

Simple but High-Accuracy Approximations for $n!$

Steven Pigeon*

Department of Mathematics, Computer Science and Engineering, Université du Québec à Rimouski, Canada.

* Corresponding author. Tel.: 418 723 1986 x1105; email: steven_pigeon@uqar.ca

Manuscript submitted February 17, 2021; accepted March 22, 2021.

doi: 10.17706/ijapm.2021.11.4.61-70

Abstract: Computing $n!$ efficiently and accurately is notoriously difficult. Many have proposed approximations, varying in accuracy and in computational complexity. Interesting and useful approximations are both accurate and computationally inexpensive, and, if possible, exact up to machine-precision floating-point numbers. In this paper, we exploit an observation on a previous approximation by Hodgman to obtain a new class of correction terms using simple, but optimal given their degree, rational functions. We show that the proposed approximations are more accurate than some of the best-known approximations while remaining computationally inexpensive.

Key words: Approximation, asymptotic error, factorial, rational function.

1. An approximation by Hodgman

The best-known series to compute $n!$ is most certainly Stirling's [1] series

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \dots\right), \quad (1)$$

where the numerators and denominators are given by sequences A001163 and A001164 from the OEIS [2], [3]. We may truncate the series to

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n}\right) \quad (2)$$

As a trade-off between accuracy and speed of computation. However, using only $1 + \frac{1}{12n}$ as a correction term makes eq. (2) underestimate $n!$. To counter this effect, a proposition, likely by Hodgman [4], p. 326, is to use

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n-1}\right) \quad (3)$$

But it now overestimates $n!$ —while yielding a larger error than eq. (2), which may explain why this approximation disappeared from ulterior editions. But if eq. (2) underestimates $n!$, eq. (3) overestimates $n!$, and $\frac{1}{12n-\alpha}$ is a continuous function, then, by the intermediate value theorem, there must be a value or a function α , with $-1 < \alpha < 0$, such that the error is zero. However, to obtain a computationally simple formula, we must limit the form α can take. For this work, we will limit ourselves to rational functions of small degree d in n , noted $\alpha(d, n)$. Let then

$$P_{\alpha,d}(n) = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n+\alpha(d,n)}\right) \quad (4)$$

be the approximation to $n!$ in which $\alpha(d, n)$ is used.

We will find the optimal rational function $\alpha(d, n)$, where the degree d is chosen to satisfy one's trade-off between accuracy and computational cost. For $d = 0$, $\alpha(d, n)$ will be a constant, while for $d > 0$, $\alpha(d, n)$ will be a non-constant rational function of degree d . In either case, it will yield a correction term of degree $d + 1$, as we will show. In this paper, we will present a new method to find the optimal rational function $\alpha(d, n)$ for any desired degree d , starting, in Section 2, with the special case $d = 0$, then, in Section 3, for arbitrary degree d . In Section 4, we will compare our results with some of the previously known approximations. Section 5 discusses the implementation and computational complexity of our proposed approximations. We conclude in Section 6.

2. Optimal Constant α

To find the optimal expression for $\alpha(d, n)$, we first notice that the squared error

$$E = (n! - P_{\alpha,d}(n))^2$$

is convex in $\alpha(d, n)$. We can therefore find $\alpha(d, n)$ by solving $\frac{\partial E}{\partial \alpha(d, n)} = 0$ for $\alpha(d, n)$. This is equivalent to solving

$$1 + \frac{1}{12n + \alpha(d, n)} = 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \dots \tag{5}$$

for $\alpha(d, n)$. Using α as a shorthand for $\alpha(d, n)$, we rework eq. (5) to obtain

$$\alpha = -\frac{1}{2} - \frac{\alpha}{24n} + \frac{139}{360n} + \frac{139\alpha}{4320n^2} + \frac{571}{17280n^2} + \frac{571\alpha}{207360n^3} - \frac{163879}{1451520n^3} + \dots \tag{6}$$

If we want α to be a constant, that is, use $\alpha(0, n)$, we keep only the constant terms from eq. (6), and we find $\alpha = -\frac{1}{2}$. Substituting $\alpha = -\frac{1}{2}$ back into eq. (4), we obtain the optimal asymptotic approximation (for α constant)

$$n! \sim P_{\alpha,0}(n) = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - \frac{1}{2}}\right) \tag{7}$$

Resulting in a first degree rational function for the correction term,

$$n! \sim P_{\alpha,0}(n) = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(\frac{24n + 1}{24n - 1}\right).$$

3. Optimal Degree-Limited Solutions

To solve for $\alpha(d, n)$ for a given degree d , we must retain from eq. (6) only terms which are of degree d or less. If we chose $d = 1$, we retain the terms of degree 0 (constants) and 1, that is,

$$\alpha = -\frac{1}{2} - \frac{\alpha}{24n} + \frac{139}{360n},$$

from which we get, by isolating α , $\alpha(1, n) = -\frac{1}{2} + \frac{293}{720n + 30}$. This yields the much better asymptotic approximation

$$n! \sim P_{\alpha,1}(n) = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - \frac{1}{2} + \frac{293}{720n + 30}}\right) \tag{8a}$$

$$= \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{360n + 15}{4320n^2 + 139}\right), \tag{8b}$$

Resulting in a second degree rational function for the correction term. Solving eq. (6) for a second degree

rational function $\alpha(2, n)$, we retain

$$\alpha = -\frac{1}{2} - \frac{\alpha}{24n} + \frac{139}{360n} + \frac{139\alpha}{4320n^2} + \frac{571}{17280n^2},$$

Giving us $\alpha(2, n) = -\frac{1}{2} + \frac{7032n+293}{17280n^2+720n-556}$. The approximation becomes

$$n! \sim P_{\alpha,2}(n) = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - \frac{1}{2} + \frac{7032n+293}{17280n^2+720n-556}}\right) \tag{9a}$$

$$= \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{17280n^2+720n-556}{207360n^3+571}\right), \tag{9b}$$

Which has now a third degree correction term. Solving eq. (6) for a third degree rational function $\alpha(3, n)$, we retain

$$\alpha = -\frac{1}{2} - \frac{\alpha}{24n} + \frac{139}{360n} + \frac{139\alpha}{4320n^2} + \frac{571}{17280n^2} + \frac{571\alpha}{207360n^3} - \frac{163879}{1451520n^3},$$

Yielding $\alpha(3, n) = -\frac{1}{2} + \frac{1181376n^2+49224n-331755}{2903040n^3+120960n^2-93408n-7994}$, and therefore

$$n! \sim P_{\alpha,3}(n) = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n - \frac{1}{2} + \frac{1181376n^2+49224n-331755}{2903040n^3+120960n^2-93408n-7994}}\right) \tag{10a}$$

$$= \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1451520n^3+60480n^2-46704n-3997}{17418240n^4-163879}\right) \tag{10b}$$

Resulting in a fourth degree rational function for the correction term.

We could find $\alpha(4, n)$, $\alpha(5, n)$, etc., to yield increasingly more accurate approximations, but each new solution would be also increasingly computationally demanding. For a good trade-off between accuracy and computation, we are likely to retain only low-degree approximations, such as eqs. (7) to (10). Complexity of evaluation is discussed in Section 5.

4. Compared Accuracy

To compare the accuracy of the proposed approximations, we will use the absolute relative error

$$E(n!, a) = \left| \frac{n! - a}{n!} \right|,$$

where a is the approximation to $n!$ being evaluated. Since we are interested in simple approximations, we will exclude from this study costly approximations involving trigonometric functions, infinite series, (infinite) continued fractions, or Bernoulli numbers [5]-[7]. We will, however, compare our solutions to Mohanty's and Rummens' formula [8], which is a response to Weissman's [9] comment on Feller's and Kasper's approximation [10], itself derived from an identity by Burnside [11]. We will also compare our solutions to Nemes' [12], Mortici's [13], and Gosper's approximations [14].

In Fig. 1 and Table 1, we compare the relative error of eqs. (2), (3), and (7) to $n!$ computed exactly using arbitrary-precision arithmetic. While Hodgman's correction, eq. (3), does worse than eq. (2), the Stirling series truncated to its first two terms, the added error is asymptotically negligible. However, the correction with $\alpha = -\frac{1}{2}$ does much better than either preceding approximations. We notice, in the lower-right corner of Fig. 1, the effects of the approximation reaching machine-precision floating point accuracy. We will use arbitrary-precision arithmetic in subsequent figures.

We show the absolute relative errors of eqs. (7) to (10) in Fig. 2 and Table 1, using arbitrary-precision arithmetic. In Fig. 2, one notices immediately "dips" in the absolute relative errors of eqs. (8) and (10). These correspond to values of n where the approximation is equal to $n!$. Solving explicitly, one finds $n \approx$

5.88037 for eq. (8), and $n \approx 0.311977$, $n \approx 1.21606$, and $n \approx 15.2894$ for eq. (10).

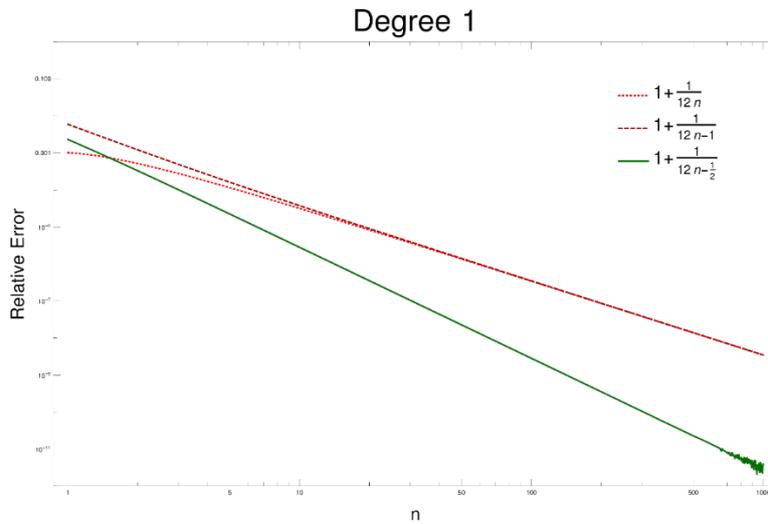


Fig. 1. Comparison of correction terms of degree 1.

Table 1. Numerically Compared Absolute Relative Errors for Constant Corrections, eq. (2), eq. (3), and eq. (7)

n	$1 + \frac{1}{12n}$	$1 + \frac{1}{12n-1}$	$1 + \frac{1}{12n-\frac{1}{2}}$
1	0.00102	0.00597	0.00232
10	0.00003	0.00004	2.81813×10^{-6}
20	8.30951×10^{-6}	9.05176×10^{-6}	3.53003×10^{-7}
30	3.74804×10^{-6}	3.96804×10^{-6}	1.04634×10^{-7}
40	2.12373×10^{-6}	2.21656×10^{-6}	4.41486×10^{-8}
50	1.36513×10^{-6}	1.41265×10^{-6}	2.26055×10^{-8}
60	9.50755×10^{-7}	9.78260×10^{-7}	1.30823×10^{-8}
70	6.99957×10^{-7}	7.17278×10^{-7}	8.23860×10^{-9}
80	5.36733×10^{-7}	5.48337×10^{-7}	5.51930×10^{-9}
90	4.24595×10^{-7}	4.32745×10^{-7}	3.87641×10^{-9}
100	3.44252×10^{-7}	3.50193×10^{-7}	2.82592×10^{-9}
500	1.38651×10^{-8}	1.39127×10^{-8}	2.26080×10^{-11}
1000	3.46925×10^{-9}	3.47519×10^{-9}	2.82600×10^{-12}
5000	1.38865×10^{-10}	1.38913×10^{-10}	2.26080×10^{-14}
10000	3.47193×10^{-11}	3.47252×10^{-11}	2.82600×10^{-16}

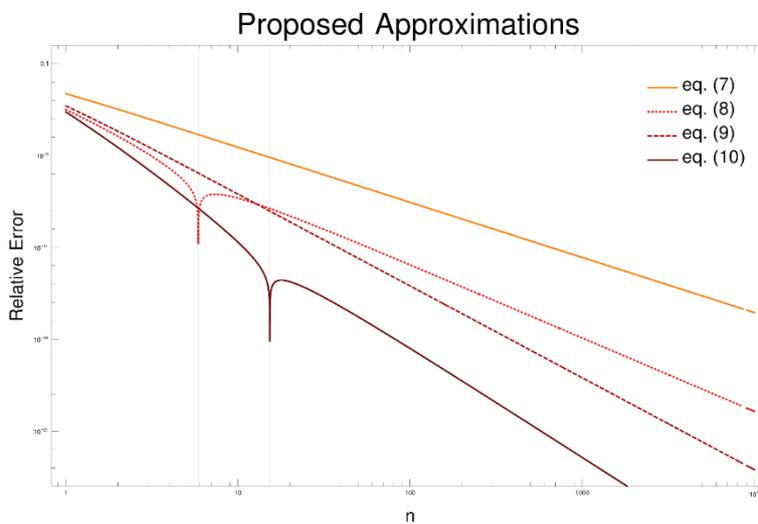


Fig. 2. Proposed approximations and their absolute relative error to $n!$.

Table 2. Numerically Compared Absolute Relative Errors for the Proposed Approximations

n	eq. (7)	eq. (8)	eq. (9)	eq. (10)
1	0.00232	0.00031	0.00050	0.00023
10	2.81813×10^{-6}	4.75012×10^{-9}	7.87352×10^{-9}	1.91489×10^{-11}
20	3.53003×10^{-7}	5.15163×10^{-10}	2.47478×10^{-10}	1.35261×10^{-13}
30	1.04634×10^{-7}	1.16269×10^{-10}	3.26261×10^{-11}	2.48111×10^{-14}
40	4.41486×10^{-8}	3.90880×10^{-11}	7.74545×10^{-12}	5.57148×10^{-15}
50	2.26055×10^{-8}	1.65760×10^{-11}	2.53852×10^{-12}	1.64257×10^{-15}
60	1.30823×10^{-8}	8.17576×10^{-12}	1.02028×10^{-12}	5.90763×10^{-16}
70	8.23860×10^{-9}	4.48322×10^{-12}	4.72080×10^{-13}	2.45804×10^{-16}
80	5.51930×10^{-9}	2.65882×10^{-12}	2.42145×10^{-13}	1.14196×10^{-16}
90	3.87641×10^{-9}	1.67487×10^{-12}	1.34377×10^{-13}	5.78184×10^{-17}
100	2.82592×10^{-9}	1.10674×10^{-12}	7.93503×10^{-14}	3.13602×10^{-17}
500	2.26080×10^{-11}	1.86136×10^{-15}	2.53949×10^{-17}	2.29887×10^{-21}
1000	2.82600×10^{-12}	1.17043×10^{-16}	7.93596×10^{-19}	3.64898×10^{-23}
5000	2.26080×10^{-14}	1.88174×10^{-19}	2.53952×10^{-22}	2.36453×10^{-27}
10000	2.82600×10^{-15}	1.17679×10^{-20}	7.93600×10^{-24}	3.70028×10^{-29}

While Fig. 2 and Table 2 show encouraging results, we will now compare our proposed approximations to known, and oft-cited, approximations. The approximation retained are of comparable computational complexity. We will compare:

- The truncated Stirling series (often ambiguously referred to as "Stirling's Approximation"),

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \tag{11}$$

- Burnside's approximation [11],

$$n! \sim \sqrt{2\pi} \left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \tag{12}$$

- Gosper's [14],

$$n! \sim \sqrt{\pi} \sqrt{2n + \frac{1}{3}} \left(\frac{n}{e}\right)^n \tag{13}$$

- Mohanty's and Rummens' [8],

$$n! \sim \sqrt{2\pi}(n+1)^{n+\frac{1}{2}} e^{\frac{1}{12(n+1)}} \tag{14}$$

- Mortici's [13],

$$n! \sim \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} \tag{15}$$

- and Nemes' [12],

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n^2 - \frac{1}{10}}\right)^n \tag{16}$$

Fig. 3 and Table 3 present the relative errors of our proposed approximation, eq. (7), against those from Burnside, Mortici, and Gosper. From Table 3, we see that Mortici's approximation, eq. (15), does marginally better than Stirling's truncated series, eq. (11). Burnside's approximation has a relative error approximately half of the truncated Stirling series, eq. (11). Compared to the other results, one can see that Burnside's approximation is of historical interest at best. Gosper's approximation fares much better, but still worse than Mohanty's and Rummens'. Mohanty's and Rummens' approximation does better than eq. (7), but just so.

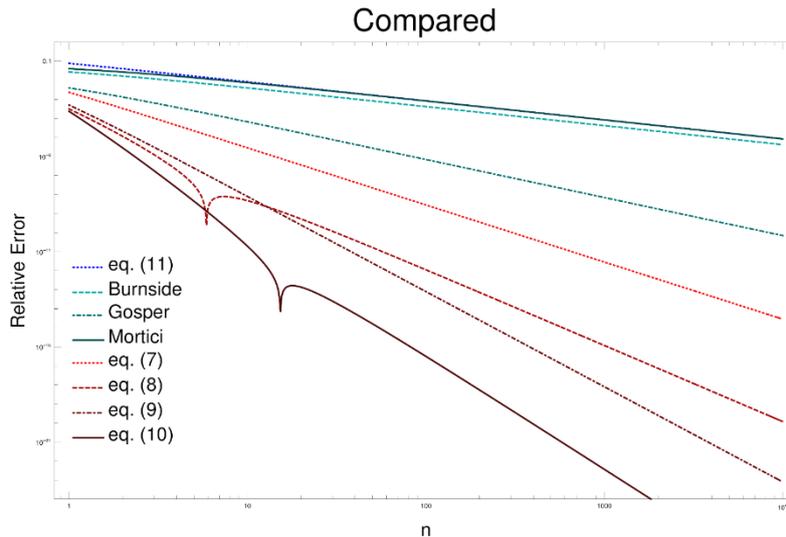


Fig. 3. Absolute relative error of proposed approximation compared to other approximations to $n!$

Table 3. All Approximation Compared, Absolute Relative Error, First Part

n	Stirling eq. (11)	Mortici eq. (15)	Burnside eq. (12)	Gosper eq. (13)	M & R eq. (14)	eq. (7)
1	0.07786	0.04050	0.02751	0.00398	0.00033	0.00232284
10	0.00830	0.00755	0.00397	0.00007	2.08209×10^{-6}	2.81813×10^{-6}
20	0.00416	0.00396	0.00203	0.00002	2.99750×10^{-7}	3.53003×10^{-7}
30	0.00277	0.00268	0.00137	7.58471×10^{-6}	9.32145×10^{-8}	1.04634×10^{-7}
40	0.00208	0.00203	0.00103	4.28485×10^{-6}	4.02969×10^{-8}	4.41486×10^{-8}
50	0.00167	0.00163	0.00083	2.74940×10^{-6}	2.09382×10^{-8}	2.26055×10^{-8}
60	0.00139	0.00137	0.00069	1.91259×10^{-6}	1.22370×10^{-8}	1.30823×10^{-8}
70	0.00119	0.00117	0.00059	1.40689×10^{-6}	7.76065×10^{-9}	8.23860×10^{-9}
80	0.00104	0.00103	0.00052	1.07814×10^{-6}	5.22665×10^{-9}	5.51930×10^{-9}
90	0.00093	0.00092	0.00046	8.52471×10^{-7}	3.68603×10^{-9}	3.87641×10^{-9}
100	0.00083	0.00082	0.00041	6.90896×10^{-7}	2.69601×10^{-9}	2.82592×10^{-9}
500	0.00017	0.00017	0.00008	2.77494×10^{-8}	2.20894×10^{-11}	2.26080×10^{-11}
1000	0.00008	0.00008	0.00004	6.94090×10^{-9}	2.76946×10^{-12}	2.82600×10^{-12}
5000	0.00002	0.00002	8.33253×10^{-6}	2.77749×10^{-10}	2.22089×10^{-14}	2.26080×10^{-14}
10000	8.33330×10^{-6}	8.33247×10^{-6}	4.16647×10^{-6}	6.94409×10^{-11}	2.77694×10^{-15}	2.82600×10^{-15}

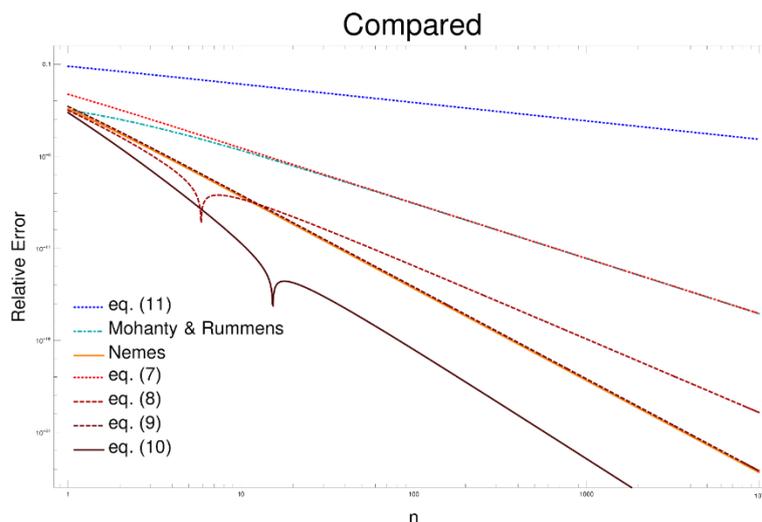


Fig. 4. Absolute relative error of proposed approximations compared to other approximations to $n!$

Fig. 4 presents the relative absolute errors of our proposed approximations against Nemes', eq. (16), and Mohanty's and Rummens', eq. (14). Note that Nemes' approximation and eq. (9) overlap on the figure, but Nemes' does better as detailed in Table 4. Table 4 presents the same results, but this time including

Stirling's series, eq. (1), with its first ten terms. Nemes' approximation does better than eq. (9), but eq. (10) does much better than either, while the ten term Stirling series shows errors tens of orders of magnitude smaller.

Table 4. All Approximations Compared, Absolute Relative Error, Second Part. * With the First then Terms

n	eq. (7)	eq. (8)	eq. (9)	eq. (10)	Nemes eq. (16)	Stirling eq. (1)*
1	0.00232284	0.000311662	0.000501953	0.000234244	0.000372486	0.000467142
10	2.81813×10^{-6}	4.75012×10^{-9}	7.87352×10^{-9}	1.91489×10^{-11}	6.47042×10^{-9}	1.13891×10^{-14}
20	3.53003×10^{-7}	5.15163×10^{-10}	2.47478×10^{-10}	1.35261×10^{-13}	2.03553×10^{-10}	2.26443×10^{-18}
30	1.04634×10^{-7}	1.16269×10^{-10}	3.26261×10^{-11}	2.48111×10^{-14}	2.68387×10^{-11}	1.40009×10^{-20}
40	4.41486×10^{-8}	3.90880×10^{-11}	7.74545×10^{-12}	5.57148×10^{-15}	6.37174×10^{-12}	2.30179×10^{-21}
50	2.26055×10^{-8}	1.65760×10^{-11}	2.53852×10^{-12}	1.64257×10^{-15}	2.08831×10^{-12}	3.44988×10^{-22}
60	1.30823×10^{-8}	8.17576×10^{-12}	1.02028×10^{-12}	5.90763×10^{-16}	8.39339×10^{-13}	6.62692×10^{-23}
70	8.23860×10^{-9}	4.48322×10^{-12}	4.72080×10^{-13}	2.45804×10^{-16}	3.88358×10^{-13}	1.58004×10^{-23}
80	5.51930×10^{-9}	2.65882×10^{-12}	2.42145×10^{-13}	1.14196×10^{-16}	1.99201×10^{-13}	4.47552×10^{-24}
90	3.87641×10^{-9}	1.67487×10^{-12}	1.34377×10^{-13}	5.78184×10^{-17}	1.10546×10^{-13}	1.45461×10^{-24}
100	2.82592×10^{-9}	1.10674×10^{-12}	7.93503×10^{-14}	3.13602×10^{-17}	6.52774×10^{-14}	5.28507×10^{-25}
500	2.26080×10^{-11}	1.86136×10^{-15}	2.53949×10^{-17}	2.29887×10^{-21}	2.08906×10^{-17}	6.98451×10^{-32}
1000	2.82600×10^{-12}	1.17043×10^{-16}	7.93596×10^{-19}	3.64898×10^{-23}	6.52832×10^{-19}	7.01285×10^{-35}
5000	2.26080×10^{-14}	1.88174×10^{-19}	2.53952×10^{-22}	2.36453×10^{-27}	2.08907×10^{-22}	7.33848×10^{-42}
10000	2.82600×10^{-15}	1.17679×10^{-20}	7.93600×10^{-24}	3.70028×10^{-29}	6.52833×10^{-24}	7.18569×10^{-45}

Another interesting measure of accuracy is the number of correct leading digits, shown in Table 5. Indeed, an approximation is "perfect" if it gives as many correct digits as the number representation can store, since any additional correct digits will be lost to quantization. If one uses single-precision IEEE 754 floating point numbers, one expects about 7 significant digits, because its mantissa is 24 bits long (of which only 23 are explicitly stored [15]), and we have $\log_{10} 2^{24} \approx 7.22$. With double-precision floating point, the mantissa is 53 bits long, and we expect $\log_{10} 2^{53} \approx 15.95$, or about 16, significant digits. At 7 significant digits, eqs. (8) to (10), as well as eq. (16), are basically equivalent in terms of accuracy. With 16 digits, only eq. (10) is cromulent.

Table 5. Number of Correct Digits for the Different Approximations. *With the First then Terms

n	n!	Stirling eq. (11)	Gosper eq. (13)	M & R eq. (14)	eq. (7)	eq. (8)	eq. (9)	eq. (10)	Nemes eq. (16)	Stirling eq. (1)*
1	1	0	0	0	0	0	0	0	0	0
10	7	1	4	6	5	6	6	6	6	6
20	19	2	4	7	7	9	9	13	9	18
30	33	2	5	7	7	10	10	13	10	20
40	48	2	5	6	6	10	11	14	11	20
50	65	2	6	8	8	10	11	15	11	21
60	82	2	5	7	7	10	11	14	11	21
70	101	3	6	8	8	9	13	15	13	23
80	119	2	5	8	8	11	12	14	12	23
90	139	3	6	9	8	12	13	17	13	24
100	158	2	5	8	8	10	12	15	12	23
500	1135	2	7	9	9	15	17	19	17	30
1000	2568	4	7	11	11	15	18	21	18	34
5000	16326	5	9	13	13	19	19	25	21	41
10000	35660	5	10	14	14	20	22	28	22	43

5. Implementation and Complexity

To be interesting, an approximation formula must, in addition of being accurate, be easily and efficiently computed. Some of the approximations we used for comparison raise the correction term to the n -th (or $(n + \frac{1}{2})$ -th) power, possibly requiring both raising to the n -th power and extracting a square root. If we have $n \in \mathbb{N}$, we can evaluate the exponentiation in $O(\log n)$ steps using the successive squaring method (an old idea, see [16], p. 76, and [17]), but this seems too restrictive, as we will want $n \in \mathbb{R}$. Rational

functions of degree d , on the other hand, can be evaluated in $O(d)$ products, whether n is a natural or a real number, and independently of its magnitude—although it will become costlier when n grows if exact arithmetic is used.

Eqs. (7) to (10) are best evaluated using reduced rational functions in order to minimize the number of multiplications and divisions required. The correction term of eq. (7) can be rewritten in many ways,

$$1 + \frac{1}{12n - \frac{1}{2}} = \frac{12n + \frac{1}{2}}{12n - \frac{1}{2}} = \frac{24n + 1}{24n - 1} = 1 + \frac{2}{24n - 1}.$$

Eq. (8) is probably best expressed as eq. (8b), resulting in three products and one division, while eq. (8a) needs two products and two divisions, a gain if we consider the divisions to be much more expensive than products. For eq. (9), we find that, expressed as eq. (9a), the correction term requires five products and two divisions. Expressed as eq. (9b), we notice that if we let $t = n^2$, we can rewrite the correction term as

$$\frac{17280t + 720n - 556}{207360tn + 571},$$

Which now requires the same number of products, but with only one division. If we apply the same type of simplification to eq. (10b), again with $t = n^2$, we find that

$$\frac{1451520n^3 + 60480n^2 - 46704n - 3997}{17418240n^4 - 163879} = \frac{(1451520n + 60480)t - 46704n - 3997}{17418240t^2 - 163879}$$

Can now be evaluated with six products and one division instead of ten products and two divisions in its original form, eq. (10a). As the degree d grows, the resulting rational functions may be subject to other optimization strategies [18], [19], but the evaluation will require, at worse, $O(d)$ multiplications, if only by using Horner's method for evaluating polynomials [20], [21].

6. Conclusion

Hodgman's correction to eq. (2), eq. (3), lead us to the observation that since these functions can be seen as two special cases of a continuous function, by the intermediate value theorem, there must be a correction for which the error is zero, that is, there must exist a function $\alpha(d, n)$ in eq. (4) that yields $n!$ exactly. From this observation, we proposed a novel family of approximations, each approximation being the optimal rational function $\alpha(d, n)$ of degree d in n . We then solved exactly for a few special cases, first for $\alpha(0, n)$, a constant, then for small degrees $1 \leq d \leq 3$. In doing so, we showed that $\alpha = -\frac{1}{2}$ is the optimal constant correction. We also showed the rational functions of small degrees fare quite well compared to oft-cited approximations for $n!$. We remarked that if the special case $\alpha(2, n)$ is quite comparable to Nemes' approximation, with the latter being a still better; $\alpha(3, n)$ is at least three orders of magnitude better than any of the approximation used for comparison. Lastly, we discussed computational complexity for the correction term, remarking that our solutions based on rational functions of (small) degree d will require at most $O(d)$ products and one division, while many of the other approximations will require $O(\log n)$ products, or require exponentiation by an arbitrary number, making our proposal also attractive computation-wise.

Conflict of Interest

The author declares no conflicts of interest regarding the contents of this article.

Author Contributions

The author declares being sole contributor.

References

- [1] James, S. (1730). *Methodus Differentialis, Sive Tractatus de Summatione et Interpolatione Serierum Infinitarum*.
- [2] Sloane, N. J. A. (1964). *Stirling's Formula: Numerators of Asymptotic Series for Gamma Function. Formerly M5400 N2347*. Retrieved from <http://oeis.org/A001163>
- [3] Sloane, N. J. A. (1964). *Stirling's Formula: Denominators of Asymptotic Series for Gamma Function. Formerly M4878 N2091*. Retrieved from <http://oeis.org/A001164>
- [4] Hodgman, C. D. (1955). *CRC Standard Mathematical Tables* (10th ed.). Chemical Rubber Publishing Company.
- [5] Lu, D. (2014). A new sharp approximation for the gamma function related to burnside's formula. *The Ramanujan Journal*, 35(1), 121–129.
- [6] Mortici, C. (2013). A continued fraction approximation of the gamma function. *J. of Mathematical Analysis and Applications*, 402(2), 405–410.
- [7] Yang, Z.-H., & Tian, J.-F. (2018). An accurate approximation formula for gamma function. *J. of Inequalities and Applications*, 56–63.
- [8] Mohanty, S., & H. A. Frans. (1984). Comment on: An improved analytical approximation to $n!$ *J. of Chemical Physics*, 80(1), 591.
- [9] Weissman, Y. (1983). An improved analytical approximation to $n!$ *American Journal of Physics*, 51(9), 778–779.
- [10] Feller, S. A., & Kasper, J. E. (1982). How good are the common approximations used in physics? *American Journal of Physics*, 50(8), 682–683.
- [11] Burnside, W. (1917). A rapidly convergent series for $\log N!$ *Messenger of Mathematics*, 46, 157–159.
- [12] Nemes, G. (2010, June). New asymptotic expansion for the gamma function. *Archiv. der Mathematik*, 95, 161–169.
- [13] Mortici, C. (2011). On Gosper's formula to the gamma function. *J. of Mathematical Inequalities*, 5(4), 611–614.
- [14] Gosper, R. W. (1978). Decision procedure for indefinite hypergeometric summation. *Procs. Nat. Acad. Science USA*, 75(1), 40–42.
- [15] Zuras, D., & Cowlshaw, M. (2008). IEEE 754-2008 – Standard for floating point arithmetic. *The Institute of Electrical and Electronics Engineers*.
- [16] Datta, B., & Singh, A. N. (1935). *History of Hindu Mathematics: A Source Book, Part I*.
- [17] Saidan, A. S. (1978). *The Arithmetic of Al-Uqlidisi: The Story of Hindu-Arabic Arithmetic as Told in the Kitāb al-Fuṣū fī al-Ḥisāb al-Hindī*. D. Reidel Publishing Co.
- [18] Belaga, E. G. (1961). On computing polynomials in one variable with initial preconditioning of the coefficients. *Problemi Kibernetiki*, 5, 7-15.
- [19] Munro, I., & Paterson, M. (1973, April). Optimal algorithms for parallel polynomial evaluation. *J. of Computer and System Sciences*, 7(2), 189–198.
- [20] Horner, W. G. (1819). A new method of solving numerical equations of all orders, by continuous approximations. *Philosophical Transactions of the Royal Society of London*, 109, 308–335.
- [21] Pan, V. Y. (1966). Methods of computing values of polynomials. *Russian Mathematical Surveys*, 21, 105–136.

Copyright © 2021 by the authors. This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited ([CC BY 4.0](https://creativecommons.org/licenses/by/4.0/)).



Steven Pigeon is an associate professor of mathematics and computer science in the Department of Mathematics, Computer Science, and Engineering at the Université du Québec à Rimouski. He received his PhD. from the Université de Montréal for his thesis on data compression in 2001. His post-doctoral studies were conducted at Western Ontario then McMaster Universities during 2002 and 2003. From 2003 to 2005, he developed laser-based 3D range finding equipment for industrial applications in the private sector. He returned to academia in 2006 where he joined an ÉTS research team on video adaptation and transcoding. In 2011-2012, he joined the Ubisoft industrial chair on artificial intelligence as principal systems architect. Since 2012, he teaches mathematics and computer science at the Université du Québec à Rimouski, where he also chairs the mathematics and computer science section. His current research interests are, in addition to data compression, machine-efficient implementations of algorithms and memory-efficient data structures.