Development of Mathematical Formulas Correlating the Normal and Tangential Components of Acceleration with Its Rectangular (x-y) Components

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Abstract: In this study, a topic in Engineering Mechanics was examined. The mathematical formulas correlating the normal and tangential components, a_n and a_t, of acceleration with its rectangular components, a_x and a_y, were derived. This study will provide engineering practitioners, teachers, and students with additional tools in solving problems in curvilinear motions.

Key words: Curvilinear motion, normal and tangential components of acceleration, rectangular components.

1. Introduction

In curvilinear motions, it is very useful to resolve the acceleration into components, either into its normal and tangential components, which are normal and tangent to the path of the motion, or into its rectangular components, commonly known as *x* and *y* components. These components separate and denote respectively the rate of change of magnitude and of direction of velocity. They are particularly useful when it is needed to relate velocity and acceleration directly with the path itself [1].

In the context of this study, acceleration is the rate at which the velocity of a body changes per unit of time. Velocity is the rate of linear motion of a body in a particular direction. The magnitude of velocity, known as speed, is usually expressed in terms of distance covered per unit of time. The mathematical definition of velocity is, $v = \frac{ds}{dt}$, and for acceleration, $a = \frac{dv}{dt}$, where v is velocity, a is acceleration, s is position, and t is time [1]–[5].

The main objective of this study is to derive the mathematical formulas correlating the normal and tangential components, a_n and a_t, of acceleration with its rectangular components, a_x and a_y. The specific objectives are:

1) To develop mathematical formulas correlating the normal and tangential components of acceleration with its rectangular components when the angle θ is measured between a_x and a_t.
2) To develop mathematical formulas correlating the normal and tangential components of acceleration with its rectangular components when the angle θ is measured between a_x and a_n.

For the scope and limitation, the study covered the derivation of mathematical formulas correlating the normal and tangential components of acceleration with its rectangular components. All discussions referred to curvilinear motion in a plane, but may also be applicable to spatial motion.

The significance of this study is to provide alternative solutions to problems in curvilinear motions. These formulas will serve as additional tools for engineering practitioners, teachers, and students.

2. Curvilinear Motion

In curvilinear motion, the position of the particle is specified by the coordinate *s*, which is the distance measured along the path from a fixed reference point. As the particle moves from *A* to *B*, see Fig. 1, during an infinitesimal time interval *dt*, it traces an arc of radius$\rho$and infinitesimal length *ds* [2]–[8].

where, $ds = \rho\, d\theta$

hence, $s = \rho\theta$

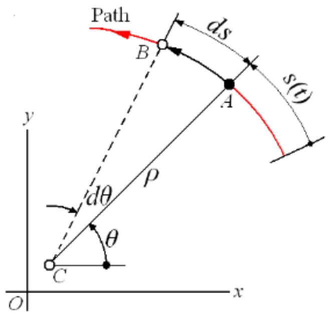


Fig. 1. Motion path of particle.

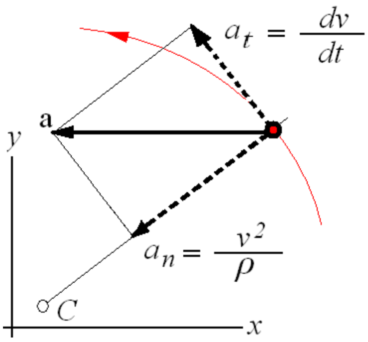


Fig. 2. Normal and tangential components of acceleration.

where θ is in radians, and ρ is called the radius of curvature of the path at *A*.

If the equation of the path is known, its radius of curvature can be computed from:

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} \quad \text{or} \quad \rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|}.$$

The normal and tangential components of acceleration, see in Fig. 2, are expressed as follows,

Normal component of acceleration *a*, $a_n = \frac{v^2}{\rho}$.

Tangential component of acceleration a, $a_t = \frac{dv}{dt}$.

Also, $a_t = v\frac{dv}{ds}$

It is very useful to resolve the acceleration into components, either into its normal and tangential components, which are normal and tangent to the path of the motion, or into its rectangular components, commonly known as x and y components. In this regard, it is important to establish the correlation between these components.

2.1. When the Angle θ Is Measured between a_x and a_t

Derivation of mathematical formulas correlating the normal and tangential components of acceleration with its rectangular components when the angle θ is measured between a_x and a_t, see in Fig. 3.

where:

a_t is the tangential component of acceleration

a_n is the normal component of acceleration

a_x is the x-component of acceleration

a_y is the y-component of acceleration

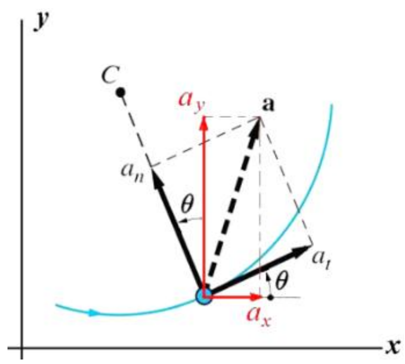
a is the resultant acceleration

See Fig. 4, we get

$$a_t = d_1 + d_2 . \tag{1}$$

$$a_y = c_1 + c_2 . \tag{2}$$

$$d_2 = \frac{a_x}{\cos\theta} . \tag{3}$$



(a) (b)

Fig. 3. Motion path of particle.

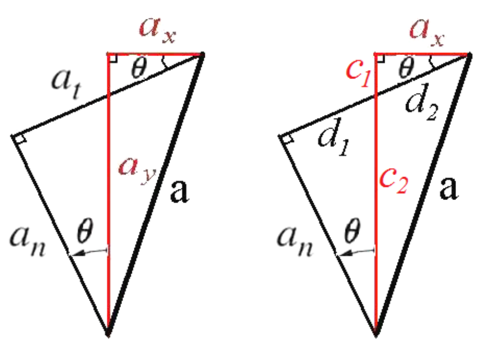


Fig. 4. Acceleration components polygon.

$$c_1 = a_x \tan\theta . \tag{4}$$

$$c_2 = a_y - c_1 . \tag{5}$$

$$c_2 = a_y - a_x \tan\theta . \tag{6}$$

$$d_1 = c_2 \sin\theta . \tag{7}$$

$$d_1 = (a_y - a_x \tan\theta)\sin\theta . \tag{8}$$

$$a_t = a_y \sin\theta - a_x \sin\theta \tan\theta + \frac{a_x}{\cos\theta} . \tag{9}$$

$$a_t = a_y \sin\theta - a_x\left(\sin\theta\tan\theta - \frac{1}{\cos\theta}\right) . \tag{10}$$

$$a_t = a_y \sin\theta - a_x\left(\frac{\sin^2\theta}{\cos\theta} - \frac{1}{\cos\theta}\right) . \tag{11}$$

$$a_t = a_y \sin\theta - a_x\left(\frac{-\cos^2\theta}{\cos\theta}\right) . \tag{12}$$

$$\boxed{a_t = a_y \sin\theta + a_x \cos\theta} . \tag{13}$$

$$\mathrm{a}^2 = a_t{}^2 + a_n{}^2 . \tag{14}$$

$$\mathrm{a}^2 = a_x{}^2 + a_y{}^2 . \tag{15}$$

$$a_n{}^2 = \mathrm{a}^2 - a_t{}^2 . \tag{16}$$

$$a_n{}^2 = a_x{}^2 + a_y{}^2 - (a_y \sin\theta + a_x \cos\theta)^2 . \tag{17}$$

$$a_n{}^2 = a_y{}^2 - a_y{}^2 \sin^2\theta - 2a_x a_y \sin\theta\cos\theta + a_x{}^2 - a_x{}^2\cos^2\theta . \tag{18}$$

$$a_n{}^2 = a_y{}^2(1-\sin^2\theta) - 2a_x a_y \sin\theta\cos\theta + a_x{}^2(1-\cos^2\theta) . \tag{20}$$

$$a_n{}^2 = a_y{}^2\cos^2\theta - 2a_x a_y \sin\theta\cos\theta + a_x{}^2\sin^2\theta . \tag{21}$$

$$a_n{}^2 = (a_y\cos\theta - a_x\sin\theta)^2 . \tag{22}$$

$$\boxed{a_n = a_y\cos\theta - a_x\sin\theta} . \tag{23}$$

$$c_2 = \frac{a_n}{\cos\theta} . \tag{24}$$

$$d_1 = a_n \tan\theta . \tag{25}$$

$$d_2 = a_t - a_n \tan\theta . \tag{26}$$

$$c_1 = d_2 \sin\theta . \tag{27}$$

$$c_1 = (a_t - a_n\tan\theta)\sin\theta . \tag{28}$$

$$c_1 = a_t\sin\theta - a_n\sin\theta\tan\theta . \tag{29}$$

$$a_y = a_t \sin\theta - a_n \sin\theta \tan\theta + \frac{a_n}{\cos\theta}. \tag{30}$$

$$a_y = a_t \sin\theta - a_n\left(\sin\theta\tan\theta - \frac{1}{\cos\theta}\right). \tag{31}$$

$$a_y = a_t \sin\theta - a_n\left(\frac{\sin\theta\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right). \tag{32}$$

$$a_y = a_t \sin\theta - a_n\left(\frac{\sin^2\theta - 1}{\cos\theta}\right). \tag{33}$$

$$a_y = a_t \sin\theta - a_n\left(\frac{-\cos^2\theta}{\cos\theta}\right). \tag{34}$$

$$\boxed{a_y = a_t \sin\theta + a_n \cos\theta}. \tag{35}$$

$$a_x{}^2 = \mathrm{a}^2 - a_y{}^2. \tag{36}$$

$$a_x{}^2 = a_t{}^2 + a_n{}^2 - (a_t \sin\theta + a_n \cos\theta)^2. \tag{37}$$

$$a_x{}^2 = a_t{}^2 - a_t{}^2 \sin^2\theta - 2a_t a_n \sin\theta\cos\theta + a_n{}^2 - a_n{}^2\cos^2\theta. \tag{38}$$

$$a_x{}^2 = a_t{}^2(1-\sin^2\theta) - 2a_t a_n \sin\theta\cos\theta + a_n{}^2(1-\cos^2\theta). \tag{39}$$

$$a_x{}^2 = a_t{}^2\cos^2\theta - 2a_t a_n \sin\theta\cos\theta + a_n{}^2\sin^2\theta. \tag{40}$$

$$a_x{}^2 = (a_t\cos\theta - a_n\sin\theta)^2. \tag{41}$$

$$\boxed{a_x = a_t\cos\theta - a_n\sin\theta}. \tag{42}$$

2.2. When the Angle θ Is Measured between a_x and a_n

Derivation of mathematical formulas correlating the normal and tangential components of acceleration with its rectangular components when the angle θ is measured between a_x and a_n, see in Fig. 5.

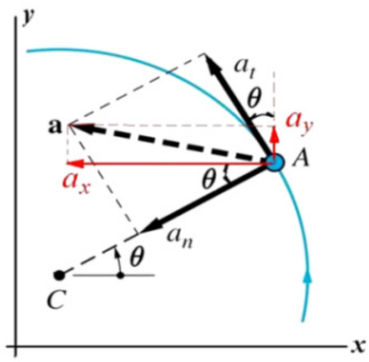


Fig. 5. Motion path of particle.

See Fig. 6, we get

$$a_t = c_1 + c_2. \tag{43}$$

$$a_x = d_1 + d_2. \tag{44}$$

$$c_1 = \frac{a_y}{\cos\theta}. \tag{45}$$

$$d_1 = c_1 \sin\theta. \tag{46}$$

$$d_1 = \frac{a_y}{\cos\theta}\sin\theta. \tag{47}$$

$$d_1 = a_y \tan\theta. \tag{48}$$

$$d_2 = a_x - a_y \tan\theta. \tag{49}$$

$$c_2 = d_2 \sin\theta. \tag{50}$$

$$a_t = \frac{a_y}{\cos\theta} + (a_x - a_y \tan\theta)\sin\theta. \tag{51}$$

$$a_t = a_x \sin\theta + \frac{a_y}{\cos\theta} - a_y \tan\theta \sin\theta. \tag{52}$$

$$a_t = a_x \sin\theta + \frac{a_y}{\cos\theta} - a_y \tan\theta \sin\theta. \tag{53}$$

$$a_t = a_x \sin\theta + a_y\left(\frac{1-\sin^2\theta}{\cos\theta}\right). \tag{54}$$

$$a_t = a_x \sin\theta + a_y\left(\frac{\cos^2\theta}{\cos\theta}\right). \tag{55}$$

$$\boxed{a_t = a_x \sin\theta + a_y \cos\theta}. \tag{56}$$

$$a_n{}^2 = \mathrm{a}^2 - a_t{}^2. \tag{57}$$

$$a_n{}^2 = a_x{}^2 + a_y{}^2 - (a_x \sin\theta + a_y \cos\theta)^2. \tag{58}$$

$$a_n{}^2 = a_x{}^2 - a_x{}^2 \sin^2\theta - 2a_x a_y \sin\theta\cos\theta + a_y{}^2 - a_y{}^2\cos^2\theta. \tag{59}$$

$$a_n{}^2 = a_x{}^2(1-\sin^2\theta) - 2a_x a_y \sin\theta\cos\theta + a_y{}^2(1-\cos^2\theta). \tag{60}$$

$$a_n{}^2 = a_x{}^2\cos^2\theta - 2a_x a_y \sin\theta\cos\theta + a_y{}^2\sin^2\theta. \tag{61}$$

$$a_n{}^2 = (a_x \cos\theta - a_y \sin\theta)^2. \tag{62}$$

$$\boxed{a_n = a_x \cos\theta - a_y \sin\theta}. \tag{63}$$

$$d_2 = \frac{a_n}{\cos\theta}. \tag{64}$$

$$c_2 = a_n \tan\theta. \tag{65}$$

$$d_1 = c_1 \sin\theta. \tag{66}$$

$$c_1 = a_t - a_n \tan\theta . \tag{67}$$

$$d_1 = (a_t - a_n \tan\theta)\sin\theta . \tag{68}$$

$$d_1 = a_t \sin\theta - a_n \sin\theta \tan\theta . \tag{69}$$

$$d_1 = a_t \sin\theta - a_n \frac{\sin^2\theta}{\cos\theta} . \tag{70}$$

Substitute (64) and (70) in(44), hence

$$a_x = a_t \sin\theta - a_n \frac{\sin^2\theta}{\cos\theta} + \frac{a_n}{\cos\theta} . \tag{71}$$

$$a_x = a_t \sin\theta + a_n \left(\frac{1-\sin^2\theta}{\cos\theta}\right) . \tag{72}$$

$$\boxed{a_x = a_t \sin\theta + a_n \cos\theta} . \tag{73}$$

$$a_y{}^2 = a_t{}^2 + a_n{}^2 - (a_t \sin\theta + a_n \cos\theta)^2 . \tag{74}$$

$$a_y{}^2 = a_t{}^2 - a_t{}^2 \sin^2\theta - 2a_t a_n \sin\theta\cos\theta + a_n{}^2 - a_n{}^2 \cos^2\theta . \tag{75}$$

$$a_y{}^2 = a_t{}^2 (1-\sin^2\theta) - 2a_t a_n \sin\theta\cos\theta + a_n{}^2 (1-\cos^2\theta) . \tag{76}$$

$$a_y{}^2 = a_t{}^2 \cos^2\theta - 2a_t a_n \sin\theta\cos\theta + a_n{}^2 \sin^2\theta . \tag{77}$$

$$a_y{}^2 = (a_t \cos\theta - a_n \sin\theta)^2 . \tag{78}$$

$$\boxed{a_y = a_t \cos\theta - a_n \sin\theta} . \tag{79}$$

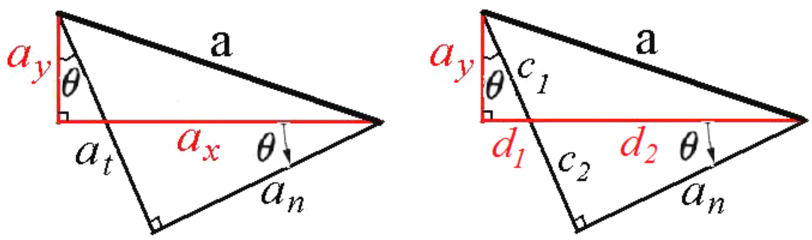


Fig. 6. Acceleration components polygon.

3. Conclusion and Directions for Future Use

The mathematical formulas derived correlating the normal and tangential components of acceleration with its rectangularcomponents when the angle θ is measured between a_x and a_t, see Fig. 3, are the following:

$$a_t = a_y \sin\theta + a_x \cos\theta . \tag{80}$$

$$a_n = a_y \cos\theta - a_x \sin\theta. \quad (81)$$

$$a_x = a_t \cos\theta - a_n \sin\theta. \quad (82)$$

$$a_y = a_t \sin\theta + a_n \cos\theta. \quad (83)$$

And the mathematical formulas correlating the normal and tangential components of acceleration with its rectangularcomponents when the angle θ is measured between a_x and a_n, see Fig. 5, are as follows:

$$a_t = a_x \sin\theta + a_y \cos\theta. \quad (84)$$

$$a_n = a_x \cos\theta - a_y \sin\theta. \quad (85)$$

$$a_x = a_t \sin\theta + a_n \cos\theta. \quad (86)$$

$$a_y = a_t \cos\theta - a_n \sin\theta. \quad (87)$$

The researcher recommends the use of the above mentioned formulas as alternative method in solving problems involving accelerations in curvilinear motions.

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