Existence of the Solution for the Problem in Subdiffusive Medium with a Moving Concentrated Source

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Abstract: This paper investigates the problem of a fractional differential equation with moving concentrated source in an infinite rod. This fractional differential operator is used to formulate the diffusion problem in a subdiffusive medium. The existence of the solution is established, the finite time blow-up criteria for the solution of the problem is given. The critical speed of the concentrated sources on the behavior of the solution is investigated.

Keywords: Green's function, fractional diffusion equations, fractional derivatives, moving concentrated source.

1. Introduction

Let $\alpha < 1$, T, a be positive real numbers, $L_\alpha u = u_t - (D_t^{1-\alpha} u)_{xx}$, where $D_t^{1-\alpha} u$ denotes the Riemann-Liouville fractional derivative.

We consider the problem

$$L_\alpha u(x,t) = \delta(x-at)\, f(u(x,t)) \text{ in } (-\infty,\infty) \times (0,T], \tag{1}$$

subject to initial and boundaries conditions

$$u(x,0)=0 \text{ in } (-\infty,\infty),$$

$$u(x,t) \to 0 \text{ as } |x|\to\infty \text{ for } 0<t<T \tag{2}$$

where $\delta(x)$ is the Dirac Delta function, $f(u) > 0$, $f'(u) > 0$, $f''(u) \geq 0$ for $u \geq 0$, $\lim_{u\to\infty} f(u) = \infty$ and $\int_0^\infty \frac{1}{f(u)} du$ is bounded above. In this paper, we will study the existence and non-existence of the solution of problem (1)-(2). The blow-up behavior of the solution related with the speed of the moving source is given. In particular, the increasing nature of the solution with respect to time will be shown. Furthermore, the location of the blow-up point is established.

The operator in this paper is used to model the diffusive problem in a subdiffusive medium c.f. [1]–[3] and [4]. In order to obtain local existence of problem (1)-(2), by using the corresponding Green's function in [5], we construct the integral representation form of the solution as in [6], which can be shown the local existence of the solution as in [7]. Different from the work of Olmstead and Roberts [8], to discuss the blow-up properties of the solution, we consider the problem (1)-(2) directly in this paper.

Our main result on the existence of the solution follows from a similar argument as in the proofs of Theorems 3 and 4 of Liu and Huang [9], and Theorem 2.2 of Chan and Treeyaprasert [10] that we obtain the following Theorem.

Theorem 1.1 *The problem (1)-(2) has a unique nonnegative continuous solution for* $0 \le t < t_b$. *If* t_b *is finite, then u is unbounded in* $[0,t_b]$.

In the next section, we show the increasing nature of the solution with respect to time in Theorem2.2, and then the decreasing nature with respect to the velocity of the moving source in Theorem2.3. Combing these results, we show that the solution blows up in a finite time when the speed is slow in Theorem2.5. This can be interpreted as the concentrated energy source moves in a slow speed, such that the energy inside the medium accumulates fast enough for blow-up to occur. On the other hand, we show that when the speed of the source is fast enough, the energy will not accumulate high enough, and hence the solution exists for all time, see Theorem 2.6. Therefore, from Theorems 2.3, 2.5 and 2.6, we conclude that there is a critical speed for problem (1)-(2), and the blow-up occurs at $x = aT$.

2. Blow-up of the Solution

Let $u(x,t)$ be the solution of the problem (1)-(2), we need the following lemma to show that u is increasing with respect to time t.

Lemma 2.1 $D_t^{2-\alpha}u > 0$ *for* $(x,t) \in (-\infty,\infty) \times (0,T)$.

Proof:

Let $w(x,t) = D_t^{-\alpha}u(x,t)$. Then $D_t^{\alpha}w(x,t) = D_t^{\alpha}D_t^{-\alpha}u(x,t) = u(x,t)$ for any $(x,t) \in (-\infty,\infty) \times (0,T)$.

By using the continuity of w, we get $D_t^{\alpha-1}w(x,t)|_{t=0} = 0$. It follows from the properties of fractional differential equations (cf. Podlubny [3]) that $u_t = D_t(D_t^{\alpha}w) = D_t^{1+\alpha}w$. Then, $(D_t^{1-\alpha}u)_{xx} = (w_t)_{xx}$.

Thus w satisfies the problem:

$$D_t^{1+\alpha}w=(w_t)_{xx}+\delta(x-at)\, f(u(x,t)) \tag{4}$$

$$w(x,0) = 0,\ w(x,t) \to 0 \text{ as } |x|\to\infty.$$

Let us take D^{-1} on both sides of (4). Then $D_t^{\alpha}w(x,0) = u(x,0) = 0$, and $D_t^{-1+\alpha}w(x,0) = 0$, we get,

$$D^{-1}(D_t^{1+\alpha}w) = D_t^{\alpha}w.$$

Similarly, it follows from $D_t^{-1}w(x,0) = 0$,

$$D^{-1}((w_t)_{xx}) = w_{xx}.$$

For the forcing term, we get

$$D^{-1}\delta(x - at)f(u(x,t)) = \frac{1}{\Gamma(1)}\int_0^t \delta(x-as)\, f(u(x,s))ds$$
$$= \frac{1}{a} f(u(x,\frac{x}{a}))$$

For $0 < x < at$, and zero otherwise. Hence w satisfies

$$D_t^{\alpha}w = w_{xx} + \begin{cases} \frac{1}{a} f\left(u\left(x,\frac{x}{a}\right)\right), 0 < x < at \\ 0, \text{otherwise.} \end{cases}$$

Since $u(x,t)$ is continuous, we have $w(x,t)$ is differentiable. Upon differentiation with respect to t on the above equation, we get

$$D_t^{\alpha} w_t = (w_t)_{xx}. \tag{5}$$

Since $u(x,t) > 0$ for $(x,t) \in (-\infty,\infty) \times (0,T)$, we have $w(x,t) > 0$ for $(x,t) \in (-\infty,\infty) \times (0,T)$. With $w(x,0) = 0$ for $x \in (-\infty,\infty)$, we get $w_t(x,0) \geq 0$ for $x \in (-\infty,\infty)$. Thus, it follows from the maximum principle that $w_t(x,t) > 0$ for $(x,t) \in (-\infty,\infty) \times (0,T)$. As a consequence, $D_t^{1-\alpha}u > 0$.

Next, we differentiate (5) with respect to t again, we get

$$D_t^{\alpha} w_{tt} = (w_{tt})_{xx}.$$

From $w_{tt}(x,0) \geq 0$ for $x \in (-\infty,\infty)$, by the maximum principle, c.f. [11], [12] we have $w_{tt}(x,t) > 0$ for $(x,t) \in (-\infty,\infty) \times (0,T)$. Note that $D_t^{2-\alpha}u = w_{tt}$ and hence the lemma follows.

Theorem 2.2 *The solution u of the problem (1)-(2) is increasing with respect to time t.*

Proof: Note that

$$D_t^{-(1-\alpha)}\left(D_t^{(2-\alpha)}u(x,t)\right) = D_t u(x,t) - \left(D_t^{(1-\alpha)}u(x,0)\right)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

We get

$$u_t(x,t) = D_t^{-(1-\alpha)}\left(D_t^{(2-\alpha)}u(x,t)\right) + \left(D_t^{(1-\alpha)}u(x,0)\right)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Furthermore, $D_t^{-(1-\alpha)}\left(D_t^{(2-\alpha)}u(x,t)\right) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} D_t^{(2-\alpha)}u(x,s)ds > 0$. From Lemma 2.1, and $D_t^{1-\alpha}u(x,0) > 0$, we have $u_t(x,t) > 0$. This completed the proof.

Note that the speed of the moving source will affect the blow-up behavior. In particular, if the source moves slowly, then energy accumulate faster, and hence blow-up will more likely to occur. Therefore, we have the following result on the comparison of the solutions with respect to the speed of the source.

Theorem 2.3 *The solution $u(x,t)$ of the problem (1)-(2) decreases with respect to a for $(x,t) \in (-\infty,\infty)\times(0,T)$.*

Proof: By taking the operator $D_t^{-1+\alpha}$ on both sides of (1), we have

$$D_t^{-1+\alpha}(u_t) = D_t^{-1+\alpha}\left((D_t^{1-\alpha}u)_{xx} - \delta(x-at) f(u(x,t))\right).$$

It follows from $D_t^0 u|_{t=0} = 0$ and $D_t^{-1}u|_{t=0} = 0$, we have $D_t^{-1+\alpha}(u_t) = D_t^{\alpha}u$. Therefore, the solution $u(x,t)$ satisfies

$$D_t^{\alpha}u(x,t) = u_{xx}(x,t) + \frac{1}{\alpha\Gamma(1-\alpha)}\left(1-\frac{x}{a}\right)^{-\alpha} f\left(u\left(x,\frac{x}{a}\right)\right)$$

For $x \in (0,at)$ and $0 < t < T$.

Note that $\frac{1}{\alpha\Gamma(1-\alpha)}\left(t-\frac{x}{a}\right)^{-\alpha} f\left(u\left(x,\frac{x}{a}\right)\right)$ decreases with respect to a, it follows from the maximum principle that $u(x,t)$ is decreasing with respect to a.

Let $R_T = \{(x,t) : \text{for any } 0 < t \leq T, -\infty < x < at\}$, $B_t = \{(x,t) : \text{for any } 0 < t \leq T, x = at\}$. Let us consider the solution u on the subdomain R_T, then u satisfies

$$u_t = (D_t^{1-\alpha}u)_{xx} \text{ in } R_T \tag{6}$$

With $u(x,0) = 0$, $u(x,t) \to 0$ as $x \to -\infty$, and $u(at,t) > 0$. For any fixed $t_0 > 0$, it follows from the maximum principle, c.f. [12], that $\max_{R_{t0}} u(x,t) = u(at_1,t_1)$ for some $(at_1,t_1) \in B_{t0}$. Since $u_t > 0$, we have $u(at_1.t_1) \leq u(at_1,t_0) \leq u(at_0,t_0)$. Thus we get $\max_{R_{t0}} u(x,t) = u(at_0,t_0)$. Next result shows the behavior of spatial derivative of the solution.

Theorem 2.4 *The solution u of the problem (1)-(2) is nonnegative, and for* $0 < t < T$, *the solution u obtains* *its maximum value at $x = at$. Furthermore, if u blows up in the finite time T, then $x = aT$ is the only blow-up point.*

Proof:

It follows from the maximum principle [12] that $u > 0$ for $(x,t) \in (-\infty,\infty) \times (0,T)$. Furthermore, for $t > 0$, and any fixed $\zeta \in (-\infty,at)$, we consider the problem

$$\begin{cases} L\alpha u(x,t) \; = \delta(x-at)\, f(u(x,t)) \text{ in } (-\infty,\zeta) \times (0,T] \\ u(x,0) = 0 \text{ in } (-\infty,\zeta) \text{ for } t > 0 \\ u(x,t) \to 0 \text{ as } x \to -\infty. \end{cases}$$

From maximum principle, the maximum value of u must occurs at $x = \zeta$. This gives that $u_x(\zeta,t) > 0$ for $t \in (0,T)$. Since ζ is arbitrary in $(-\infty, at)$ for $t > 0$, we have $u_x(x,t) > 0$ for $(x,t) \in (-\infty,at) \times (0,T)$. A similar argument gives that $u_x(x,t) < 0$ for $(x,t) \in (at,\infty) \times (0,T)$. Hence the result follows.

Then an argument similar to that in the proof of Theorem 4 of Liu [7] that $x = aT$ is the only blow-up point if the solution blows up at time $t = T$.

Let $u_1(x,t)$ and $u_2(x,t)$ be the solution of the problem (1)-(2) corresponding to a_1 and a_2 respectively with $a_1 < a_2$. Then, according to Theorem 2.3, $u_1(x,t) > u_2(x,t)$ for $(x,t) \in (-\infty,\infty) \times (0,\tilde{t}\,)$ with $\tilde{t}= \min\{t_1,t_2\}$ where t_i denotes the existence time of the solution of the problem with respect to a_i for $i = 1,2$. If $\max_{x\in(-\infty,\infty)}u_2(x,t) \to\infty$ as $t \to T$, we must have $\max_{x\in(-\infty,\infty)}u_1(x,t) \to\infty$ as $t \to \hat{T}$ for some $\hat{T} \leq T$. Therefore, the existence time $t_1 \leq t_2$. The previous theorems give that the solution accumulates energy at $x = at$ as time goes by. Hence, when the source is moving slower, the energy may large enough so that the solution blows up in a finite time T. Thus, we have the following result.

Theorem 2.5 *The solution $u(x,t)$ of the problem (1)-(2) blows up for a is small.*

Proof: Assume that $\int_0^\infty \frac{1}{f(u)}du \leq K$ for some $K > 0$. To investigate the blow-up behavior of the solution u of the problem, let $F(t) = \int_{-\infty}^{\infty} e^{-x^2} u(x,t)dx$. It follows from a direct computation that,

$$F'(t) = \int_{-\infty}^{\infty} e^{-x^2} u_t\,(x,t)dx,$$

And

$$\int_{-\infty}^{\infty} e^{-x^2}\, D_t^{1-\alpha} u_{xx}(x,t)dx = \int_{-\infty}^{\infty} (4x^2 \; - \; 2)e^{-x^2}\, D_t^{1-\alpha}u(x,t)dx.$$

Since $D_t^{1-\alpha}u(x,t) > 0$, we get

$$F'(t) \geq -2D_t^{1-\alpha}F(t) + \int_{-\infty}^{\infty} \delta(x-at)e^{-x^2} f(u(x,t))dx$$

$$= -2D_t^{1-\alpha}F(t) + e^{-a^2t^2} f\big(u(at,t)\big). \tag{7}$$

Since $u(at,t) \geq u(x,t)$ for any $(x,t) \in (-\infty,\infty)\times(0,T)$, we have $\sqrt{\pi}u(at,t) = \int_{-\infty}^{\infty} e^{-x^2} u(at,t)dx \geq \int_{-\infty}^{\infty} e^{-x^2} u(x,t)dx = F(t)$. It follows from the increasing nature of f, we get $e^{-a^2t^2} f(u(at,t)) \geq e^{-a^2t^2} f(\frac{1}{\sqrt{\pi}}F(t))$ for $0 < t < T$. Hence (7) becomes

$$F'(t) + 2D_t^{1-\alpha}F(t) \geq e^{-a^2t^2} f(\tfrac{1}{\sqrt{\pi}}F(t))$$

Or

$$\frac{F'(t)}{f(\frac{1}{\sqrt{\pi}}F(t))} + 2\frac{D_t^{1-\alpha}F(t)}{f(\frac{1}{\sqrt{\pi}}F(t))} \geq e^{-a^2t^2}.$$

Let us integrate above inequality with respect to time from 0 to $\tilde{t}$ and get

$$\sqrt{\pi}\int_0^{F(\tilde{t})}\frac{1}{f(u)}du+\frac{2}{\Gamma(\alpha)}\int_0^{\tilde{t}}\frac{\int_0^{s}(s-\tau)^{\alpha-1}F\prime(\tau)\mathrm{d}\tau}{f(\frac{1}{\sqrt{\pi}}F(s))}ds\geq\frac{\sqrt{\pi}erf\ (a\tilde{t}\ )}{2a}. \tag{8}$$

It follows from interchanging the order of integration and the fact that $F(t)$ is increasing with respect to time t, we have

$$\frac{2}{\Gamma(\alpha)}\int_0^{\tilde{t}}\frac{\int_0^{s}(s-\tau)^{\alpha-1}F'(\tau)d\tau}{f(\frac{1}{\sqrt{\pi}}F(s))}ds=\frac{2}{\Gamma(\alpha)}\int_0^{\tilde{t}}\int_0^{\tilde{t}}\frac{(s-\tau)^{\alpha-1}F'(\tau)}{f(\frac{1}{\sqrt{\pi}}F(s))}dsd\tau$$

$$\leq\frac{2}{\alpha\Gamma(\alpha)}\int_0^{\tilde{t}}\frac{(\tilde{t}-\tau)^{\alpha}F'(\tau)}{f\left(\frac{1}{\sqrt{\pi}}F(\tau\ )\right)}d\tau$$

$$=\frac{2}{\alpha\Gamma(\alpha)}\sqrt{\pi}\tilde{t}^{\alpha}\int_0^{F(\tilde{t}\ )}\frac{1}{f(u)}du.$$

Then (8) becomes

$$K\left(1+\frac{2\tilde{t}^{\alpha}}{\alpha\Gamma(\alpha)}\right)\geq\int_0^{F(\tilde{t}\ )}\frac{1}{f(u)}du\left(1+\frac{2\tilde{t}^{\alpha}}{\alpha\Gamma(\alpha)}\right)\geq\frac{erf\ (a\tilde{t}\ )}{2a}$$

For $\tilde{t}\in(0,T)$ where T is the existence time for the solution u.

For a fixed $a_1>0$, let $u_{a1}(x,t)$ be the solution corresponding to $a=a_1$. Assume that $u_{a1}(x,t)$ exists for $t\in(0,t_1)$. Note that if $\tilde{a}<a_1$ and $\tilde{a}<\frac{1}{2}\cdot\frac{1}{K\left(1+\frac{2\tilde{t}^{\alpha}}{\alpha\Gamma(\alpha)}\right)}$, then the existence time for its corresponding solution $u_{\tilde{a}}(x,t)$ must be less than or equal to t_1. Otherwise,

$$K\left(1+\frac{2\tilde{t}^{\alpha}}{\alpha\Gamma(\alpha)}\right)\geq\frac{erf\ (a\tilde{t}\ )}{2a}>K\left(1+\frac{2\tilde{t}^{\alpha}}{\alpha\Gamma(\alpha)}\right)$$

A contradiction. This shows that the solution exists in a finite time only when a is small.

On the other hand, if a is large, the source moves too fast so that the energy cannot accumulate large enough for blow-up to occur, we have the following criteria for the solution exists for all time.

Theorem 2.6 *The solution of the problem (1)-(2) exists for all time t when a is large.*

Proof: Let $G(t)=\int_0^{\infty}e^{-ax}u(x,t)dx$ for $t>0$. Let $k(t)=aD_t^{1-\alpha}u(0,t)+D_t^{1-\alpha}u_x(0,t)$, since $x=0$ is not a blow-up point, we get $D_t^{1-\alpha}u(0,t)$ and $D_t^{1-\alpha}u_x(0,t)$ are bounded for all $t\geq 0$, and hence $k(t)$ is bounded above for $t\geq 0$. Note that $G(t)$ satisfies

$$G'(t)=-k(t)+a^2(D_t^{1-\alpha}G(t))+e^{-a^2t}f(u(at,t)).$$

Assume that $u(at,t)\to\infty$ as $t\to T^-$. Then for $M>0$, there is $t_0>0$ such that $f(u(at,t))>M$ for $t\geq t_0$. Note that $D_t^{1-\alpha}G(t)=\frac{1}{\Gamma(\alpha)}\tilde{t}^{-1+\alpha}G(t)\geq\frac{1}{\Gamma(\alpha)}t^{-1+\alpha}G(t)$. Therefore

$$G'(t)-\frac{a^2}{\Gamma(\alpha)}t^{-1+\alpha}G(t)\geq -k(t)+e^{a^2t}M,$$

Then

$$\frac{G\prime(t)}{G(t)}-\frac{a^2}{\Gamma(a)}t^{-1+\alpha}\geq-\frac{k(t)}{G(t)}+\frac{e^{a^2t}M}{G(t)}.$$

Upon integration from t_0 to T, we have

$$\int_0^{\infty}\frac{1}{f(\nu)}d\nu-\frac{a^2(T^{\alpha}-t_0^{\alpha})}{\alpha\Gamma(\alpha)}\geq\int_{t_0}^{T}\frac{k(t)}{G(t)}dt+\int_{t_0}^{T}\frac{e^{a^2t}M}{G(t)}dt\,.$$

The right hand side of the above inequality remains positive and bounded above while the left hand side becomes negative when $a^2 > \frac{\alpha\Gamma(\alpha)\int_0^\infty \frac{1}{f(v)}dv}{T^\alpha - t_0^\alpha}$. This contradiction shows that the solution remains bounded for all $t > 0$.

By combining the Theorems 2.3, 2.5, and 2.6, we conclude the follow result:

Theorem 2.7 *There is a^* such that when $a > a^*$ the solution of (1)-(2) remains bounded for all $t > 0$; while for $a < a^*$ the solution blows up in a finite time T.*

3. Conclusion

In this study, the moving source problem of in a subdiffusive medium was studied. We showed the increasing nature of the solution with respect to time, so that the solution of the problem might become unbounded in a finite time. In particular, we showed that if the speed of the source moved too fast, the energy cannot be accumulated large enough, so that the solution exists for all time. Conversely, when speed is slow, the energy will grow fast enough for the blow-up to occur. These results advance the blow-up properties of the fractional differential equation, and deepen the understanding of the role of fractional derivatives and the singularity of solution. It also provided another analytical tool for the study of concentrated sources' problem in subdiffusive medium. The linear movement of the source was considered here, it is worth for further discussion on the nonlinear type of movement situation.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

The author conducted the research and prepared the paper all by himself. The author had approved the final version.

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