = \sum_{c=1}^{n} \phi_c(s) u_c, \quad v(s) = \sum_{c=1}^{n} \phi_c(s) v_c \tag{4}$$

where $\varphi_c(s)$, are the boundary element basis functions.

Now, S, the actual boundary was replaced by $\bar{S}$ (union of straight elements). Hence by applying the variable approximation and considering the BEM for the exterior form, equation (3) was simplified to by:

$$\sum_{c=1}^{n}\left[[\int_{S^c} G(x;s)ds] v_c - [\int_{\bar{S}^c} \frac{\partial G}{\partial n}(x;s)ds] u_c \right] = \begin{cases} -u(x) & if\ x \in D \\ -0.5u(x) & if\ x \in S \\ 0\ if & x \in E \end{cases} \tag{5}$$

These variables were assumed to be constant on each element. For this case we chose one node per element, hence the same number of nodes and elements. As earlier noted, two steps are used in the method: Obtaining $\underline{u}$ and $\underline{v}$ or using $\underline{u}$ and $\underline{v}$ to obtain u at some general point. For the former, the collocation method was applied. It is at the collocation point that the relation between $\underline{u}$ and $\underline{v}$ is forced to hold. With this method the variable point x is brought to each of the boundary nodes in turn. The point *q* was let to be the mid-point of element r. this led to the following equivalent equations:

$$\sum_{c=1}^{n}\left[[\int_{S^c} G(q_r;s)ds] v_c - [\int_{\bar{S}^c} \frac{\partial G}{\partial n}(q_r;s)ds] u_c \right] = 0.5u_r$$

$$\sum_{c=1}^{n}\left[L_{rc}v_c - M_{rc}u_c\right] = 0.5u_r, \quad L_r v - M_r u = 0.5u_r \tag{6}$$

where *r* = 1, ..., *n*, and *n* is the number of nodes. The exterior formulation simply negates the RHS of equation (6).

It was noted that equation (6) could be written in matrix form as $L\underline{v} - (M - 0.5)\underline{u} = 0$ for the exterior case [2]. For a Dirichlet problem where $\underline{u}$ is given, $\underline{v}$ can be found. We also noted on the other hand if it is a mixed problem, the resulting matrices have to be ordered and re-partitioned. The matrices L_{rc} and M_{rc} were resolved using the equations below for both the straight and curved elements.

$$L_{rc} = \int_{\bar{S}} G(q_r;s)ds \ and \ M_{rc} = \int_{\bar{S}} \frac{\partial G}{\partial n}(q_r;s)ds \tag{7}$$

To resolve these integrals, we assumed the availability of the coordinates of the collocation point *q* and the end points *a*, *b*. The diagonal entries arise when *q* oincides with the element node. This accomplished the first step, because at this stage $\underline{v}$ could be calculated for a Dirichlet problem [2]. In the second step $\underline{u}$ at a general point could be found by

$$u(x) = M(x)\underline{u} - L(x)\underline{v} \tag{8}$$

Resolving the matrix and integral forms for the curved elements can be found in the findings of this manuscript.

2. Methodology

This study involved a review of the BEM, and a simple comparison of the BEM with the FEM. The Dirac-Delta and the Green's functions were also briefly reviewed in relation to the Laplace problem in the finite space. The Laplace equation with its boundary integral formulation was done and the BEM applied. The algebra behind the straight and curved elements was analyzed with more emphasis put on the curved elements. Finally, three model problems were tested for analysis of the BEM and curved elements in relation to solving the Laplace problem. Several MATLAB programs were used in the analysis section. They included among others:

- *Beexcp.m*: This uses the BEM for solving the Laplace's equation in the exterior of the Ellipse for a Dirichlet type problem. The same program was used in computing the capacitance between the concentric cylinders. The element data for this program and *Beextcp.m* is generated by a file *eltsp.m* which requires only the number of nodes and ordinates. The values of the problem variable at the element mid-points i.e. the $\underline{u}$ data, is calculated from the exact solution. This in turn is calculated in a function sub-program *ansep.m* (this returns the analytic value of a function at a sequence of points $(x_1, y_1), \ldots, (x_n, y_n)$).
- *Beextc.m*: It uses the BEM for solving the Laplace's equation in the exterior of the unit circle for a Dirichlet type problem (*Beextcp5.m* on the other hand uses a circle of radius 0.4). The element data, is calculated from the exact solution. This is calculated in a function sub-program *anse1.m*. The function *Eltsp5.m* discretizes the boundary of the circle of radius 0.4 and stores such coordinates.
- *Beintc.m*: This uses the BEM for solving the Laplace's equation in the interior of the unit circle as applied to curved elements for a Dirichlet type problem. The element data (discretization), is done by the file *Elts.m* while the $\underline{u}$ data is calculated from the exact solution by using the sub-program *Mant2.m*. This function *Mant2.m* computes vectors L_r, M_r which are the rows of the boundary element matrices L and M. The L_x, M_x ectors in the second stage of the BEM also use this function. It also uses a function called *Trapez.m* to solve quadratic-in-sine integrals.

3. Findings

3.1. Computationof the Matrix Terms for Curved Elements

Some element-discretization procedures and variable approximation techniques like those above remained the same as for the straight elements. We therefore began the computation of the terms for the matrices L_{rc} and M_{rc} already considered in the previous section. A curved boundary element, as an arc of a circle, AB was considered. Thus Fig. 1 illustrates the setting for computing matrix terms for the curved element (see Fig. 1 below).

From Fig. 1, the following expressions were resolved:

$$\underline{a} = [a_1, a_2] \text{ , } \underline{b} = [b_1, b_2] \text{, } \underline{d} = 0.5(\underline{b} - \underline{a}) \text{, } \theta = \arctan 2[d(2) - d(1)] \ ms = \sqrt{d.d} \ \ \alpha^e = \arcsin\left(\frac{ms}{\rho}\right),$$
$$\varphi_A = \frac{\pi}{2} + \alpha^e - \theta \ \varphi_c = \frac{\pi}{2} - \theta \text{, } \ \underline{n}_c = [\cos\varphi_c - \sin\varphi_c] \ q_c = 0.5(\underline{a} + \underline{b}) + \rho(1 - \cos\alpha^e)\underline{n} \quad (9)$$

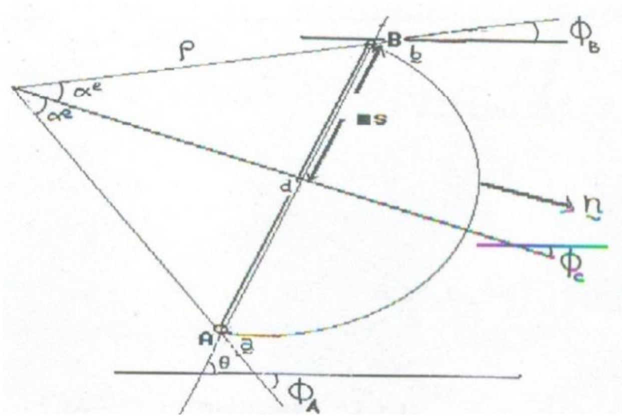

Fig. 1. The setting for computing the matrix terms for curved element.

The integrals M and L were evaluated by mapping the curved element on to [-1, 1] as seen in Fig. 2 below:

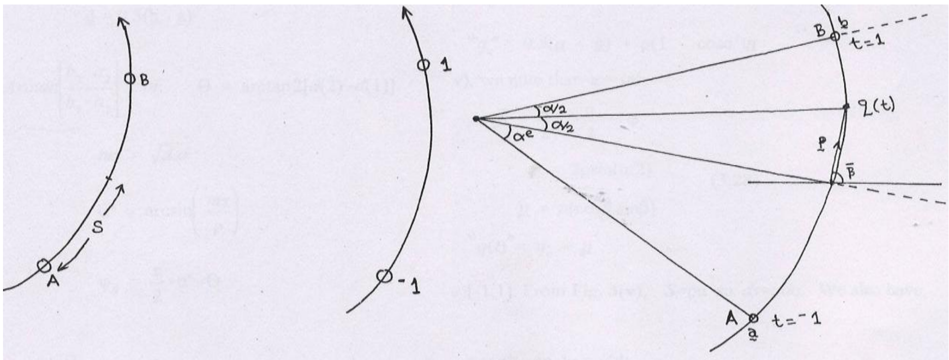

Fig. 2. Introducing $t \in$ [-1, 1].

In Fig. 2, t is introduced to the interval [-1, 1] that bounds the element under consideration. Similarly, we note that:

$$\alpha = t_{\alpha}{}^{e}), \ \ \overline{\beta} = \frac{\pi}{2} + \frac{\alpha}{2} - \phi_{c}, \ \ p = 2\rho\sin(\alpha/2) \text{, } \ \underline{p} = p[\cos\overline{\beta}, \sin\overline{\beta}], \ \ q(t) = q_c + \underline{p} \tag{10}$$

$S \in$ [0, 1] maps to $t \in$ [-1, 1], and from Fig. 2, $S = \rho\alpha$ so that $dS = \rho d\alpha$. Similarly, we have $\alpha = \alpha^e t$ so that $d\alpha = \alpha^e dt$, implying that $ds = \rho\alpha^e dt$.

Assuming the availability of the end points a and b and the coordinates of the collocation point q_r, the computation of the matrix terms could be done [1]. We first of all made the following simplifications;

$$\underline{r}.\underline{n} = (\underline{c} + \underline{p}).\underline{n} \ \text{ and } \ r^2 = \lambda \sin^2\left(\frac{\alpha^e t)}{2}\right) + \cos\left(\frac{\alpha^e t)}{2}\right) + \gamma \tag{11}$$

where $\lambda = [2\rho[\cos\overline{\beta}, \sin\overline{\beta}]][2\rho[\cos\overline{\beta}, \sin\overline{\beta}]]$, $\beta = 2[\underline{c}.(2\rho[\cos\overline{\beta}, \sin\overline{\beta}])]$, $\gamma = \underline{c}.\underline{c}$

Equations (7) are simplified to:

$$M_{rc} = -\frac{l_c}{4\pi}\int_{-1}^{1}\frac{(\underline{c}+\underline{p}).\underline{n}\,dt}{\lambda\sin^2(\frac{\alpha^e t)}{2}) + \cos(\frac{\alpha^e t)}{2}) + \gamma}, \quad L_{rc} = -\frac{l_c}{8\pi}\int_{-1}^{1}\ln\left(\lambda\sin^2(\frac{\alpha^e t)}{2}) + \cos(\frac{\alpha^e t)}{2}) + \gamma\right) \tag{12}$$

The integrals in equations (12) were elevated using the trapezoidal rule with a varying number of ordinates. The number of ordinates is denoted **no** in the computer program *Trapez.m*. However the diagonal elements given by the integrals M_{rr} and L_{rr} equired a different approach because a singularity cannot be avoided. The following work simplifies the terms for the singularity [10]. For the integral M_{rr} the vectors r and n are orthogonal and therefore it is zero. So when $q_c = q_r$ $r(t) = \underline{p} = 2\rho\sin(\frac{\alpha^e t)}{2})$. Therefore the integral L_{rr} as found to have a log type singularity and consequently needed the Gaussian quadrature approach for the improper integrals. The integral has a vertical asymptote on [-1, 1]. In this approach we tried to isolate the singularity so that

$$\frac{-l_c}{4\pi}\int_{-1}^{1}\ln(\frac{\alpha^e t}{2})dt = \frac{-l_c}{4\pi}\left(\ln(\frac{l_r}{2}) - 2\right) \quad r = 1, 2, \ldots, n. \quad r = c \;\; and \;\; \rho_{\alpha}^{e} = \frac{l_c}{2} \tag{13}$$

3.2. Model Problems

Problem 1

It was required to recover the solution of

$$u(x) = \sin\pi x \sinh\pi y \tag{14}$$

at (0,0) on a unit circle, where the Dirichlet conditions are excluded in the solution. At the four nodes, $\underline{u}$ = [3.6252 -3.6252 3.6252 -3.6252]^T. The singular integrals were solved by use of the trapezium rule and Gaussian quadrature. The following matrices show the evaluated entries:

$$M = \begin{pmatrix} -0.0000 & -0.0118 & -0.0053 & 0.0118 \\ -0.0118 & 0.0000 & 0.0118 & -0.0053 \\ -0.0053 & 0.0118 & 0.0000 & -0.0118 \\ 0.0118 & -0.0053 & -0.0118 & 0.0000 \end{pmatrix} \quad L = \begin{pmatrix} 0.2802 & -0.0632 & -0.1135 & -0.0632 \\ -0.0632 & 0.2802 & -0.0632 & -0.1135 \\ -0.1135 & -0.0632 & 0.2802 & -0.0632 \\ -0.0632 & -0.1135 & -0.0632 & 0.2802 \end{pmatrix}$$

$\underline{v}$ = [6.4137 -6.4137 6.4137 -6.4137]^T compared to the exact $\underline{v}$ = [2.1148 -2.1148 2.1148 -2.1148]^T, $L(x)$ = [-0.0409 -0.1299 -0.1223 0.2802]^T and $M(x)$ = [-0.0145 -0.0043 -0.0053 0.0000]^T. Therefore $u(0,0)$ = -1.9545 compared to the exact value $u(0,0) = 0$. Table 1 gives the error bounds with varying nodes (element number) as seen below:

Table 1. The Error Bounds as n Varies on a Unit Circle

Nodes (n)	*8*	*16*	*32*	*64*	*128*
Error $\lVert e^{\infty} \rVert$	$0.1097*10^{1}$	$0.4323*10^{0}$	$0.1286*10^{0}$	$0.3820*10^{-1}$	$0.1100*10^{-1}$

Problem 2

We were required to find the potential outside an ellipse with radii 0.1*m*, 0.2m such that

$$u:\nabla^{2}u(x),\ x\in E, u(s)=10\ \ and\ \ s\in S \tag{15}$$

The wave (velocity potential) equation was used to fit this problem by assigning the sound element to zero so as to simplify it to $\nabla^{2}u(x)=0.$ This can be reformulated by use of equation (3) so that $\underline{v}$ is the normal component of the surface velocity and ξ is a point on the boundary. The above procedure remains and the potential could then be given by equation (8). The ellipse's boundary was divided in to circular arcs (elements), which were varied with the programs *beextcp.m* and *eltsp.m*to give the corresponding potential in Table 2. It was assumed that $\underline{u}$ and $\underline{v}$ are constant over each element and that on $S_j, \underline{u}(\xi)=\underline{u}(\xi_j),\ and\ \underline{v}(\xi)=\underline{v}(\xi_j).$

Table 2. The Potential of an Ellipse

Nodes (n)	*4*	*8*	*16*
Potential	-0.4323	-18.8639	129.1129

Problem 3

Finally, a problem from electro-statics required us to find the capacitance between two concentric cylinders, the elliptic inside the circular one. The circle has radius R = 0.4*m*, the ellipse with radii 0.2*m*, and 0.1*m*. Two boundary conditions were given: $u(s_e)=0\ and\ u(s_c)=1.$

The capacitance is the ratio of the charge on either plate Q to the potential difference, V between the plates. It is determined by the area/distance between the plates, and the nature of the dielectric.

We first of all did general calculations, without considering the boundary conditions. First, the ellipse was assumed to be a sphere/circle with an average radius r_1= 0.15*m* in order to establish the charge on the inner plate. Similarly, the potential of the circle V_c is zero since the circle is assumed earthed. The approximate capacitance C, is given by

$$C=\frac{Q}{V}=\frac{4\pi\varepsilon_o r_1 R}{R-r_1},\ \ \therefore\ \ C\approx\ 26.6910pF \tag{16}$$

(The permittivity of free space if the charges are in a volume, ε_o = 8.854 $\times$ 10^{-12}).

Secondly, we assumed the cylinders to be a cable of some length *L*. We neglected any supporting disks, and used Gauss' law. Thus

$$\int_S \overline{E}.da=\frac{Q}{\varepsilon_o}$$

where $\overline{E}$, is the electric field. The potential difference *V* is

$$V=-\int_S \overline{E}\,dF=\frac{Q}{2\pi L\varepsilon_o}\int_{r1}^{R}\ln(\frac{R}{r1})\ \ \therefore\ \ C=\frac{2\pi\varepsilon_o L}{\ln(R/r1)} \tag{17}$$

Table 3 below gives the capacitance as lengths of the cylinders vary from 0.5m to 5m.

Table 3. The Capacitance with Varying Lengths

Length, L (m)	*0.5*	*0.7*	*1.1*	*2.0*	*5.0*
Capacitance, **C** (pF)	28.332	39.665	62.331	113.328	283.321

We finally applied the BEM to the problem by simply treating the situation as a connected domain and finally apply the exterior form to this Dirichlet problem. The capacitance could be given by the integral over either boundary of the normal derivative. Equation (17) for the potential difference could therefore be given by

$$V = \frac{Q}{2\pi L_{\varepsilon o}} \int_C V^3(x)dx \quad x \in E \tag{18}$$

Equation (18) has varying values as n changes and therefore gives varying capacitance values as seen in the Table 4 below:

Table 4. The Capacitance with Varying Number of Nodes

Length, L (m)	*n = 8*	*n = 64*	*n =128*
	C (pF)	C (pF)	C (pF)
0.5	36.512	33.529	31.067
2.0	146.047	134.117	124.266
5.0	365.116	335.292	310.665

4. Discussion

Upon recovering the solution of equation (14) on the unit circle at its center(0,0), the results were not good with four nodes. For example $\underline{u}$ calculated had greatest absolute value of 3.6252. The exact $\underline{v}$ compared with the calculated one gave an absolute difference of only 4.2989. However a better accuracy was achieved through the program *Beintc.m*. The accuracy increases with more elements. For instance when $n \geq 64$ the exact and approximate solutions are almost similar away from the accuracy when $n = 8$. Noted also was that the maximum absolute errors in the approximation become smaller as you move from the left to the right of the boundary (see Table 1, Fig. 3 and Fig. 4 below).

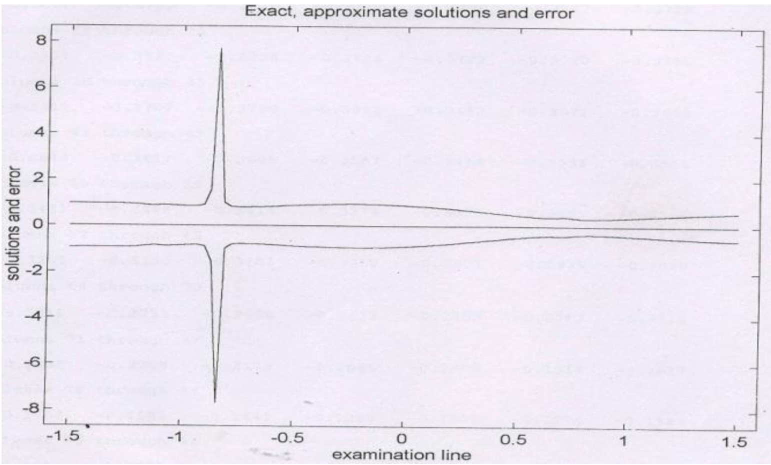

Fig. 3. *Beintc.m* results for n = 8.

The potential given by the equation (8) above is approximate. The programs *Beextcp.m* and *elts.m* helped in the generation of the potential of this ellipse. For the condition $u(s)=10$, the potential was seen to be increasing as n increases as seen in Table 2 above. This was conformable with the fact that the more the boundary of the ellipse is discretized, the better it is approximated and the better the accuracy of the solution. The same ellipse was made to be the inner cylinder and the circle of radius 0.4, the outer cylinder. It is hardly possible to find the capacitance if the charges on the inner and outer cylinders is different. The BEM with curved elements can cater for this through the approximation of the boundary more closely with increased discretization. Therefore the capacitance that follows equation (16) could be compared to that in Table 3 for the smallest length considered. The capacitance in Table 3 was observed to be better because the cylinder was assumed some length L. As this length increased, the capacitance was increasing as well because it should be proportional to its length.

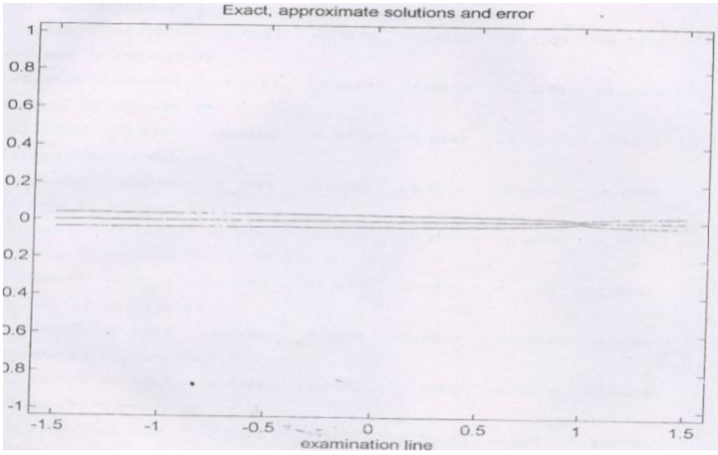

Fig. 4. *Beintc.m*results for *n* = 64.

Using the numerical analysis techniques, we observed that this was a Dirichlet problem [2] which could also be handled using the BEM for curved elements. Such elements can be used to avoid the assumption about the radius of the ellipse. This capacitance was seen to be proportional to the integral over either boundary of v, which could be found after applying the BEM using the given u. So n was varied on the circle and ellipse to give some of the results in Table 4 above. In fact when *n* = 256, *C* = 29.9 pF and *C* = 298.8 pF for the smallest and largest length considered.

The above discussions show that curved elements and the BEM applied to a two-dimensional exterior Laplace problem were better than the straight elements. Moreover, better solutions are obtained if such elements are applied to cases that have curvatures like the potential and capacitance problems a discussed above. In fact when straight elements are applied to such cases, they are either too much of approximations or they completely fail to work. For example, unless we use the curved elements, it becomes very difficult to establish the charge on the elliptic problem as the problem above.

Acknowledgments

This work was made possible through project funding by Busitema University where am employed.

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