

On the Finite Amplitude Oscillations in a Viscous Two-Dimensional Fluid

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Abstract: The characteristics of wave motion in a two-dimensional viscous fluid medium which is described by a bounded solution is obtained for the case in which the depth is essentially high. The distortion rising from the mixing of lamina irregular and turbulent mode of flow under a common transverse gradient is represented by the complex viscous term and relates to the Reynolds number Re , so that when the coefficient of viscosity ν is larger, the Reynolds number is Zero. The governing equation include terms relating to the non-linear coupling in the plane for which viscosity is the dominant factor influencing the eddy driven current in the medium. The solution suggests a system of wave motion generated by gradual transition from the linear law of resistance for lamina flow to that of quadratic law for turbulent flow.

Keywords: Viscosity, amplitude, finite, lamina, turbulent.

1. Introduction

The characteristics of wave motion in a two-dimensional viscous fluid flow medium is discuss in this paper, for which the depth is essentially high. Shallow water processes which accounts for the linear long waves will be neglected in favor of the non-linear coupling in the plane for which viscosity is a dominant factor influencing the Eddy driven current in the medium Holland [1].

Makarious [2] noted that evolution of the long progressive waves and the surface of shallow water has been an intense successful research activity for a long time. Phan *et al* [3] noted the existence of finite amplitude vibrations of the cantilever beam of rectangular cross section immersed into a viscous fluid, while a two- dimensional flow caused by the plate oscillation and their hydrodynamic influence on the plate is well understood and reported by Muaiev *et al.* [4]. A thorough computation on the evolution of the periodic surface wave was done by Longnet-Higgins [5]. From their result the waves progress towards a coast line which often steepen and over turn. This work is in agreement with Galvin [6], who observed features of the plunging breakers.

Okeke [7] in his work presented a model for weakly non-linear waves on the surface of the shallow water where he analyzed the development and propagation of singularities along the wave crest and eventual breaking using the Bousinesq equation for periodic wave trains in shallow water, without viscosity.

In this research, we considered the dominant non-linear oscillations in the deep fluid medium with operating influence of viscosity. Our concern is the development of non-periodic bounded solutions that yield finite amplitude oscillations as the leading asymptotic flow approaches shallow water and while at the

same time the wave action undergoes gradual attention of the trailing end at a great depth.

2. Basic Equations

The mutually orthogonal axes of the Cartesian coordinate system are the x and z axes forming the plane for which the transverse motion of the stream takes place. The x axes is taken horizontal and normal to the shoreline while the z -axes is vertically upward.

The instructions rise of fluid representing the wave profile given by

$$z = \eta(x, t) \quad (1)$$

Given these specifications in a medium of infinitesimal disturbances, the evolution of dominant non-linear wave in deep water is familiar Stokes kinematic equations for instantaneous wave motion confined to the dimensions of x and w and given by

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} = \frac{-1}{\rho} \left(\frac{\partial p}{\partial x} + \frac{\partial p}{\partial z} \right) + rv^2(u + w) \quad (2)$$

where v is the kinematics viscosity of water, u , w are the components of particle velocity in the x and z directions with $t > 0$ as the time, ρ is the density of the medium, and P is Pressure of the fluid medium.

For mass flow, the equation of continuity is given by

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (3)$$

2.1. Boundary Condition

The following boundary conditions are applied to our model:

$$\frac{\partial \eta}{\partial t} = w \quad \text{at } z = \begin{cases} 0 & t=0 \\ \eta & \eta=0 \end{cases} \quad (4)$$

Which are the kinematic boundary conditions.

The dynamic boundary conditions due to the balance of pressure P with friction expressed as

$$P - 2\mu \frac{\partial w}{\partial z} = \lambda j \quad (5)$$

where λ is a measure of curvature of the fluid surface due to non-linear coupling in the plane and given by.

$$\lambda = \frac{\partial^2 \eta}{\partial x^2} \quad (5a)$$

j is the surface Tension and μ is the dynamic coefficient of frictions, therefore equation (5) becomes

$$P - 2\mu \frac{\partial w}{\partial z} = j \frac{\partial^2 \eta}{\partial x^2} \quad (6)$$

$z = \eta$ and given by

$$\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \quad (7)$$

Equations (6) and (7) are continuous at $z = \eta$.

Let $\psi(x, z, t)$ be the conservative stream function describing the transverse motion of the fluid such that

$$u = -\frac{\partial \psi}{\partial z} \quad (8)$$

And

$$w = \frac{\partial \psi}{\partial x} \quad (9)$$

To obtain the equation of transverse motion, we relate the horizontal motion to the perpendicular motion by considering the one-dimensional Navier Stokes equations without body force given by

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \nabla^2 u = 0 \quad (10)$$

Differentiate (10) with respect to z and equation (8) we have

$$\frac{\partial}{\partial t} \left(-\frac{\partial^2 \psi}{\partial z^2} \right) = -\frac{1}{\rho} \frac{\partial^2 P}{\partial x \partial z} + \nu \nabla^2 \left(-\frac{\partial^2 \psi}{\partial z^2} \right) \quad (11)$$

Similarly, the transverse motion of equation (2) is given by

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \nabla^2 w = 0 \quad (12)$$

Differentiating (12) with respect to x and substitute in equation (9)

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \psi}{\partial x^2} \right) = -\frac{1}{\rho} \frac{\partial^2 P}{\partial x \partial x} + \nu \nabla^2 \left(\frac{\partial^2 \psi}{\partial x^2} \right) = 0 \quad (13)$$

Subtracting equation (11) from equation (13) we will obtain

$$\frac{\partial}{\partial t} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \psi = \nu \nabla^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \psi \Rightarrow \frac{\partial}{\partial t} \nabla^2 \psi = \nu \nabla^4 \psi \quad (14)$$

In equation (14) if $\psi \rightarrow 0$ as $z \rightarrow 0$; the wave profile will gradually diminish at great depth

The kinematic boundary condition of equation (4) will yield

$$\frac{\partial \eta}{\partial t} = \left. \frac{\partial \psi}{\partial x} \right|_{z=0} = w|_{z=0} \quad (15)$$

For the dynamic boundary condition of equation (5) we have

$$P - 2\mu \frac{\partial^2 \psi}{\partial x \partial z} = j \frac{\partial^2 \eta}{\partial x^2} \Big|_{z=\eta} \quad (16)$$

Therefore cross-lateral displacement will yield

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (17)$$

At $z = 0$ or $z = \eta$

3. Solution of the Problem

We shall look at the oscillatory solution built up by the complex exponential growth of horizontal wave motion with stream function given by;

$$\psi(x, y, t) = f(z) e^{i(kx + \delta t)} \quad (18)$$

Taking equations (14) to (17), we will find a mixed B.V.P expressed as 4th order PDE governing the plane wave motion expressed as;

$$\frac{\partial}{\partial t} \nabla^2 \psi = \nu \nabla^4 \psi \quad (19)$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \psi}{\partial x} = w \text{ at } z = 0 \quad (19a)$$

And

$$p - 2\mu \frac{\partial^2 \psi}{\partial x \partial z} = j \frac{\partial^2 \eta}{\partial x^2} = at \text{ at } z = \eta$$

And

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} = 0 \text{ at } z = 0 \text{ or } z = \eta$$

Our interest here is only in the oscillatory solution built up by the complex exponential growth of horizontal wave motion with stream function given by equation (19a).

To simplify the treatment, we rewrite Equation (19) as

$$\begin{aligned}\left(\frac{\partial}{\partial t} \nabla^2 - \nu \nabla^4\right) \psi &= \left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla^2 \psi = 0 \\ \Rightarrow \left(\frac{1}{\nu^2} \frac{\partial}{\partial t} - \nabla^2\right) \nabla^2 \psi &= 0\end{aligned}\quad (20)$$

The stream function equation (19) introduces into equation (20), we will obtain the equation of motion describing the vertical profile of the flow

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right) \left(\frac{\partial^2}{\partial z^2} - k^2 - i \frac{\partial}{\partial t}\right) f(z) = 0 \quad (21)$$

Simplifying the non-linear process, we introduce a function $g(z)$ representation of the vertical variation of the fluid as the wave motion which is equivalent to equation (21) given by

$$\left(\frac{\partial^2}{\partial z^2} - k^2 - i \frac{\partial}{\partial t}\right) f(z) = g(z) \quad (22)$$

where $g(z) = 0$ as $z \rightarrow -\infty$

Substituting equation (22) into equation (21) we will obtain

$$\left[\frac{\partial^2}{\partial z^2} - k^2\right] g(z) = 0 \quad (23)$$

Which yields the

$$g(z) = Ae^{kz} + Be^{-kz} \quad (24)$$

Ae^{kz} is the dominant part of the solution which is bounded and representing the asymptotic flow upstream with a finite amplitude.

Therefore,

$$g(z) = Ae^{kz} \quad (25)$$

To simplify equation (22) we introduce the factor

$$k^2 + \frac{i\delta}{\nu} = m^2 \quad (26)$$

$$\left[\frac{\partial^2}{\partial z^2} - m^2\right] f(z) = Ae^{kz} \quad (27)$$

$$f(z) = \beta_0 e^{mz} + \frac{Ae^{kz}}{k^2 - m^2} \quad (28)$$

$$\text{let } \alpha_0 = \frac{A}{k^2 - m^2} \quad (29)$$

Equation (27) reduces to

$$f(z) = \alpha_0 e^{kz} + \beta_0 e^{mz} \quad (30)$$

Then, we obtain a stream function

$$\psi = [\alpha_0 e^{kz} + \beta_0 e^{mz}] e^{i(kz - \delta t)} \quad (31)$$

To determine the non-shallow water rise of elevation, we use the following boundary condition

$$\frac{\partial \mu}{\partial t} = \frac{\partial \psi}{\partial t} \Big|_{z=0} \quad (31a)$$

Now differentiating equation (31) with respect to t , results to

$$\frac{\partial \mu}{\partial t} = i\delta(\alpha_0 + \beta_0) e^{i(kz - \delta t)} \quad (32)$$

Integrating equation (32) with respect to t , we will have

$$\mu = (\alpha_0 + \beta_0) e^{-i\delta t} \quad (33)$$

From $\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} = 0$ at $z = 0$, if μ is substituted in equation (31) we will have

$$k^2(\alpha_0 + \beta_0) + (k^2\alpha_0 + m^2\beta_0) = 0 \quad (34)$$

Eliminating m from equation (34) using equation (26) we have

$$\begin{aligned} k^2(\alpha_0 + \beta_0) + (k^2\alpha_0 + k^2\beta_0 + \frac{i\delta}{v}\beta_0) &= 0 \\ \Rightarrow 2k^2(\alpha_0 + \beta_0) &= -\frac{i\delta}{v}\beta_0 \\ \Rightarrow \frac{(\alpha_0 + \beta_0)}{\beta_0} &= -\frac{i\delta}{2k^2v} \\ \Rightarrow \frac{\alpha_0}{\beta_0} + 1 &= -\frac{i\delta}{2k^2v} \\ \Rightarrow \frac{\alpha_0}{\beta_0} &= -1 - \frac{i\delta}{2k^2v} \end{aligned} \quad (35)$$

$\frac{i\delta}{k^2v}$ has the representation of Reynolds number R and express as a ratio of the dynamical measure of stress over fluid inertia that essentially generates the transition from linear law of resistance of lamina flow Stuart [8] to that of the quadratic law for turbulence flow due to the action of the eddies in continuous horizontal and transverse motion.

Hence the distortion arising from the mixing of lamina, irregular and turbulent modes of flow with a common transverse gradient justifies the complex nature of viscosity and thus introduces an imaginary part in the propagation frequency, δ , which can be express as

$$\delta = \delta_o e^{i\delta} \quad (36)$$

$$\delta = \delta_o [\cos \delta + i \sin \delta] \quad (37)$$

Thus, the Reynolds number becomes

$$R = -\frac{\delta_o \sin \delta}{vk^2} \quad (38)$$

where δ_o is the characteristic frequency of flow motion

$$\delta = -\delta_o \sin \delta \text{ is the im } (\delta_o)$$

We now determine the exact expression of the component of u, v, η and p

Since we know the exact expression of the stream function

$$\psi(x, z, t) = [\alpha_o e^{kz} + B_o e^{mz}] e^{(kx - \delta t)} \quad (39)$$

With $\alpha_o = \frac{A}{k^2 - m^2}$

First taking equation (33), we see that the non-periodic rise of wave height in deep water is given by:

$$\eta(x, t) = -\frac{k}{\delta} (\alpha_o + \beta_o) e^{(kx - \delta t)}$$

The quantity

$$\eta_o = -\frac{k}{\delta} (\alpha_o + \beta_o) \quad (40)$$

is the characteristic amplitude of the wave elevation and is finite since:

$$|\eta_o| = \left| \frac{k}{\delta} (\alpha_o + \beta_o) \right| < \infty \text{ as}$$

$$\left| \frac{k}{\delta} \right| < \infty \text{ and } |\alpha_o + \beta_o| = |\alpha_o| + |\beta_o| < \infty$$

Hence

$$\eta(x, t) = \eta_0 e^{i(kx - \delta t)} \quad (41)$$

$$\text{at } z = \eta_0, \quad x = 0 \quad \& \quad t = 0$$

To determine u , w and p . we note that

$$\psi(x, z, t) = [\alpha_o e^{kz} + \beta_o e^{mz}] e^{i(kx - \delta t)} \quad (42)$$

If

$$u = -\frac{\partial \psi}{\partial z} \Rightarrow u = -\frac{\partial}{\partial z} [\alpha_o e^{kz} + \beta_o e^{mz}] e^{i(kx - \delta t)}$$

Hence

$$u = -[\alpha_o k e^{kz} + \beta_o m e^{mz}] e^{i(kx - \delta t)} \quad (43)$$

Now letting $\alpha = \alpha_o k$ and $\beta = \beta_o m$

$$\psi(x, z, t) = -[\alpha e^{kz} + \beta e^{mz}] e^{i(kx - \delta t)} \quad (44)$$

Also from equation (9), we have

$$\begin{aligned} \omega = \frac{\partial \psi}{\partial x} &\Rightarrow \omega = \frac{\partial}{\partial x} [\alpha_o e^{kz} + \beta_o e^{mz}] e^{i(kx - \delta t)} \\ &\Rightarrow \omega(x, z, t) = ik[\alpha_o e^{kz} + \beta_o e^{mz}] e^{i(kx - \delta t)} \end{aligned} \quad (45)$$

Now $u(x, z, t)$ in equation (44) represents the velocity component driving the horizontal flow, which $\omega(x, z, t)$ is the component due to transverse flow. Hence the current system generating the waves in the stream is giving by velocity potential, $\tilde{v}$ and expressed as

$$v = \begin{pmatrix} u \\ w \end{pmatrix} \quad (45a)$$

$$\begin{aligned} \tilde{v} &= \left[\begin{pmatrix} -\alpha_o k \\ ik\alpha_o \end{pmatrix} e^{kz} + \begin{pmatrix} -\beta_o m \\ ik\beta_o \end{pmatrix} e^{mz} \right] e^{i(kx - \delta t)} \\ &= [\alpha_o k \begin{pmatrix} -1 \\ i \end{pmatrix} e^{kz} + \beta_o \begin{pmatrix} -m \\ ik \end{pmatrix} e^{mz}] e^{i(kx - \delta t)} \end{aligned} \quad (46)$$

Letting $\alpha_o k = \tau_0$ then equation (46) yields the current system in the stream as

$$\tilde{v} = \left[\begin{pmatrix} -1 \\ i \end{pmatrix} \tau_0 e^{kz} + \beta_o \begin{pmatrix} -m \\ ik \end{pmatrix} e^{mz} \right] e^{i(kx - \delta t)} \quad (47)$$

This shows that the oscillations are due to Turbulence resulting from the non-linear coupling in the plane (x, t) which the transverse motion interplay with horizontal frictional flow. For finite amplitude oscillations, the boundary data is defined for the values at $x = 0, t = 0, z = \eta_0$

Hence, $\tilde{v}$ is the current system defining the free surface and given by

$$\tilde{v}_o = \begin{pmatrix} u_o \\ w_o \end{pmatrix} = \left[\begin{pmatrix} -1 \\ i \end{pmatrix} \tau_0 e^{kz} + \beta_o \begin{pmatrix} -m \\ ik \end{pmatrix} e^{mz} \right] e^{i(kx - \delta t)} \quad (48)$$

To find the pressure distribution $p(x, z, t)$, we use equation (16) as

$$p - 2\mu \frac{\partial^2 \psi}{\partial x \partial z} = j \frac{\partial^2 \eta}{\partial x^2} \Rightarrow p = 2\mu \frac{\partial^2 \psi}{\partial x \partial z} + j \frac{\partial^2 \eta}{\partial x^2} \quad (49)$$

$$\text{If } \eta(x, t) = \eta_0 e^{i(kx - \delta t)}$$

$$\Rightarrow \frac{\partial \eta}{\partial x} = ik\eta_0 e^{i(kx - \delta t)} \quad \text{and} \quad \frac{\partial^2 \eta}{\partial x^2} = -k^2 \eta_0 e^{i(kx - \delta t)} \quad (50)$$

where $x = 0, t = 0, z = \eta_0$

$$\frac{\partial^2 \eta}{\partial x^2} = -k^2 \eta_0 \quad (51)$$

$$w(x, z, t) = [\alpha_o e^{kz} + \beta_o e^{mz}] e^{i(kx - \delta t)}$$

Therefore

$$\frac{\partial \psi}{\partial z} = -[k\alpha_0 e^{kz} + m\beta_0 e^{mz}]e^{i(kx-\delta t)}$$

Hence,

$$\frac{\partial^2 \psi}{\partial x \partial z} = \frac{\partial}{\partial x} \frac{\partial \psi}{\partial z} = ik[k\alpha_0 e^{kz} + m\beta_0 e^{mz}] \quad (52)$$

Substitute equation (50) and (52) into (49), we will obtain

$$\begin{aligned} P &= 2\mu \frac{\partial^2 \psi}{\partial x \partial z} + j \frac{\partial^2 \eta}{\partial x^2} \\ &= 2\mu(ik\alpha_0 e^{kz} + ikm\beta_0 e^{mz})e^{i(kx-\delta t)} + j(-k^2 \eta_0 e^{i(kx-\delta t)}) \\ P(x, z, t) &= [2ik\mu(k\alpha_0 e^{kz} + m\beta_0 e^{mz}) - k^2 j\eta_0] e^{i(kx-\delta t)} \end{aligned} \quad (53)$$

Which is the pressure distribution along the flow, and at the boundary $x = 0$, $t = 0$, $z = \eta_0$.

We obtain the bounded pressure distribution about the elevation $z = \eta_0$ as

$$P(0, \eta_0, 0) = 2ik^2 \mu \alpha_0 e^{kz} - k^2 j\eta_0 \quad (54)$$

4. Discussion of Result

The bounded solution of equations (48) to (54) of equation (2) and equation (3) represent a system of wave motion for which the flow that is initially lamina increasing sensitive to small disturbances due to an irregular eddying motion in the horizontal layer of the fluid against the transverse motion that generally cause interlacing and constantly varying streams, crossing and re-crossing the flow field. The eddying motion is caused by viscous driven current, and the quantity $\frac{\delta}{vk^2}$ in (35) has the representation of the Reynolds number R , and in this case expresses as a ratio, the dynamical measure of stress over fluid inertia that essentially generate the transition from the linear law of resistance for lamina flow to that of quadratic law for turbulence flow due to the action of the eddies in continuous horizontal and transverse motion.

Hence the distortion arising from the mixing of lamina, irregular and turbulent modes of flow with the common transverse gradient justifies the complex nature of the viscosity and thus introduces an imaginary part in the propagation frequency, δ , which can be expressed as

$$\delta = \delta_o e^{i\delta}$$

Which is equivalent to

$$\delta = \delta_o [\cos\delta + i \sin\delta]$$

Thus the Reynolds number for bounded finite amplitude oscillations now becomes

$$R = -\frac{\delta_o \sin\delta}{vk^2} = \frac{\delta}{vk^2}$$

where δ is the characteristics frequency, and

$$\delta_1 = \delta_o \sin\delta = \text{Im}(\delta)$$

Thus if the coefficient of viscosity is large the Reynolds number will be zero. In inviscid fluid, $\delta = 0$ and $R = 0$, hence the operating waves now oscillates in shallow water.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

The first author conceived the idea, produced a draft copy and discussed with the second author who also made relevant contribution. Both authors approve the final result of the manuscript.

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