A Family of 3-Manifolds Arising from the Plumbing Construction

Yifei Zhao
Phillips Exeter Academy, Exeter, NH, USA.

Email: zhaoy10086@gmail.com
Manuscript submitted August 10, 2023; revised September 1, 2023; accepted September 15, 2023.
doi: 10.17706/ijapm.2023.13.4.62-79

Abstract: Inspired by Kirby's work in which constructed the Poincare 3-sphere $\mathbf{S^3/2I}$ by an $\mathbf{E_8}$-plumbing, we worked out a general and systematic argument to show that other spherical 3-manifolds $\mathbf{S^3/2O}$, $\mathbf{S^3/2T}$ and $\mathbf{S^3/2D_n}$ can be obtained from a plumbing construction by the Dynkin diagrams $\mathbf{E_7}$, $\mathbf{E_6}$, and $\mathbf{D_{n+2}}$, respectively.

Keywords: 3-Manifolds, vector bundles, Seifert fiber spaces, invariant theory

1. Introduction

Topologists have a construction called the plumbing of vector bundles that was used in Milnor's work [1] on the group of homotopy spheres Θ_{4k-1}. Precisely, when $k > 1$, consider the cotangent bundle over $\mathbf{S}^{2k}$ and plumb eight copies of it according to the E_8 diagram's structure. The boundary forms a $(4k-1)$-manifold which generates Θ_{4k-1}. When $k = 1$, the same recipe recovers the famous Poincare homology 3-sphere $\mathbf{S}^3/2I$ [2], where $2I$ denotes the binary icosahedral group, which is the lift of the icosahedral group I along the double cover $p: SU(2) \to SO(3)$. There is a canonical isomorphism

$$SU(2) = \left\{\begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} : z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\right\} \cong \mathbf{S}^3 \tag{1}$$

The Poincare 3-sphere belongs to a family called spherical 3-manifolds. Those are quotients of $\mathbf{S}^3$ by finitely many rotations. In particular, because there is a group structure on $\mathbf{S}^3 = SU(2)$ coming from the multiplication of quaternions, the group $SU(2)$ embeds into $SO(4)$ by the self action of $\mathbf{S}^3$. Thus, given a finite subgroup $G \subset SO(3)$, denote $2G = p^{-1}(G)$, then $2G$ can be viewed as a finite subgroup of $SO(4)$, and we can form $\mathbf{S}^3/2G$, with $2G$ being its fundamental group. Equivalently we can understand $\mathbf{S}^3/2G$ as a coset space

$$\begin{array}{ccc} 2G \subset \mathbf{S}^3 = & SU(2) & \hookrightarrow SO(4) \\ \uparrow & \downarrow p & \\ G \subset & SO(3) & \end{array} \tag{2}$$

Given integers $p, q, r > 1$, following Coxeter and Moser [3], we let $\langle p, q, r\rangle$ denote the group defined by the presentation

$$\langle \mathrm{a}, \mathrm{b}, \mathrm{c} : \mathrm{a}^{\mathrm{p}} = \mathrm{b}^{\mathrm{q}} = \mathrm{c}^{\mathrm{r}} = \mathrm{abc}\rangle \tag{3}$$

It is a finite group if and only if $p^{-1}+q^{-1}+r^{-1}>1$. The only solutions are $(2,3,5)$, $(2,3,4)$, $(2,3,3)$ and $(2,2,n)$, $n>1$. In these cases, abc is of order 2, and the groups correspond to

(1) $G=I$ the icosahedral group of order 60, $2G=\langle 2,3,5\rangle$. The generators a,b,c are sent by p to the rotations of an icosahedron whose axis contains a pair of midpoints of edge, centers of face and vertices, respectively.

(2) $G=O$ the octahedral group of order 24, $2G=\langle 2,3,4\rangle$.

(3) $G=T$ the tetrahedral group of order 12, $2G=\langle 2,3,3\rangle$.

(4) $G=D_n$ the dihedral group of order $2n$, $2G=\langle 2,2,n\rangle$. Here we are using the obvious embedding $O(2)\hookrightarrow SO(3)$.

Remark. Strictly speaking the above subgroups $G\subset SO(3)$ are only defined up to conjugacy. The resulting spaces $\mathbf{S}^3/2G$ are then well-defined up to homeomorphism.

A parallel to this story is, the theory of *Dynkin diagrams*, originally from the Lie theory. The goal of this paper is to explore this beautiful connection. Fig. 1 below shows the five simply laced Dynkin diagrams:

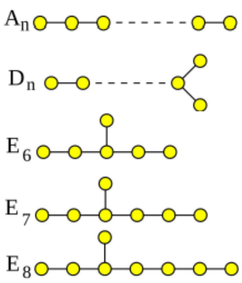

Fig. 1. Dynkin diagrams. The subscript indicates the number of vertices.

They are related to our 3-manifolds via the core result of this paper:

Main theorem. Consider the cotangent bundle E over $\mathbf{S}^2$. A plumbing construction of E according to the diagrams A_n, D_n, E_6, E_7, E_8 gives a 4-manifolds whose boundary is homeomorphic to $L(n+1,-1)$ (the lens space), $\mathbf{S}^3/2D_{n-2}$, $\mathbf{S}^3/2T$, $\mathbf{S}^3/2O$ and $\mathbf{S}^3/2I$, respectively.

The rest of the paper is organized as follows. Section 2 is preparatory, it introduces all the relevant topological objects such as Seifert fibered spaces, which serve as a bridge in our proof via the classification theorem. The key step is in Section 3, where it carries out an explicit invariant theoretic computation that enables us to apply the method of Kirby [4] to the remaining cases. The main theorem can be viewed as a manifestation of the McKay correspondence. Author includes this further background as well as the thought processes in a brief commentary chapter at the end.

2. Topological Theory

2.1. Plumbing

Throughout, the topological object we plumb together will be an oriented plane bundle E over $\mathbf{S}^2$. Up to isomorphism, such a bundle is in 1-1 correspondence with an integer n, its Euler number. We can define the Euler number of E without the notion of cohomology or self-intersection number: note that restricted over the two hemispheres, E is the trivial bundle $D^2\times\mathbf{R}^2$. The datum of E is just a transition map $t\colon\mathbf{S}^1\to SO(2)$ at the equator.

Definition 2.1. We say that E is of Euler number n, if E is isomorphic to the bundle with the transition

$$t(\theta)=\begin{pmatrix}cos(n\theta) & -sin(n\theta)\\ sin(n\theta) & cos(n\theta)\end{pmatrix},\theta\in[0,2\pi] \tag{4}$$

That is, E is obtained by gluing two $D^2 \times \mathbf{R}^2$ along the equator by a $2n\pi$ twist.

For example, inverting the orientation of E changes n to $-n$. The Euler number of the tangent bundle $T\mathbf{S}^2$ is 2, the Euler characteristic of $\mathbf{S}^2$.

Next, we look at two different equivalent descriptions of the E_8-plumbing explained in Kirby's work [4].

1) Description 1: The plumbing process, the bulk of the first description, is conducted through the following steps. Given 2 copies of E (e.g., tangent bundle over $\boldsymbol{S}^2$ of Euler number 2 or cotangent bundle of Euler number −2), we can perform the following construction:

a) Take the disk bundle $D(E)$, which is taking a unit 2-disk centered at the points on the base space of the plane bundle.

b) In each $\boldsymbol{S}^2$ take a small disk D^2, and consider the restricted bundle $D(E)|_{D^2} = D^2 \times D^2$.

c) Glue two $D(E)$, $D_1(E)$ and $D_2(E)$, along $D^2 \times D^2$ by the map $\tau(x, y) = (y, x)$. In other words, we are identifying a point $(x, y) \in D^2 \times D^2 \subseteq D_1(E)$ with $(y, x) \in D^2 \times D^2 \subseteq D_2(E)$.

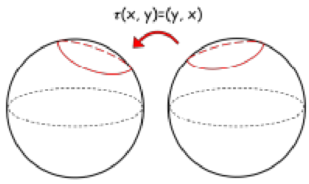

Fig. 2. Red lines enclose the disks whose bundle is plumbed.

Fig. 2 shows that the points in one enclosed region will be identified with their corresponding point in the other; this process producing a compound space containing the two bundle spaces.

Definition 2.2. The resulting space of 2.1 is the plumbing of E according to the A_2 diagram (i.e., two vertices joined by an edge). Clearly the space is independent of the small disks chosen.

We now explain how to plumb E according to an arbitrary simple graph: each vertex in the graph represents a copy of $D(E)$ and each edge represents a construction as above. Using this same method and plumbing 8 copies of E together according to the E_8 Dynkin diagram, we construct the E_8-plumbing. Notice that the choice of the neighborhood, or the 2-disk in $\boldsymbol{S}^2$ to do the plumbing does not change the homotopy type.

2) Description 2: In the second description, we will consider, as an essential component, the construction of A_2 diagram closely with the space $\boldsymbol{S}^3 - \Lambda$, where Λ is a link of circles. Specifically, we will constantly consider the boundaries of the plumbed and manipulated parts to create $\boldsymbol{S}^3 - \Lambda$ and utilize its properties. For the A_2 diagram, Λ is just two linked circles, or solid tori to be specific. We will refer back to the beginning of Section 2.1 and Definition 2.1 extensively for this description.

We look at the A_2-plumbing, again, and give it an equivalent description [4]. Note that $D(E)$ is obtained by gluing the disk bundles over the two hemispheres, two $D^2 \times D^2 = D^4$ along a solid torus $\boldsymbol{S}^1 \times D^2 \subset \partial D^4$ by a $2n\pi$ twist. This solid torus is the 2-disk bundle over the common boundary (i.e., equator) of the two hemispheres. A more clearly presented relationship between the different components involved in the plumbing description is given below:

$$S^3 = \partial D^4 = \partial(D^2 \times D^2) = \partial(D(E)|_{D^2}) = (\partial(D^2) \times D^2) \cup (\partial(D^2) \times D^2) \tag{5}$$

We first take the 2-disk bundle of ONE of the two hemispheres, equivalent to picking a disk on $\boldsymbol{S}^2$ in description 1, $D^2 \times D^2 = D^4$ in each $D(E)$ to do the plumbing. Recall the map τ we use for plumbing is in Eq. (3) of Section 2.1. The resultant space/object here through the first step merges into one 4-disk. We will look at how the map acts on the disk bundles over the common boundary, as those are the parts that yield constructive results for our purposes. We must track where the two tori go. Through τ, the second solid torus $\boldsymbol{S}^1 \times D^2$ is identified with $D^2 \times \boldsymbol{S}^1$ in the first ∂D^4, which is linked with the first solid torus $\mathbf{S}^1 \times D^2$. The plumbing sends the base space coordinates to the fiber space, and vise versa. Note that even if those two solid tori don't "seem" connected like in a link, torus 2 can be continuously deformed into torus 3 that forms a link with torus 1.

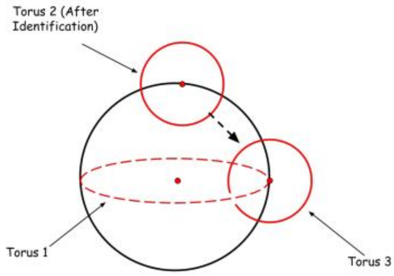

Fig. 3. A family of 3-manifolds arising from the plumbing construction.

In the compound space we now have, if we take the boundary of the resultant plumbed object, we will get $\mathbf{S}^3 - \Lambda$. The solid tori are 3-dimensional, homotopic equivalent to solid tori residing in the boundary of D^4, and only considering the boundaries of the solid tori, $\mathbf{T}$ results in the removal of links from $\mathbf{S}^3$.

Then we glue back the remaining unplumbed D^4, the other hemisphere's disk bundle, in each $D(E)$ to the boundary 3-sphere, filling up the hollowed link. When we glue back the 2-disk bundle, we are applying the naturally induced transtion map $t: \mathbf{S}^1 \times D^2 \to \mathbf{S}^1 \times D^2$, as defined in Eq. (2).

In conclusion, take a 4-disk and dig off two linked solid tori at its boundary, the 3-sphere, then in each place glue back a 4-disk by the transition map of E, t, this recipe produces a space homotopy equivalent to the A_2-plumbing of E. Notice that the solid tori glued back carries a $2 \cdot 2\pi$ twist, and this twist will affect the homotopy type of the space.

This description up to homotopy equivalence works for an arbitrary simple graph, and will be used in the computation of the fundamental group in the next section. Take the E_8 diagram for example, we have a D^4 minus 8 solid tori forming an E_8 link in ∂D^4, and in each place a D^4 is attached. Again, note that we can move and/or change the size of small plumbing disk in each $\mathbf{S}^2$ without changing the final homotopy type, so all plumbing areas merge into one single D^4.

2.2. Compute the Fundamental Group

This section motivates our main theorem. Through an argument of computing the fundamental group in Kirby's paper [4], we provide some evidence of the main results before actually getting to the proof.

Take again the E_8 diagram as example in the second description in the last section, consider the boundary 3-manifold M. We have $M = U \cup V_1 \cup \ldots \cup V_8$,
where $U \simeq \mathbf{S}^3 - E_8$, the 3-sphere minus 8 solid tori, and $V_k \simeq \mathbf{S}^3 - \mathbf{S}^1 \simeq \mathbf{S}^1$ $(1 \leq k \leq 8)$, the 3-sphere minus one solid tori. Each V_k is attached to U by the transition map of E, and $U \cap V_k \simeq \mathbf{S}^1 \times \mathbf{S}^1$, a torus. We thus apply the Van Kampen theorem to compute $\pi_1(M)$.

We use a standard algorithm of computing π_1 of a link complement [5]: every crossing provides a conjugacy relation among the generators. So $\pi_1(U)$ is generated by 8 elements as in the picture below, and every pair of adjacent generators commute.

Now say $\pi_1(V_k) = \mathbf{Z}$ generated by b_k, and $\pi_1(U \cap V_k) = \mathbf{Z}^2$ generated by x_k and y_k, where $i_k(x_k) = b_k$, $j_k(x_k) = a_k$, $i_k(y_k) = 0$, and $j_k(y_k)$ is the loop in the boundary torus that wraps n times (the Euler number of E) along the longitude, and one time along the latitude. It follows that, for E the cotangent bundle $T{\times}\mathbf{S}^2$ (i.e., the tangent bundle with the orientation inverted) in particular, we have 8 further relations in $\pi_1(M)$:

$$1 = a_1^2a_2 = a_1a_2^2a_3 = a_2a_3^2a_4 = a_3a_4^2a_5 = a_4a_5^2a_6a_8 = a_5a_6^2a_7 = a_6a_7^2 = a_5a_8^2 \quad (6)$$

An algebraic manipulation then yields

$$\pi_1(M) = \langle a_1, a_7 : (a_1a_7)^2 = a_7^3 = a_1^5\rangle = 2I \quad (7)$$

The above argument has full generality to apply to an arbitrary graph. Play with the A_n diagram, the resulting 3-manifold M has $\pi_1(M)$ generated by n elements $a_1, \dots, a_n$ that are subject to

$$1 = a_1^2a_2 = a_1a_2^2a_3 = \cdots = a_{n-2}a_{n-1}^2a_n = a_{n-1}a_n^2 \quad (8)$$

It follows that

$$a_k = a_1^{(-1)^{k-1}k} \quad (9)$$

The relation $a_{n-1}a_n^2 = 1$ implies that $a_1^{n+1} = 1$. So $\pi_1(M) = C_{n+1}$, the cyclic group of order $n+1$. This suggests that M might be a lens space. Indeed, in a literature by Orlik we found:

Theorem 2.3. Plumb n bundles over $\mathbf{S}^2$ whose Euler numbers are $-b_1$, ..., $-b_n$ in sequence according to the A_n diagram, the resulting 4-manifold has boundary homeomorphic to $L(p,q)$ [6].
where

$$\frac{q}{p} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots}} \quad (10)$$

In our case all $b_k = 2$, the continued fraction is $\frac{n}{n+1}$. So, we conclude that $M \cong L(n+1, -1)$.

Remark. Unlike the case $2I$, the fundamental group does not determine a lens space. So, the notion $\mathbf{S}^3/C_{n+1}$ is ambiguous, while $\mathbf{S}^3/2I$ and so on are well defined

We then tried a general (p,q,r) graph as in Fig. 3. The intermediate steps will be similar to the steps above but will be explained in a more detailed and nuanced manner. Consider an arbitrary link with circles representing torus labeled, representing a diagram (only the special cases discussed in this paper are types of Dynkin diagram), that comes in the following form with exactly one junction point:

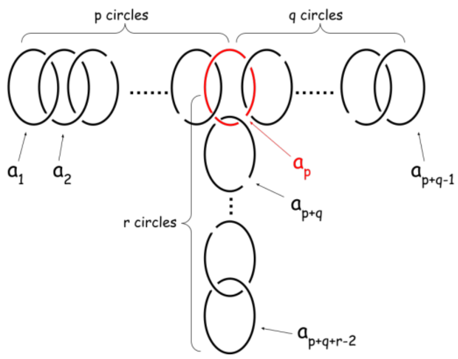

Fig. 3. General link with loops assigned to each circle/solid torus.

Proposition 2.4. Notice the following. $\pi_1(\mathbf{S}^3 - \Lambda) \cong \pi_1(\mathbf{R}^3 - \Lambda)$, $\Lambda \in \{E_6, E_7, E_8, D_n\}$

Proof: We know that $\mathbf{S}^2 = \mathbf{R}^2 \cup \{\infty\}$, so $(\mathbf{S}^3 - \Lambda) = (\mathbf{R}^3 - \Lambda) \cup \{\infty\}$. Then, by the general form of the Van Kampen Theorem:

$$\pi_1(\boldsymbol{S}^3 - \Lambda) = \pi_1\big((\boldsymbol{R}^3 - \Lambda) \cup \{\infty\}\big) = \pi_1(\boldsymbol{R}^3 - \Lambda) *_{(\boldsymbol{R}^3 - \Lambda)\cap(\{\infty\})} \pi_1(\{\infty\}) = \pi_1(\boldsymbol{R}^3 - \Lambda) \quad (11)$$

Proposition 2.5. The fundamental group of a link's complement considered as the (p, q, r) graph $(p, q, r \geq 2)$ yields a group

$$\langle a, b, c : a^p = b^q = c^r = abc \rangle \tag{12}$$

Proof: By looking at the knot as represented in the links of tori represented above, we can establish the following relationships between the loops around the longitude seen as elements in the group. To have a uniform definition, we define a single loop starting from the "outside" (in 2D representation given in the diagram) of torus a_k as "a_k" if the loop, first goes on top across and then below the circle when drawn on the general diagram above, while it is defined as a_1^{-1} if a loop first goes under then above and across the circle. If a loop is considered with a start point within the "hole" of a torus, then a loop's definition will be the opposite from the one above.

$$\langle a_1, a_2 \dots a_{p+q+r-2} : a_n a_{n+1} = a_{n+1} a_n, a_p a_{p+q} = a_{p+q} a_p, 1 \leq n \neq p+q-1 \leq p+q+r-2 \rangle \tag{13}$$

We also have a series of other relationships between elements by considering a loop around each of the circles in the link. Notice that because of the way we have glued back the tori under the transition map. So, one loop around the latitude line of a torus also comes with two loops out of that torus around the longitude line of the torus because of the $2 \times 2\pi$ twist brought by the transition map at the boundaries of the removed tori. There are three types of circles in the link in terms of the loops associated with them.

(1) For tori on an end, like a_1, one loop around its latitude line also equals to a loop around the torus next to them because of the way the tori are linked. So, we have, $a_1^2 a_2$ in the case for a_1.

(2) For the most special torus a_p, one loop around its latitude line also equals to a loop around the three tori attached to it because of the way the tori are linked. After the partial commutative relationships, we have $1 = a_{p-1} a_p^2 a_{p+1} a_{p+q}$.

(3) For any other tori a_n in the link, each one is linked to two other tori. On the same note, we have $a_{n-1} a_n^2 a_{n+1}$.

We thus have the following relationships.

$$1 = a_1^2 a_2 = a_{p+q+1}^2 a_{p+q-2} = a_{p+q+r-2}^2 a_{p+q+r-3}$$

$$1 = a_{p-1} a_p^2 a_{p+1} a_{p+q} = a_{k-1} a_k^2 a_{k+1}, \text{ for } 2 \leq k \leq p+q+1 \text{ and } p+q+1 \leq k \leq p+q+r-3 \tag{14}$$

$$1 = a_{p-1} a_p^2 a_{p+1} a_{p+q}$$

The whole chain of equalities can be simplified into a neat form.

Summarizing everything, after doing all the manipulations, we have:

$$\langle a, b : a^p = b^q = (ab)^r \rangle \tag{15}$$

which is an alternative form for the group presented in 2.8. $(p, q, r) = (2,3,5), (2,3,4), (2,3,3), (2,2,n-2)$ corresponds to the groups $2I, 2O, 2T, 2D_n$, which are preimages of I, O, T, D_n under the 2-cover $\boldsymbol{S}^3 = SU(2) \to SO(3)$. The cases where it is finite correspond exactly to the four Dynkin diagrams: $E_8 = (2,3,5)$, $E_7 = (2,3,4)$, $E_6 = (2,3,3)$, $D_n = (2,2,n-2)$. This suggests our main theorem, that the resulting 3-manifold might be $\boldsymbol{S}^3/2I$ and so on. Indeed, it follows directly from a result of Perelman which states that all compact 3-manifolds with finite π_1 are spherical. But of course, this paper will not end here.

Remark. During our algebraic manipulation, we came across some crucial steps. First, we have the relationships $1 = a_1^2 a_2 = a_{p+q-1}^2 a_{p+q-2} = a_{p+q+r-2}^2 a_{p+q+r-3}$, which are what we get by analyzing the two tori at the end of each arm. By arms here, we mean taking the parts of the link starting with a_p and all the tori in the same direction until the end of the link (conveniently, the tori enclosed by the bracket). WLOG we can take $a_1^2 a_2 = 1$ and get that $a_2 = a_1^{-2}$, and down the arm, we get that $a^3 = a_1^3$. Notice that the sign of the exponents alternates and the number increasing by 1 whenever it moves 1 torus close to the central a_p. Indeed, every other loop generated by the other tori in an arm can be written in terms of one of a_1, a_{p+q-1}, or $a_{p+q+r-2}$, the single loops generated by an end torus. More importantly, around the intersection of three arms, a_p. we have the relationships:

$$a_p = a_1^{p\cdot(-1)^{p-1}} = a_{p+q-1}^{q\cdot(-1)^{q-1}} = a_{p+q+r-2}^{r\cdot(-1)^{r-1}} \tag{16}$$

Notice here that in the case of a sign change, we simple change the originally defined generator to its inverse to create the final group presentation. For example, if exponent component p (q or r for the other two cases) is even, then we will take the generator $a = a_1^{-1}$, otherwise, generator $a = a_1$. Conveniently, we redefine a_{p+q-1} and $a_{p+q+r-2}$ with b and c using the same way.

Further, observe that with the relationship $a_{p-1}a_p^2a_{p+1}a_{p+q} = 1$, a_p generator may be written in terms of a, b, c. Notice the parity of p is different from that of $p-1$ and $p+1$, and with some simple testing, we can get that

$$\begin{gathered} a_{p-1}a_p = a_1^{p\cdot(-1)^{p-1}+(p-1)\cdot(-1)^{p-2}} = a \\ a_p a_{p+1} = a_1^{r\cdot(-1)^{r-1}+(r-1)\cdot(-1)^{r-2}} = b \\ a_{p-1}a_p^2a_{p+1}a_{p+q} = abc^{1-r} = 1;\ abc = c^r \end{gathered} \tag{17}$$

Then, we have the relationship

$$a_p = a^p = b^q = c^r = abc \tag{18}$$

which matches the presentation given in Eq. (12).

2.3. Seifert Fibered Space

We need the notion of Seifert fibered space as an intermediate tool for our proof. So, we will not present its whole theory here but rather it is towards the application [6, 7].

A Seifert fibered space is, roughly speaking, a variation of $\boldsymbol{S}^1$ *fiber bundle*. Consider the trivial $\boldsymbol{S}^1$ bundle over the 2-disk, that is, a solid torus $D^2 \times \boldsymbol{S}^1$. Perform a Dehn surgery: cut along a longitude (tube of the torus shape), twist a cross section by a $\frac{2\pi q}{p}$ rotation (where p, q are coprime integers) then paste it back to the other cross section. The $\boldsymbol{S}^1$ action on this twisted torus is then altered accordingly: an orbit circle wraps p times along the longitude and q times along the latitude, except that the central circle $\{0\} \times \boldsymbol{S}^1$ becomes an orbit of isotropy group C_p. The cyclic C_p group's special properties also allow it to have a set group of non-trivial stabilizers. This makes the twisted torus into a $\boldsymbol{S}^1$ bundle over the orbitfold D^2/C_p, which is a fundamental example of Seifert fibered space. The Seifert fibered spaces considered in this paper are constructed from the following recipe:

(1) Take a closed oriented surface X to be the base space.

(2) Take an oriented $\boldsymbol{S}^1$ bundle E over X.

(3) Take m trivializing small disks in X disjoint with each other, replace the m solid tori over them by some twisted tori. That is, we cut off a solid torus and glue back a (p_i, q_i)- twisted torus, in a way that the $\boldsymbol{S}^1$ actions at the boundary are compatible. That is, we are not only gluing the points but also preserving the group action results on the boundary. In this case, only the central circle is said to be an exceptional fiber of type (p_i, q_i), $1 \le i \le m$.

Example. Consider the unit sphere $\boldsymbol{S}^3 \subset \mathbb{C}^2$. Define a circle action (rotation action) on $\boldsymbol{S}^3$:

$$\gamma \cdot (z_1, z_2) = \left(\gamma^{p} z_1, \gamma^{q} z_2\right),\ \gamma \in S^1 \tag{19}$$

Fig. 4 shows that by stretching and twisting the cylinder to match the same numbers on the two faces, a Seifert fibered space is formed.

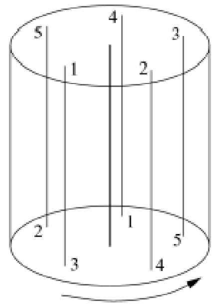

Fig. 4. The example of type (5,2). Extracted from https://commons.wikimedia.org/w/index.php?curid=4406471.

for a pair of coprime integers $0 < q < p$. Then the map $\mathbf{S}^3 \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ given by

$$(\mathrm{z}_1, \mathrm{z}_2) \mapsto \frac{\mathrm{z}_1^{\mathrm{q}}}{\mathrm{z}_2^{\mathrm{p}}} \tag{20}$$

is the resulting Seifert fibering (note that when $p = q = 1$ we recover the Hopf fibration). The exceptional fibers are those with non-trivial isotropy group, which are $\{z_1 = 0\}$ with C_q and $\{z_2 = 0\}$ with C_p. At $\{z_2 = 0\}$ there is a local $\boldsymbol{S}^1$ fibering with the base $D^2 = \left\{|z_2| \leq \frac{1}{2}\right\}$. This solid torus is twisted by $\frac{2\pi q}{p}$ since $\gamma = e^{\frac{2\pi i}{p}}$ sends (z_1, z_2) to $\left(z_1, e^{\frac{2\pi q i}{p}} z_2\right)$. Then it is glued into the global fibering, making an exceptional fiber of type (p, q). Similarly, $\{z_1 = 0\}$ is an exceptional fiber of type (q, p).

Example. Here is another example concerning the A_2-plumbing of a bundle E of Euler number n. Denote its boundary by M. Recall that the construction M involves gluing two $D^2 \times \mathbf{S}^1$ along the boundary $\mathbf{S}^1 \times \mathbf{S}^1$ through the identification map $\tau(x, y) = (y, x)$. To define a Seifert fiber structure, the $\mathbf{S}^1$ rotation action on one $D^2 \times \mathbf{S}^1$ is given by fiber rotation, namely acting only on the $\mathbf{S}^1$ component of the solid torus. The action goes through a natural transformation to the other $D^2 \times \mathbf{S}^1$ through τ. Near the fiber bundle of the base space's boundary $\mathbf{S}^1 \times \mathbf{S}^1$ the $\mathbf{S}^1$ action becomes a rotation in the base space. Through the transition map t of E defined the same way as the one given in equation 1.3, near $\{0\} \times \mathbf{S}^1$ the $\mathbf{S}^1$ orbit wraps n times along the longitude (the base D^2) and one time along the latitude (the fiber $\mathbf{S}^1$). Eventually it degenerates to a mere fiber rotation at the center of the base space $\{0\} \times \mathbf{S}^1$ continuously.

We will apply the following classification theorem presented in Orlik's or Jankins' books to proof the ultimate homeomorphism result (see [6], section 1.10 or [7], section 1.5):

Theorem 2.6. Given a 3-manifold M with Seifert fiber structure. The homeomorphism type of M is determined by

(1) The genus g of the base space X.

(2) The Euler number (defined by a self-intersection number).

(3) The type of exceptional fibers, i.e., pairs of coprime numbers (p_i, q_i) with $0 < q_i < p_i$.

Remark. Our notations are different from [6]. The type (p, q) there is the type (p, q^{-1}) here, and the invariant b there (called the cross-section obstruction) is minus the Euler number here.

2.4. Method of the Proof

To prove the main theorem, the strategy is to give Seifert fiber structures to the spaces $\mathbf{S}^3/2G$ and those from the plumbing, then use the classification theorem to conclude.

Denote by M the boundary of a plumbing of $T\times\mathbf{S}^2$ according to the (p,q,r) graph. This side is completely analogous to [4], where the case $(2,3,5)$ is shown.

Proposition 2.7. When $p^{-1}+q^{-1}+r^{-1}>1$, there is a Seifert fiber structure on M with the base $\mathbf{S}^2$, three exceptional fibers of type $(p,1)$, $(q,1)$, $(r,1)$ and the Euler number 1.

Proof: We briefly summarize Kirby's arguments. The Seifert fiber structure is produced by first doing a fiber rotation at the central vertex X, then extending it to the three arms. At the end of an arm, we might need the procedure described in the last section to deal with the base rotation. The determination of Seifert invariants is scattered in [4]: the base space has genus 0 as a consequence of finite $\pi_1(M)$ (page 132); each arm makes an exceptional fiber of type $(p,1)$, $(q,1)$ or $(r,1)$ (page 134) and increases the Euler number by 1 (page 135). The cotangent bundle has Euler number −2 so the final Euler number is 1.

Now to compare with the other side, we need some computational works.

Theorem 2.8. For some polynomial function $\varphi_G:\mathbb{C}^3\to\mathbb{C}$, there is a homeomorphism

$$S^3/2G\cong\Phi_G^{-1}(0)\cap S^5 \tag{21}$$

where $\mathbf{S}^5\subset\mathbb{C}^3$ is the unit sphere. We have $\varphi_G:\mathbb{C}^3\to\mathbb{C}$ as a polynomial function. For $G\in\{D_n,T,O,I\}$, we get:

(1) $\varphi_{D_n}=z_1^2+z_2^2z_3+z_3^{n+1}$.

(2) $\varphi_T=z_1^2+z_2^3+z_3^4$.

(3) $\varphi_O=z_1^2+z_2^3+z_2z_3^3$.

(4) $\varphi_I=z_1^2+z_2^3+z_3^5$.

Remark. Algebraic geometers know these as Du Val singularities. However, we will prove it in the next chapter by a very down-to-earth computation in the flavor of invariant theory based on what is called group involution in Du Val [8], chapter 5.

The main theorem then follows from the following.

Proposition 2.9. There is a Seifert fiber structure on $\varphi_G^{-1}(0)\cap\boldsymbol{S}^5$ defined by

$$\gamma\cdot(z_1,z_2,z_3)=(\gamma^a z_1,\gamma^b z_2,\gamma^c z_3),\gamma\in S^1 \tag{22}$$

such that the Seifert invariants coincide with those in the previous proposition. We have

(1) $G=D_n$, $(a,b,c)=(n+1,n,2)$.

(2) $G=T$, $(a,b,c)=(6,4,3)$.

(3) $G=O$, $(a,b,c)=(9,6,4)$.

(4) $G=I$, $(a,b,c)=(15,10,6)$.

It is a well-known fact that the quotient, $\mathbb{C}^2/2G$, represents the orbit space of $2G$ acting on $\mathbb{C}^2$. An orbit, then, is the set of points generated by the group action acting on one point in $\mathbb{C}^2$. Per the groups ($2G$) we are looking at, they are all subgroups of $SU(2)$. So, the elements in the orbit space $\mathbb{C}^2/2G$ are of the form:

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{pmatrix}u\\ v\end{pmatrix}\ u,v\in\mathbb{C},\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in 2G\in SU(2) \tag{23}$$

3. Invariant Theory

Recall that, we have a subgroup $2G\subset SU(2)$, $G\in\{D_n,T,O,I\}$, which acts on $\mathbb{C}^2$, and also on $\mathbb{CP}^1=\mathbb{C}\cup\{\infty\}$ by linear fractional transformation. The latter is compatible with the action of $G\subset SO(3)$ on $\mathbf{S}^2$ with respect to the double cover $SU(2)\to SO(3)$.

Properties $\mathbb{CP}^1$ is used only to calculate the elements in the orbit that we are specifically doing calculations on, which depends on the group G we look at. Recall the description of an orbit from the end of Section 2, the properties of the double cover map allow us to consider the resultant orbits of the $2G$

action on $\mathbb{CP}^1$ as sets of points on a regular polygon (whose symmetry group is D_n) or some regular polyhedrons (whose symmetry groups are T, O, I). For our purpose, we will use three special orbits $\mathcal{O}$: vertex (V), midpoint of edges (E), certers of faces (F).

Our ultimate goal is to construct an invariant function under the group action $2G$, and to do that, we first define an intermediate function to ease our construction.

For $t \in \mathbb{CP}^1$, define a function from $\mathbb{C}^2$ to $\mathbb{C}$:

$$r_t(u,v) = \begin{cases} u - tv, t = [p,q] = \frac{p}{q} \in \{\infty\} \subseteq \mathbb{CP}^1 \\ -v, t = [p,q] = \frac{p}{q} \in \mathbb{C} \subseteq \mathbb{CP}^1 \end{cases} \tag{24}$$

which can be viewed as a regular function on $\mathbb{CP}^1$ with $\mathcal{O}$ the set of zeros. For any $g \in 2G$, since $g \cdot \mathcal{O} = \mathcal{O}$, the functions $f_{\mathcal{O}}(g \cdot (u,v))$ and $f_{\mathcal{O}}(u,v)$ must differ by a constant multiple. That is

$$f_{\mathcal{O}}(au + bv, cu + dv) = k \cdot f_{\mathcal{O}}(u,v), \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in 2G \tag{25}$$

where $k = k(g, \mathcal{O}) \in \mathbb{C}$ is a constant not depending on (u,v). It follows immediately that $k(-,\mathcal{O})$ is a character of $2G$ for every orbit $\mathcal{O}$. The value for $k(g,\mathcal{O})$ will be calculated and utilized in each case.

Definition 3.1. A function satisfying the above equation is called an almost invariant form of $2G$. An invariant form is an almost one with $k \equiv 1$.

So invariant forms are functions on $\mathbb{C}^2/2G$. In what follows, we will construct them using almost invariant forms associated to some orbit $\mathcal{O}$ with non-trivial stabilizer. To this end, we compute $k(g,\mathcal{O})$ explicitly. Firstly, we have

$$\frac{r_{\frac{at+b}{ct+d}}(au+bv, cu+dv)}{r_t(u,v)} = \begin{cases} \frac{1}{ct+d}, t \neq \infty, t \neq -\frac{d}{c} \\ d, t = \infty, c = 0 \\ \frac{1}{c}, t = \infty, c \neq 0 \\ -c, t = -\frac{d}{c}, c \neq 0 \end{cases} \tag{26}$$

The numerator of the expression above comes from the fact that an element in the complex projective plane, after group action map, satisfy:

$$g \cdot (u,v) = (au + bv, cu + dv)$$

$$g \cdot t = g \cdot \left(\frac{p}{q}\right) = g \cdot ([p,q]) = [ap + bq, cp + dq] = \frac{ap+bq}{cp+dq} = \frac{a\frac{p}{q}+b}{c\frac{p}{q}+d} = \frac{at+b}{ct+d} \tag{27}$$

Taking the product over $t \in \mathcal{O}$ yields

$$k(g,\mathcal{O}) = \begin{cases} \prod_{t\in\mathcal{O}} \frac{1}{ct+d}, \quad \infty \notin \mathcal{O} \\ d \cdot \prod_{t\in\mathcal{O}\setminus\{\infty\}} \frac{ad-bc}{ct+d} = d \cdot a^{|\mathcal{O}|-1} = d^{2-|\mathcal{O}|} = a^{|\mathcal{O}|-2} \quad \infty \in \mathcal{O}\ c = 0 \\ \frac{1}{c} \cdot (-c) \cdot \prod_{t\in\mathcal{O}\setminus\{-\frac{d}{c},\infty\}} \frac{1}{ct+d} = -\prod_{t\in\mathcal{O}\setminus\{-\frac{d}{c},\infty\}} \frac{1}{ct+d} \quad \infty \in \mathcal{O}\ c \neq 0 \end{cases} \tag{28}$$

The lengthy derivation of the results is shown as the following, starting with the expression itself. All notations carried over from above. The functions will be re-notated in the specific calculations for the different types of orbit we look at.

$$\mathrm{k}(\mathrm{g},\mathcal{O}) = \frac{\mathrm{f}_{\mathcal{O}}(\mathrm{u,v})}{\mathrm{f}_{\mathcal{O}}(\mathrm{g}\cdot(\mathrm{u,v}))} = \prod_{\mathrm{t}\in\mathcal{O}} \frac{\mathrm{r}_{\frac{\mathrm{at+b}}{\mathrm{ct+d}}}(\mathrm{au+bv,cu+dv})}{\mathrm{r_t(u,v)}} = \prod_{\mathrm{t}\in\mathcal{O}} \frac{(\mathrm{au+bv}) - \frac{\mathrm{at+b}}{\mathrm{ct+d}}(\mathrm{cu+dv})}{\mathrm{u-tv}} \tag{29}$$

Proposition 3.2. We give detailed calculations to proof equation 3.3, in which we show that the value of

$$Q = \frac{(\mathrm{au+bv}) - \frac{\mathrm{at+b}}{\mathrm{ct+d}}(\mathrm{cu+dv})}{\mathrm{u-tv}} \tag{30}$$

equals with different values of t and c:

Proposition 3.3. For $t \neq \infty$ or $-\frac{d}{c}$, $Q = \frac{1}{ct+d}$

Proof:

$$Q = \frac{(\mathrm{au+bv})(\mathrm{ct+d}) - (\mathrm{at+b})(\mathrm{cu+dv})}{(\mathrm{u-tv})(\mathrm{ct+d})}$$
$$= \frac{\mathrm{auct+aud+bvct+bvd-atcu-atdv-bcu-bdv}}{(\mathrm{u-tv})(\mathrm{ct+d})}$$
$$= \frac{\mathrm{aud+bvct-atdv-bcu}}{(\mathrm{u-tv})(\mathrm{ct+d})}$$
$$= \frac{(\mathrm{ad-bc})(\mathrm{u-tv})}{(\mathrm{u-tv})(\mathrm{ct+d})} = \frac{\mathrm{ad-bc}}{\mathrm{ct+d}} \tag{31}$$

Because a, b, c, d are entrances of a matrix in $SU(2)$, their determinant $ad - bc = 1$.

Proposition 3.4. For $c = 0$, $t = \infty$, so $\frac{at+b}{ct+d} = \infty$. $Q = d$

Proof: From the definition of $\gamma_t(u, v)$ function when $t = \infty$ we can transform equation 3.8 in to the following.

$$Q = \frac{-(\mathrm{cu+dv})}{-\mathrm{v}} = \frac{-\mathrm{dv}}{-\mathrm{v}} = \mathrm{d} \tag{32}$$

Proposition 3.5. For $c \neq 0$, $t = \infty$, so $\frac{at+b}{ct+d} = \frac{a}{c}$. $Q = \frac{1}{c}$

Proof:

$$Q = \frac{\mathrm{au+bv} - \frac{\mathrm{a}}{\mathrm{c}}(\mathrm{cu+dv})}{-\mathrm{v}} = \frac{\mathrm{bv} - \frac{\mathrm{a}}{\mathrm{c}}\mathrm{dv}}{-\mathrm{v}} = \frac{(\mathrm{cb-ad})\mathrm{v}}{-\mathrm{cv}} = \frac{-\mathrm{v}}{-\mathrm{cv}} = \frac{1}{\mathrm{c}} \tag{33}$$

Proposition 3.6. For $c \neq 0$, $t = -\frac{d}{c}$, so $\frac{at+b}{ct+d} = \infty$. $Q = -c$

Proof:

$$Q = \frac{-(\mathrm{cu+dv})}{\mathrm{u-tv}} = \frac{-(\mathrm{cu+dv})}{\left(\mathrm{u} + \frac{\mathrm{d}}{\mathrm{c}}\mathrm{v}\right)} = -\mathrm{c} \tag{34}$$

Summarizing all the different cases for the product components calculated above, the total possible product of $k(\gamma, \mathcal{O})$ are

$$k(\gamma, \mathcal{O}) = \begin{cases} \prod_{t \in \mathcal{O}} \frac{1}{ct+d}, \ \infty \notin \mathcal{O} \\ d \cdot \prod_{t \in \mathcal{O} \setminus \{\infty\}} \frac{ad-bc}{ct+d} = d \cdot a^{|\mathcal{O}|-1} \Rightarrow d^{2-|\mathcal{O}|} = a^{|\mathcal{O}|-2} \ \infty \in \mathcal{O} \ c = 0 \\ \frac{1}{c} \cdot (-c) \cdot \prod_{t \in \mathcal{O} \setminus \left\{-\frac{d}{c}, \infty\right\}} \frac{1}{ct+d} = -\prod_{t \in \mathcal{O} \setminus \left\{-\frac{d}{c}, \infty\right\}} \frac{1}{ct+d} \ \infty \in \mathcal{O} \ c \neq 0 \end{cases} \tag{35}$$

(For the second case, we used the fact that $ad - bc = ad = 1$.)

We will study the cases $G = D_n, T, O, I$ individually. In each case, we will extend the above calculations to show appropriate results.

Remember, our goal, through group involution and invariant theorem is to construct a function, a map, $f_G: \mathbb{C}^3 \to \varphi_G^{-1}(0) \subseteq \mathbb{C}^3$, such that φ keeps invariant in each case of G. We will look at the values of these maps in each specific case of $G \in \{D_n, O, T, I\}$ later in this section.

3.1. Dihedral Case

Recall the formula of stereographic projection from $\mathbf{S}^2 \subset \mathbf{R}^3$ to $\mathbb{C} \cup \{\infty\}$:

$$(x, y, z) \mapsto \frac{x+yi}{1-z} \tag{36}$$

This map will be constantly referred back as "the projection" in future sections. Consider a regular n-gon $A_1 \dots A_n$ placed in the equatorial plane such that $A_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$. Say the action of $G = D_n$ on $\mathbf{S}^2$ fixes this n-gon.

There are three D_n-orbit with non-trivial stabilizer: the orbit $\mathcal{V}$ of all A_k; the orbit $\mathcal{E}$ of all midpoints of arcs $A_k A_{k+1}$; the orbit $\mathcal{F}$ of the north and south poles. We have

$$\begin{cases} \mathcal{V} = \{\mathrm{t}^\mathrm{n} - 1 = 0\}, \ \mathrm{f}_\mathcal{V}(\mathrm{u}, \mathrm{v}) = \mathrm{u}^\mathrm{n} - \mathrm{v}^\mathrm{n} \\ \mathcal{E} = \{\mathrm{t}^\mathrm{n} + 1 = 0\}, \ \mathrm{f}_\mathcal{E}(\mathrm{u}, \mathrm{v}) = \mathrm{u}^\mathrm{n} + \mathrm{v}^\mathrm{n} \\ \mathcal{F} = \left\{\mathrm{t} = \left\{\frac{1}{0}, \frac{0}{1}\right\}\right\} = \{0, \infty\}, \mathrm{f}_\mathcal{E}(\mathrm{u}, \mathrm{v}) = -\mathrm{uv} \end{cases} \tag{37}$$

An element $g \in 2D_n$, according to whether it fixes the two poles or swaps them, has one of the following forms that are shown as transformation matrices.

(1) Rotation: $g = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$. Since $g \cdot 1 = z^2 \in V$, we have $z^{2n} = 1$.

(2) Reflection: $g = \begin{pmatrix} 0 & w^{-1} \\ -w & 0 \end{pmatrix}$. Since $g \cdot 1 = -w^{-2} \in V$, we have $w^{2n} = (-1)^n$.

Remark. The entries in the above matrices satisfy the equalities because our rotation matrix, when applied n times, g^n, will belongs to the kernel of the double cover map from $2D_n \subseteq SO(2)$ to $D_n \subseteq SO(3)$. Same thing for the reflection matrix being applied an even number of times; the reflection matrix applied an odd number of times will satisfy the square of its image of the double cover map becomes the identity in $SO(3)$.

Complete Construction. We close up our construction using the explicit formula for f_V, f_E, f_F, and show the necessary slight adjustments in the process to satisfy the invariant condition.

Proposition 3.7. There is invariant form

$$(\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3) = (\mathrm{uv}(\mathrm{u}^{2\mathrm{n}} + \mathrm{v}^{2\mathrm{n}}), \mathrm{u}^{2\mathrm{n}} - \mathrm{v}^{2\mathrm{n}}, \mathrm{u}^2\mathrm{v}^2) \tag{38}$$

To be exact, we will need to modify the coefficients to construct the sum to be 0. Later on, we will rename the quantifiers to (z_1, z_2, z_3), which is (f_1, f_2, f_3) with appropriate constant coefficients, for clarity.

Proof: We prove the case $2 \nmid \mathrm{n}$ and the other case is completely analogous. We need to show $k_1 = k_2 = k_3 = 1$. One way is to use equation 3.6 and the following relationships

$$\begin{gathered} f_1^2 = f_3(f_2^2 + 4f_3^n) \\ f_3 = f_\mathcal{V} f_\mathcal{E} \\ f_3 = f_\mathcal{F}^2 \end{gathered} \tag{39}$$

to compute. But it is not hard to compute directly the k for each component. For each type of function, we first test the invariant conditions for f_F whose orbit only contains two values:

$$f_\mathcal{F}(u, v) = (u - 0 \cdot v)(u - \infty \cdot v) = -uv \tag{40}$$

By applying element $g \in 2D_n$ to (u, v)

$$\begin{cases} f_\mathcal{F}(u, v) \mapsto (zu, z^{-1}v) = -uv \quad \textit{Rotation} \\ f_\mathcal{F}(u, v) \mapsto (w^{-1}v, -wu) = uv \quad \textit{Reflection} \end{cases} \tag{41}$$

We have

$$k(g, \mathcal{F}) = \frac{(au + bv)(cu + dv)}{uv} = \begin{cases} 1, & \textit{rotation} \\ -1, & \textit{reflection} \end{cases} \tag{42}$$

So $k_3 = k(g, \mathcal{F})^2 \equiv 1$.

For the two forms that contain $f_\mathcal{V}$ and $f_\mathcal{E}$, they are to be considered together:

$$f_{\mathcal{V}}(u,v) = \prod_{t\in V}(u - tv) = u^n - t^n v^n = u^n - v^n \tag{43}$$

$$f_{\mathcal{E}}(u,v) = \prod_{t\in E}(u - tv) = u^n - t^n v^n = u^n + v^n \tag{44}$$

We derived the simplified forms of both functions using the root of unity of equations with complex roots as well as the definition of the orbits given in Eq. (37). So, we have the following:

$$\begin{cases} f_{\mathcal{V}}(u,v) \mapsto (zu, z^{-1}v) = (zu)^n - (z^{-1}v)^2 = z^n u^n - z^{-n} v^n \quad Rotation \\ f_{\mathcal{V}}(u,v) \mapsto (w^{-1}v, -wu) = (w^{-1}v)^n - (-wu)^n = w^{-n}v^n - (-w)^n (u)^n \; Reflection \end{cases} \tag{45}$$

$$\begin{array}{c} f_{\mathcal{E}}(u,v) \mapsto (zu, z^{-1}v) = (zu)^n + (z^{-1}v)^2 = z^n u^n + z^{-n} v^n \quad Rotation \\ f_{\mathcal{E}}(u,v) \mapsto (w^{-1}v, -wu) = (w^{-1}v)^n + (-wu)^n = w^{-n}v^n + (-w)^n (u)^n \; Reflection \end{array} \tag{46}$$

So $k_2 = \big(k(g,\mathcal{V})k(g,\mathcal{E})\big)^2 \equiv 1$. Because, from the relationship the matrix entries satisfy, we have:

$$k(g,\mathcal{V})k(g,\mathcal{E}) = \begin{cases} \frac{z^{2n}u^{2n} - z^{-2n}v^{2n}}{u^{2n}-v^{2n}} = 1, \quad rotation \\ \frac{w^{-2n}v^{2n} - w^{2n}u^{2n}}{u^{2n}-v^{2n}} = -1, reflection \end{cases} \tag{47}$$

coincides with $k(g,\mathcal{F})$. Because of the way f_1 is constructed from f_2 and f_3, if k_2 and k_3 are invariant, then k_1 is, too.

Because the equality $f_1^2 = f_3(f_2^2 + 4f_3^n)$ from basic algebraic calculations, we can define a map f_{D_n} from $\mathbb{C}^2/2D_n$ to $\varphi_{D_n}^{-1}(0)$ by

$$f_{D_n} = \left(\sqrt[n]{2}\cdot i \cdot f_1, f_2, \sqrt[n]{4}\cdot f_3\right) = (z_1, z_2, z_3) \tag{48}$$

Later we will show that this map f_{D_n} induces the homeomorphism as said in Theorem 2.8.

We distributed the terms into three independent complex expressions, and we have

$$(z_1,z_2,z_3)\begin{cases} z_1 = i\cdot\sqrt[n]{2}uv(u^{2n}+v^{2n}) \;\; z_1^2 = -\sqrt[n]{4}u^2v^2(u^{2n}+v^{2n})^2 \\ z_2 = f_E f_V \;\; z_2^2 = (u^{2n}-v^{2n})^2 \\ z_3 = \sqrt[n]{4}f_F^2 = \sqrt[n]{4}u^2v^2 \end{cases} \tag{49}$$

Per our equality construction our subsequent proof at the end of Section 3 involving Seifert invariant derived of it will be facilitated.

3.2. Octahedral Case

We place the octahedron in $\mathbf{S}^2$ so that its dual cube has a pair of faces parallel to the equatorial plane. So, its six vertices are sent by the projection to $\{0,\infty,\pm1,\pm i\} \subseteq \mathbb{CP}^1$. Denote this O-orbit by $\mathcal{V}$ then

$$f_{\mathcal{V}}(u,v) = \prod_{t\in\mathcal{V}}(u - tv) = uv(v^4 - u^4) \tag{50}$$

The other two orbits with non-trivial stabilizer are $\mathcal{E}$ consisting of 12 midpoints of edges and $\mathcal{F}$ consisting of 8 centers of faces. To compute $f_{\mathcal{E}}$ and $f_{\mathcal{F}}$, we may compute all these coordinates and calculate the almost-invariant function. But there is a shortcut: we can rely on the properties of the Hessian Matrix of $f_{\mathcal{V}}$ to achieve an invariant $f_{\mathcal{F}}$, which is inspired by Nash [9]. It is straightforward to check (by some lengthy calculations) that the Hessian determinant

$$\mathcal{H}(f_{\mathcal{V}}) = \begin{vmatrix} \frac{\partial^2 f_{\mathcal{V}}}{\partial u^2} & \frac{\partial^2 f_{\mathcal{V}}}{\partial u \partial v} \\ \frac{\partial^2 f_{\mathcal{V}}}{\partial u \partial v} & \frac{\partial^2 f_{\mathcal{V}}}{\partial v^2} \end{vmatrix} \tag{51}$$

is an almost invariant form.

Proposition 3.8. $\mathcal{H}(f_{\mathcal{V}}) = f_{\mathcal{F}}$ up to some constant multiple.

Proof: We know that both $\mathcal{H}(f_{\mathcal{V}})$ and $f_{\mathcal{F}}$ are degree-8 almost invariant forms under the group action $2O$ by our definition. Then, the 8 roots of $\mathcal{H}(f_{\mathcal{V}})$ must be a union of several orbits. Among the different types of orbits formed under $2O$ action, that are $\mathcal{V}$ (of 6 points), $\mathcal{E}$ (of 12 points), $\mathcal{F}$ (of 8 points), and non-special orbits. As a result, $\mathcal{H}(f_{\mathcal{V}})$ has the same set of roots as $f_{\mathcal{F}}$. This means that $\mathcal{H}(f_{\mathcal{V}}) = f_{\mathcal{F}}$ up to some constant multiple.

Explicitly, if we expand the expression for $\mathcal{H}(f_{\mathcal{V}})$, we will get $f_{\mathcal{F}}$,

$$f_{\mathcal{F}} = u^8 + 14u^4v^4 + v^8 \tag{52}$$

Similarly, the Jacobian determinant

$$\mathcal{J}(f_{\mathcal{V}}, f_{\mathcal{F}}) = \begin{vmatrix} \frac{\partial f_{\mathcal{V}}}{\partial u} & \frac{\partial f_{\mathcal{V}}}{\partial v} \\ \frac{\partial f_{\mathcal{F}}}{\partial u} & \frac{\partial f_{\mathcal{F}}}{\partial v} \end{vmatrix} \tag{53}$$

is an almost invariant form of degree 12. With the same logic as the derivation of $f_{\mathcal{F}}$, we also see that the Jacobian Matrix can be used for our construction for $f_{\mathcal{E}}$, which has 12 points in the solution set. However, there's another possibility the Jacobian Matrix's determinant can equal to: $f_{\mathcal{V}}^2$. This because the number of solutions in $f_{\mathcal{V}}$ is 6, and squared will get 12 solutions. However, the latter can be eliminated because 0 is not a root of $\mathcal{J}(f_{\mathcal{V}}, f_{\mathcal{F}})$, but it is a root of $f_{\mathcal{V}}$, so we will have a contradiction. Then, we must have the former case. Explicitly,

$$f_{\mathcal{E}} = u^{12} - 33u^8v^4 - 33u^4v^8 + v^{12} \tag{54}$$

Recall the almost invariant relationship $\varphi\big(g \cdot (u,v)\big) = k \cdot \big(\varphi(u,v)\big)$ under the group action $2O$ acting on $\mathbb{C}^2$. Because of how $k(g,\mathcal{O})$ is defined, it follows that $k(g \cdot h, \mathcal{O}) = k(g,\mathcal{O})k(h,\mathcal{O})$, which means the transformations defined by $k(-,\mathcal{O})$ is a group homomorphism from $2O$ to $\mathbb{C} - \{0\}$. Now we compute the multipliers k. Recall that

$$2O = \langle x, y, z \colon x^2 = y^3 = z^4 = xyz \rangle \tag{55}$$

and that $k(-,\mathcal{O})$ is always a character of $2O$ and a group homomorphism. It follows that

$$k(x)^2 = k(y)^3 = k(z)^4 = k(x)k(y)k(z) \in \mathbb{C} - \{0\} \tag{56}$$

Notice here that the group homomorphism maps a non-abelian group to an abelian group hence, with some manipulation, we can deduce, $k(y) = 1$, $k(x) = k(z) = \pm 1$. So, the character k may be completely determined by the value $k(z)$. This generator can be taken as

$$z = \begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix} \tag{57}$$

i.e., the 90° rotation. So, we use Eq. (28) to compute

$$\begin{gathered} k(z,\mathcal{V}) = a^4 = 1 \\ k(z,\mathcal{E}) = \left(\frac{1}{d}\right)^{12} = -1 \\ k(z,\mathcal{F}) = \left(\frac{1}{d}\right)^{8} = 1 \end{gathered} \tag{58}$$

where a represents the corresponding matrix entrance as defined in the beginning of Section 3.

Proposition 3.9. There are three invariant forms

$$(z_1, z_2, z_3) = (f_{\mathcal{V}} f_{\mathcal{E}}, f_{\mathcal{V}}^2, f_{\mathcal{F}}) \tag{59}$$

Proof: Immediately from the above results.

Proposition 3.10. There are constants $\lambda, \mu \in \mathbb{C}$ such that

$$f_{\mathcal{V}}^4 + \lambda f_{\mathcal{F}}^3 + \mu f_{\mathcal{E}}^2 = 0 \tag{60}$$

Proof: Note that $f_{\mathcal{V}}^4$, $f_{\mathcal{F}}^3$ and $f_{\mathcal{E}}^2$ are all invariant forms so they take the same value inside an orbit. We set

$$\lambda = -\frac{f_{\mathcal{V}}^4(\mathcal{E})}{f_{\mathcal{F}}^3(\mathcal{E})}, \mu = -\frac{f_{\mathcal{V}}^4(\mathcal{F})}{f_{\mathcal{F}}^3(\mathcal{F})} \tag{61}$$

So that, the zeros of $f_{\mathcal{V}}^4 + \lambda f_{\mathcal{F}}^3 + \mu f_{\mathcal{E}}^2$ contain $\mathcal{E} \cup \mathcal{F}$. But the degree is 24 and $|\mathcal{E} \cup \mathcal{F}| = 20$, there is no room for any one other whole orbit to fit in the zeros which contradicts the definition of the polynomial. This property forces the polynomial expression to be 0, not be any arbitrary polynomial of degree 24. In fact, $\mu = -\lambda = 108^{-1}$.

Corollary 3.11. Up to some constant multiples, $f_O = (z_1, z_2, z_3)$ defines a map from $\mathbb{C}^2/2O$ to $\varphi_O^{-1}(0)$.

Recall that $\varphi_O^{-1}(0)$ is the solution set of $z_1^2 + z_2^3 + z_3^3 z_2$. This φ_O we constructed always equal to 0, after we substitute in the expressions for the three quantifiers in terms of $f_{\mathcal{V}}$, $f_{\mathcal{F}}$, and $f_{\mathcal{E}}$, along with appropriate constant coefficients.

3.3. Tetrahedral Case

This case is closely related to the octahedral case. We place the tetrahedron so that the vertices are 4 of the 8 vertices of the previous dual cube. Then T is naturally viewed as a subgroup of O of index 2. Precisely,

$$\mathrm{O} = \mathrm{T} \cup \mathrm{zT} \tag{62}$$

where z is defined in 3.36. Again, we denote by $\mathcal{V}, \mathcal{E}, \mathcal{F}$ the orbits of vertices, midpoints of edges, centers of faces and use a superscript to indicate the tetrahedron or the octahedron or the cube.

Proposition 3.12. We have the relations

$$\begin{aligned} f_{\mathcal{V}^o} &= f_{\mathcal{E}^t} \\ f_{\mathcal{F}^o} &= f_{\mathcal{V}^t} f_{\mathcal{F}^t} \\ f_{\mathcal{E}^o} &= f_{\mathcal{V}^t}^3 + f_{\mathcal{F}^t}^3 \end{aligned} \tag{63}$$

and they are all $2T$-invariant.

Proof: The first one follows from the fact that $\mathcal{V}^O = \mathcal{F}^c = \mathcal{E}^t$, and the second one from $\mathcal{F}^O = \mathcal{V}^c = \mathcal{V}^t \cup \mathcal{F}^t$. For the last one, we need the group character argument. The relation

$$k(x)^2 = k(y)^3 = k(z)^3 = k(x)k(y)k(z) \in \mathbb{C} - \{0\} \tag{64}$$

implies that $\big(k(x), k(y), k(z)\big) = \left(1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\right)$ or $\left(1, e^{\frac{4\pi i}{3}}, e^{\frac{2\pi i}{3}}\right)$. Thus $f_{\mathcal{V}^t}^3$ and $f_{\mathcal{F}^t}^3$ are $2T$-invariant forms, so is their sum. Furthermore, the sum is $2O$-almost invariant. To show this, it remains to consider the action of z. Since it swaps $\mathcal{V}^t$ and $\mathcal{F}^t$, we have $f_{\mathcal{V}^t}\big(z \cdot (u,v)\big) = k \cdot f_{\mathcal{F}^t}(u,v)$ for some constant k. In fact, a slight variation of 3.6 gives

$$k = a^4 = -1 \tag{65}$$

Similarly, $f_{\mathcal{F}^t}\big(z \cdot (u,v)\big) = f_{\mathcal{V}^t}(u,v)$. We conclude that the sum is an $2O$-almost invariant form of degree 12 with the multiplier $k(z) = -1$, so it must be $f_{\mathcal{E}^o}$.

To show that the three are $2T$-invariant, the second is easy since $f_{\mathcal{F}^O}$ is already $2O$-invariant. The third one was just seen. For the first, since it is both $2O$ and $2T$ -almost invariant, its multiplier takes value in $\{\pm 1\} \cap \left\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\right\}$. Hence it must be 1.

Corollary 3.13. Up to some constant multiples, $f_T = \big(f_{\mathcal{E}^o}, f_{\mathcal{F}^o}, f_{\mathcal{V}^o}\big)$ defines a map from $\mathbb{C}^2/2T$ to $\varphi_T^{-1}(0)$, the solution set of $z_1^2 + z_2^3 + z_3^4$.

3.4. Icosahedral Case

This case has an advantage that the group character argument immediately implies that, all almost invariant forms are actually invariant. We place the icosahedron so that the orbit of vertices is

$$\mathcal{V} = \left\{0, \infty, \epsilon^{k} + \epsilon^{k+1}, \epsilon^{k} + \epsilon^{k+2}): \epsilon = e^{\frac{2\pi i}{5}}, k = 0,1,2,3,4\right\} \tag{66}$$

It follows that (by some hard work)

$$f_{\mathcal{V}} = uv(v^{10} - 11u^5v^5 - u^{10}) \tag{67}$$

The same method as in the last section will give $f_{\mathcal{E}}$ and $f_{\mathcal{F}}$. By the same reasoning in proposition 3.10 there is a relation (up to some constant multiples)

$$f_{\mathcal{E}}^2 + f_{\mathcal{F}}^3 + f_{\mathcal{V}}^5 = 0 \tag{68}$$

and we set $f_I = (f_{\mathcal{E}}, f_{\mathcal{F}}, f_{\mathcal{V}}): \mathbb{C}^2/2I \to \varphi_I^{-1}(0)$. Now we have gathered f_{D_n}, f_T, f_O and f_I.

Proposition 3.14. The map $f_G: (\mathbb{C}^2 - \{0\})/2G \to \varphi_G^{-1}(0) - \{0\}$ is a homeomorphism.

Proof: First note that the action of $2G$ on $\mathbb{C}^2 - \{0\}$ is free, so it suffices to prove that (see [10], theorem 21.10) $f_G: \mathbb{C}^2 - \{0\} \to \varphi_G^{-1}(0) - \{0\}$ is a $|2G|$-sheeted covering. We use Lee's work [10], proposition 4.46 which states that a map between connected manifolds is a finite-sheeted covering if and only if it is a proper local diffeomorphism. It is straightforward to check that the Jacobian matrix of f_G is nonsingular hence it is a local diffeomorphism. It is clearly proper since it is a polynomial map. Since f_G is $2G$ -invariant, it is at least $|2G|$-sheeted. It remains to find a point $y \in \varphi_G^{-1}(0) - \{0\}$ such that $|f_G^{-1}(y)| \leq |2G|$. For this, there is some numerical coincidence:

(1) $G = D_n$, consider $y = (0,1,0)$, equation ?? immediately implies that $\left|f_{D_n}^{-1}(y)\right| = 4n = |2D_n|$.

(2) $G = T, O, I$, recall that $f_G = (z_1, z_2, z_3)$ where $\deg(f_G) = (12,8,6), (18,12,8), (30,20,12)$ respectively. Consider $y = (1,-1,0)$. Recall that $f_3 = f_{\mathcal{V}^O}, f_{\mathcal{F}^O}, f_{\mathcal{V}^I}$ respectively, $z_3 = 0$ restricts to $deg(z_3)$ complex lines through the origin. By substituting, in each line z_i $(i = 1,2)$ has the form $u^{\deg(z_i)}$ or $v^{\deg(z_i)}$ up to some constant multiple. So, the sets of zeros of $z_1 = 1$ and $z_2 = -1$ form regular gons, hence they have at most $gcd(deg(z_1), deg(z_2))$ zeros in common. We conclude by noting that in each case

$$\deg(z_3) \cdot \gcd(\deg(z_1), \deg(z_2)) = |2G| \tag{69}$$

Finally, theorem 2.8 follows as a corollary of the proposition above. For all $(z_1, z_2) \in \mathbf{S}^3$ the unit sphere of $\mathbb{C}^2$, the function from $\mathbf{R}^{>0}$ to $\mathbf{R}^{>0}$ defined by

$$\lambda \mapsto \left|f_G(\lambda z_1, \lambda z_2)\right| \tag{70}$$

is monotonic and surjective. Hence there is a unique λ such that $f_G(\lambda z_1, \lambda z_2) \in \mathbf{S}^5$. Since the action of $2G$ is isometric,

$$f_G \circ \lambda: \boldsymbol{S}^3/2G \to \varphi_G^{-1}(0) \cap \boldsymbol{S}^5 \tag{71}$$

is a well-defined continuous map. It has a continuous inverse, which is f_G^{-1} composed with the obvious map $\mathbb{C}^2 - \{0\} \to \mathbf{S}^3$.

3.5. Compute the Seifert Invariants

In this section we prove proposition 2.9. First note that the map $\pi_G: \varphi_G^{-1}(0) \cap \mathbf{S}^5 \to \mathbf{S}^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ given by

$$\pi_G(z_1, z_2, z_3) = \begin{cases} z_3^{n+1}/z_1^2, \ G = D_n \\ z_3^4/z_1^2, \ G = T \\ z_2^3/z_1^2, \ G = O \\ z_3^5/z_1^2, \ G = I \end{cases} \tag{72}$$

Defines a Seifert fibering with the circle action given by Eq. (19) We study the exceptional fibers individually. The general principle is that we look for fibers with non-trivial isotropy group, which can only happen when $z_1 z_2 z_3 = 0$ because $\gcd(a, b, c) = 1$ in each case.

(1) $G = I$, there are three exceptional fibers. Consider $\{z_3 = 0\}$, it has the isotropy group $C_{\gcd(15,10)} = C_5$. The action of $e^{\frac{2\pi i}{5}}$ sends z_3 to $e^{\frac{2\pi i}{5}} z_3$, so the type of $\{z_3 = 0\}$ is (5,1). Similarly, $\{z_2 = 0\}$ is exceptional of type (3,1), and $\{z_1 = 0\}$ is exceptional of type (2,1).

(2) $G = O$, as usual $\{z_3 = 0\}$ is exceptional of type (3,1). The set $\{z_1 = 0\}$ breaks into two fibers: $\{z_2 = 0\}$ and $\{z_2^2 + z_3^2 = 0\}$. The action on the first one is $\gamma \cdot (0,0,z_3) = (0,0,\gamma^4 z_3)$. Because $e^{\frac{\pi i}{2}}$ sends z_1 to $e^{\frac{\pi i}{2}} z_1$, this fiber has type (4,1). The second one has the isotropy group $C_{\gcd(6,4)} = C_2$ so it has type (2,1).

(3) $G = T$, as usual $\{z_3 = 0\}$ is exceptional of type (2,1). The difference with the first case is that, $\{z_1 = 0\}$ is a regular fiber (because $\gcd(4,3) = 1$), and $\{z_2 = 0\} = \{z_1^2 + z_3^4 = 0\} = \{z_1 = i z_3^2\} \cup \{z_1 = -i z_3^2\}$ breaks into two exceptional fibers. Both has the isotropy group $C_{\gcd(6,3)} = C_3$. Since the action of $e^{\frac{2\pi i}{3}}$ sends z_2 to $e^{\frac{2\pi i}{3}} z_2$, both has type (3,1).

(4) $G = D_n$, the situation depends on the parity of n. When $2|n$, the fiber $\{z_1 = 0\}$ breaks into three: $\{(0, z_2, 0)\}$, $\left\{z_2^2 = i z_3^{n/2}\right\}$ and $\left\{z_2^2 = -i z_3^{n/2}\right\}$. The first one has the isotropy group C_n. It has type $(n, 1)$ since $e^{\frac{2\pi i}{n}}$ sends z_1 to $e^{\frac{2\pi i}{n}} z_1$. The latter two have the isotropy group $C_{\gcd(n,2)} = C_2$, so both has type (2,1). When $2 \nmid n$, still $\{(0, z_2, 0)\}$ is an exceptional fiber of type $(n, 1)$. The other two exceptional fibers are $\{z_2 = 0\} = \left\{z_1^2 = i z_3^{(n+1)/2}\right\} \cup \left\{z_1^2 = -i z_3^{(n+1)/2}\right\}$, both is of type (2,1).

Denote by d the integer such that the action of γ sends φ_G to $\gamma^d \varphi_G$. The Euler number n is computed by [6].

$$n = \sum_{i=1}^{3} \frac{q_i}{p_i} - \frac{d}{abc} = \begin{cases} \frac{1}{n} + \frac{1}{2} + \frac{1}{2} - \frac{2n+2}{2n(n+1)}, G = D_n \\ \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{12}{6 \cdot 4 \cdot 3}, G = T \\ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{18}{9 \cdot 6 \cdot 4}, G = O \\ \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{30}{15 \cdot 10 \cdot 6}, G = I \end{cases} = 1 \tag{73}$$

We examine that all these results are well in correspondence with the spaces arising from plumbing, thus complete the proof of the main theorem.

4. Conclusion

One of the most elegant results in classical mathematics is Elie Cartan's classification of simple complex Lie algebras in the late 19th century, presented in the form of Dynkin diagrams. We are very delighted to find its incarnation in the topological world in this paper. In doing this program, we have fully appreciated the power of classification theorems in math. What's even more exciting is that sometimes the classifications of two different structures which are seemingly irrelevant reveal the similar pattern, and turn out to be unified in some broader theoretical framework. For example, the classification of platonic solids has been known since the ancient Greeks. Little do people expect that it can be related to Dynkin diagrams via the so-called ADE classification. In reading Kirby's work from 1987, we are soon convinced that this

E_8- icosahedron correspondence is only a tip of the iceberg. Our paper aims to generalize the calculation presented there to other orbit spaces besides $S^3/2I$, which, from our account is a relatively simple case. Originally, we thought the equation constructed for each case would resonate the group presentation of the $2G$ group that created the orbit space. This $2G$ group also has a corresponding compact lie group represented by a Dynkin Diagram. However, as our equivalency check persisted, such generalization was more complicated than we thought. Thus, slight adjustment was made to the function being constructed. Fortunately, the generalization still existed and the adjustments were relatively minor. We managed to get the right forms in the end but, as is normal in doing math, one can as well prove something or compute something down to earth without thoroughly understanding it. The deep reason why the correspondence in our paper can be established still holds an intriguing mystery for the author. We are aware that it can be explained using some modern algebraic geometry as in McKay's work [11]. This line will keep the author motivated in the future years of studies. For now, we are very excited to discover this correspondence and present our work, as math always manifest its beauty and delight people via various surprising connections.

Conflict of Interest

The author declares no conflict of interest.

References

[1] Milnor, J. (1959). Differentiable structures on spheres. *Amer. J. Math. 81*, 962–972.

[2] Poincaré, M. H. (1904). Fifth complement to the analysis located. *Rend. Circ. Matem. Palermo., 18*, 45–110. https://doi.org/10.1007/BF03014091 (in French)

[3] Coxeter, H. S. M., & Moser, W. O. J. (1980). *Generators and Relations for Discrete Groups*. Springer-Verlag.

[4] Kirby, R. C., & Scharlemann, M. G. (1979). *Eight Faces of the Poincare Homology 3-Sphere*. Academic Press.

[5] Lickorish, W. B. R. (1997). *An Introduction to Knot Theory*, Springer-Verlag.

[6] Orlik, P. (1972). *Seifert Manifolds*, Springer-Verlag.

[7] Jankins, M., & Neumann, W. D. (1983). *Lectures on Seifert Manifolds*. Brandeis University.

[8] Duval, P. (1964). *Homographies, Quaternions and Rotations*. Oxford University Press.

[9] Nash, O. (2014). On Klein's icosahedral solution of the quintic. *Expositiones Mathematicae, 32(2)*, 99–120.

[10] Lee, J. M. (2013). *Introduction to Smooth Manifolds* (2nd ed.). Springer-Verlag.

[11] McKay, J. (1980). Graphs, singularities and finite groups. *Proc. Symp. Pure Math., 37*, 183–186.