Explicit Solution of Fuzzy Differential Equations by Mean of Fuzzy Sumudu Transform

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Abstract: Differential equations has long become a very significant tool in modelling real-life problems and phenomena. Classical mathematics, via ordinary differential equations, however, failed to cope with the situations where uncertainty arise. It is well informed that in many cases, details and information of the physical phenomena are pervaded with uncertainty. Due to this, fuzzy differential equations (FDEs) become the best apparatus to model real-life phenomena. The main aim of this paper is to propose a novel procedure for solving FDEs through fuzzy Sumudu transform (FST). For this purpose, some basic concept and properties regarding to fuzzy concept and theories will be studied. The classical Sumudu transform is then extended into the fuzzy setting before solving FDEs. To make this happen, the FST will be interpreted under strongly generalized differentiability concept. Theorem on first degree derivative of FST will also be provided to describe the functionality of FST. Finally, a numerical example of solving FDEs using FST is given.

Key words: Fuzzy number, fuzzy sumudu transform, generalized differential, fuzzy differential equation.

1. Introduction

In the literature, integral transforms have been used broadly in solving ordinary differential equations. This can be seen in many fields such as mathematics, engineering, astronomy and physics (see in [1]-[3]). One of the most recent integral transform in the literature is the Sumudu transform, proposed by Watugala [4], [5]. The new transform in the field has been proven to have some advantages over previous integral transforms, like its unity property and the effortless visualization. The work on Sumudu transform is then continued by Weerakon [6], [7] who extended the use of Sumudu transform to partial differential equations and later worked on complex inversion formula for Sumudu transform. Later on, various researchers noted the functionality of Sumudu transform and give their effort in building some important basis for Sumudu transform, like theorems and properties [8]-[10].

Recently, FDEs have picked the attention of researches as FDEs overcome the limitation of ordinary differential equations on solving real-world problems which have failed to cope with the situation when the initial value of a differential equation has uncertainty. These are obvious when researchers dealt with living subjects', for example, soil, water, air, microbial populations, etc. This is where the FDEs come handy. FDEs take account of the uncertainties of the initial values, making them best in modelling real-world problems. Since Sumudu transform works efficiently in solving differential equations, it will be crucial when the transform is able to perform in FDEs. Even though there is a vast literature discussing on solving FDEs, most of them are still at the initial stages and some solving methods could be quite complicated [11], [12]. So, there is a need for another method that could provide a more straightforward solution.

In this paper, we propose a fuzzified version of Sumudu transform, termed FST. Its properties and theorems will also be discussed. Later, we will illustrate the use of FST on a first order FDE.

2. Basic Concepts

We denote R as the set of all real numbers and $\mathcal{F}(R)$ as the set of all fuzzy real numbers across this paper. In this section, we review some basic concepts on fuzzy numbers and FDEs.

Definition 2.1

By R, we denote the set of all real numbers. A fuzzy number is mapping $U : R \to [0,\ 1]$ with the following properties:

1) U is upper semi continuous,
2) U is fuzzy convex, i.e., $U(\lambda x + (1-\lambda)y) \geq min\{U(x), U(y)\}$ for all $x,\ y \in R,\ \lambda \in [0,\ 1]$,
3) U is normal, i.e., $\exists x_0 \in R$ for which $U(x_0) = 1$,
4) Supp $U = \{x \in R \mid U(x) > 0\}$ is the support of the U, and its closure cl(supp u) is compact.

Definition 2.2

Let $\mathcal{F}(R)$ be the set of all fuzzy numbers on R. The α-level set of a fuzzy number $U \in \mathcal{F}(R)$, $\alpha \in [0,\ 1]$, denoted by U_α, is defined as

$$U_\alpha = \begin{cases} \{x \in R \mid U(x) \geq \alpha, & if\, 0 \leq \alpha \leq 1, \\ cl(supp\ U), & if\ \ \alpha = 0. \end{cases} \tag{1}$$

It is clear that the α-level set of fuzzy number is closed and bounded interval, $[\underline{u}_\alpha, \overline{u}_\alpha]$ where $\underline{u}_\alpha$ denotes the lower bound of U_α and $\overline{u}_\alpha$ denotes the upper bound of U_α.

Theorem 2.1

Let $f : R \times \mathcal{F}(R) \to \mathcal{F}(R)$, and it is represented by $(\underline{f}_\alpha(x), \overline{f}_\alpha(x))$. For any fixed $\alpha \in [0,\ 1]$, assume $\underline{f}_\alpha(x)$ and $\overline{f}_\alpha(x)$ are Riemann-integrable on $[a,b]$ for every $b \geq a$, and assume there are two positive $\underline{M}_\alpha$ and $\overline{M}_\alpha$ such that $\int_a^b \ |\underline{f}_\alpha(x)|\, dx \leq \underline{M}_\alpha$ and $\int_a^b \ |\overline{f}_\alpha(x)|\, dx \leq \overline{M}_\alpha$ for every $b \geq a$. Then, $f(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and the improper fuzzy Riemann-integrable is a fuzzy number. Furthermore, we have

$$\int_a^\infty \ f(x)dx = \left(\int_a^\infty \ \underline{f}_\alpha(x)dx, \int_a^\infty \ \overline{f}_\alpha(x)dx \right). \tag{2}$$

Definition 2.3

Let $f : (a,b) \to \mathcal{F}(R)$, and $x_0 \in (a,b)$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \mathcal{F}(R)$, such that

1) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) -^h f(x_0)$, $f(x_0) -^h f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0} \frac{f(x_0+h) -^h f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) -^h f(x_0-h)}{h} = f'(x_0), \tag{3}$$

2) for all $h > 0$ sufficiently small, there exist $f(x_0) -^h f(x_0+h)$, $f(x_0-h) -^h f(x_0)$ and the limits (in the metric D)

$$\lim_{h \to 0} \frac{f(x_0) -^h f(x_0+h)}{-h} = \lim_{h \to 0} \frac{f(x_0-h) -^h f(x_0)}{-h} = f'(x_0). \tag{4}$$

(h and $-h$ at denominators mean $1/h$ and $-1/h$ respectively.)

Note. In this paper, we only consider Case (1) and (2) in the strongly generalized differentiability proposed by Bede and Gal [13] since they are more important as stated in [14].

Theorem 2.2

Let f : R→$\mathcal{F}$(R) be a continuous fuzzy-valued function and denote $f(x) = (\underline{f}_\alpha(x), \overline{f}_\alpha(x))$, foreach $\alpha \in [0,1]$. Then

1) If is (1)-differentiable, then $\overline{f}_\alpha(x)$ and $\overline{f}_\alpha(x)$ are differentiablefunctions and $f'(x) = (\underline{f}_\alpha{}'(x), \overline{f}_\alpha{}'(x))$.

2) If is (2)-differentiable, then $\overline{f}_\alpha(x)$ and $\overline{f}_\alpha(x)$ are differentiable functions and $f'(x) = (\overline{f}_\alpha{}'(x), \underline{f}_\alpha{}'(x))$.

3. Fuzzy Sumudu Transform

In this section, we study the classical Sumudu transform in the fuzzy setting called fuzzy Sumudu transform (FST).

Definition 3.1

Let f : R→$\mathcal{F}$(R) be continuous fuzzy-valued function. Suppose that $f(ux) \odot e^{-x}$ is improper fuzzy Riemann-integrable on $[0,\infty)$, then $\int_0^\infty f(ux) \odot e^{-x} dx$ is called fuzzy Sumudu transform and is denoted by

$$G(u) = S[f(x)] = \int_0^\infty f(ux) \odot e^{-x} dx, \tag{5}$$

($u > 0$ and u is an integer)

where the variable u is used to factor the variable x in the transform in the argument of the fuzzy-valued function.

From Theorem2.1, we obtain

$$\int_0^\infty f(ux) \odot e^{-x} = \left(\int_0^\infty \underline{f}_\alpha(ux) e^{-x}, \int_0^\infty \overline{f}_\alpha(ux) e^{-x} \right), \tag{6}$$

and also,

$$s[\underline{f}_\alpha(x)] = \int_0^\infty \underline{f}_\alpha(ux)e^{-x} \tag{7}$$

And

$$s[\overline{f}_\alpha(x)] = \int_0^\infty \overline{f}_\alpha(ux)e^{-x}. \tag{8}$$

Then, we conclude that

$$S[f(x)] = (s[\underline{f}_\alpha(x)],\ s[\overline{f}_\alpha(x)]). \tag{9}$$

Theorem 3.1

Let $f : \mathrm{R} \to \mathcal{F}(\mathrm{R})$ be a continuous fuzzy-valued function, and f is the primitive of f' on $[0,\infty)$. Then

$$S[f'(x)] = \frac{G(u) -^h f(0)}{u} \tag{10}$$

where f is (1)-differentiable, or

$$S[f'(x)] = \frac{(-f(0)) -^h (-G(u))}{u} \tag{11}$$

where f is (2)-differentiable.

4. A Numerical Example

In this section, we provide a numerical example of solving fuzzy differential equation using FST. Consider the following initial value problem

$$\begin{cases} Y'(t) = -y(t), & 0 \le t \le T, \\ Y_\alpha(t_0) = (\underline{y}_\alpha(0), \overline{y}_\alpha(0)), & 0 < \alpha \le 1. \end{cases} \tag{12}$$

By using FST, we have

$$S\left[Y'(t)\right] = S\left[-y(t)\right], \tag{13}$$

and

$$S[Y'(t)] = \int_0^\infty Y'(ut) \odot e^{-t} dt. \tag{14}$$

First, we consider the condition where $Y'(t)$ is (1)-differentiable. So, from Theorem 2.2, we have

$$\begin{cases} \underline{y}_\alpha(t) = -\underline{y}_\alpha(t), & \underline{y}_\alpha(t_0) = \underline{y}_\alpha(0), \\ \overline{y}_\alpha(t) = -\overline{y}_\alpha(t), & \overline{y}_\alpha(t_0) = \overline{y}_\alpha(0). \end{cases} \tag{15}$$

By Theorem3.1,

$$S[Y'(t)] = \frac{S[y(t)]\text{ž } (y(0))}{u}. \tag{16}$$

Therefore,

$$S[-y(t)] = \frac{S[y(t)]\text{ž } (y(0))}{u}, \tag{17}$$

$$-S[y(t)] = \frac{S[y(t)]\text{ž } (y(0))}{u}. \tag{18}$$

Then, we have

$$-s[\overline{y}_\alpha(t)] = \frac{s[\underline{y}_\alpha(t)] - (\underline{y}_\alpha(0))}{u}, \tag{19}$$

And

$$-s[\underline{y}_\alpha(t)] = \frac{s[\overline{y}_\alpha(t)] - (\overline{y}_\alpha(0))}{u}. \tag{20}$$

Finally, we obtain

$$\underline{y}_\alpha(t) = e^{t}\left(\frac{\underline{y}_\alpha(0) - \overline{y}_\alpha(0)}{2}\right) + e^{-t}\left(\frac{\underline{y}_\alpha(0) + \overline{y}_\alpha(0)}{2}\right), \tag{21}$$

and

$$\overline{y}_\alpha(t) = e^{t}\left(\frac{\overline{y}_\alpha(0) - \underline{y}_\alpha(0)}{2}\right) + e^{-t}\left(\frac{\overline{y}_\alpha(0) + \underline{y}_\alpha(0)}{2}\right). \tag{22}$$

Next, we consider the condition where $Y'(t)$ is (2)-differentiable. So, from Theorem 2.2,

$$\begin{cases} \underline{y}_\alpha(t) = -\overline{y}_\alpha(t), & \underline{y}_\alpha(t_0) = \underline{y}_\alpha(0), \\ \overline{y}_\alpha(t) = -\underline{y}_\alpha(t), & \overline{y}_\alpha(t_0) = \overline{y}_\alpha(0). \end{cases} \tag{23}$$

By Theorem3.1,

$$S[Y'(t)] = \frac{-(y(0))\text{ž } (-S[y(t)])}{u}. \tag{24}$$

Therefore,

$$S[-y(t)] = \frac{-(y(0))\text{ž } (-S[y(t)])}{u}, \tag{25}$$

$$-S[y(t)] = \frac{-(y(0))\text{ž } (-S[y(t)])}{u}. \tag{26}$$

Then, the computation is similar as in the (1) differentiable case. Finally, we obtain

$$\underline{y}_{\alpha}(t)=\underline{y}_{\alpha}(0)e^{-t}, \tag{29}$$

and

$$\overline{y}_{\alpha}(t)=\overline{y}_{\alpha}(0)e^{-t}. \tag{30}$$

The result obtained using the FST in both cases proposed in this paper are shown in Fig. 1 and Fig. 2 respectively. We can see that for Case 1, the result diverges as t increases. While for Case 2, the result implicates that the solution converges as t increases. This results agreed with the statement made in [13] and [14].

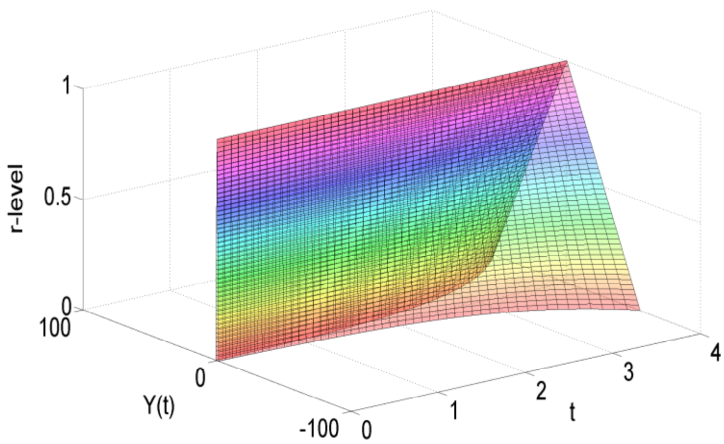

Fig. 1. The solution of $Y(t)$ for case 1.

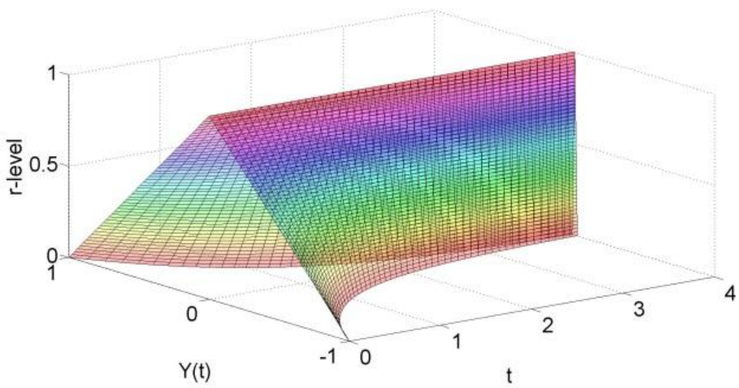

Fig. 2. The solution of $Y(t)$ for case 2.

5. Conclusion

In this paper, we have studied an extended version of the classical Sumudu transform in the fuzzy setting termed fuzzy Sumudu transform (FST). We also have demonstrated the usage of FST on solving FDEs and the final results are agreed with the conclusions made by many authors who studied fuzzy differential equations.

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