A New Okun Coefficient Based on AEPD

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Abstract: This paper proposes a new Okun Coefficient with error terms distributed as Asymmetric Exponential Power Distribution proposed by Zhu and Zinde-Walsh. Method of Maximum Likelihood Estimation is used to estimate this model. Markov Chain Monte Carlo method (MCMC) is used to generate random variables from AEPD for simulation. In empirical analysis, U.S. and Japan are studied from 1999 Q1 to 2019 Q4. Empirical results show Okun theory is partly supported by US and Japan if the error assumption is changed from Normal to AEPD. Likelihood Ratio (LR) test proves the existence of fat tailness and skewness in residuals and based on Kolmogorov-Smirnov (KS) test, it is accepted that residuals follow AEPD under 5% significance level. Finally, Okun-AEPD has better in-sample fit than Okun-Normal by Akaike Information Criterion (AIC).

Key words: Asymmetric Exponential Power Distribution (AEPD), Markov Chain Monte Carlo (MCMC), Maximum Likelihood Estimation (MLE), Okun's law.

1. Introduction

Okun's Law provides a general notion construing that the deviation of output from its natural level is inversely related to the deviation of unemployment from its natural rate, which becomes one of the important indicators in macroeconomics.

In empirical economy, the Okun coefficient calculated from Okun Law is widely used to study business cycle. Many studies exam the robustness of Okun coefficient with different datasets. For example, the relationship between unemployment and GNP or GDP varies by country. In the United States, the Okun coefficient estimates that when unemployment falls by 1%, GNP will rise by 3% and GDP will rise by 2%. Also, Industrialized nations with labor markets that are less flexible than those of the United States such as France and Germany, tend to have higher Okun coefficients. In those countries, the same percentage change in GNP has a smaller effect on the unemployment rate than it does in the United States.

Although economists broadly support Okun's law [1]-[3], but it's considered to be inaccurate and researchers try to make some extensions on Okun Law. For example, Zou proposed a dual structural decomposition of Okun's law and proved Okun's Law is still valid in China [4]. Also, Chen added oil price as external shock to improve Okun's Law and the result show with external shock, the nonlinear Okun Law seems more efficient in China [5].

Different from previous research, in this paper, we suggest a new Okun coefficient by adding Asymmetric Exponential Power Distribution. Asymmetric Exponential Power Distribution (AEPD) is first suggested in Zhu and Zinde-Walsh [6] and it is one extension of Normal distribution with both skewness and tail parameters added to capture fat tails and asymmetric effects in financial data. We use AEPD as residual

distribution instead of Normal distribution and figure out whether AEPD can provide a more accurate Okun coefficient.

<div align="center">Table 1. Extensions of Normal Distribution</div>

Authors	Distributions and Applications
Subbotin, 1923 [7]	EPD
Aitchison J. and Brown J.A.C., 1957[8]	Lognormal distribution
Azzalini, 1986 [9]	SEPD
Fernandez et al., 1995 [10]	Modified SEPD
Ayebo and Kozubowski , 2004 [11]	SEPD in finance
DiCiccio and Kozubowski, 2004 [12]	Properties of MLE of the SEPD
Zhu and Zinde-Walsh, 2009 [6]	AEPD

Notes: EPD= Exponential Power Distribution; SEPD= Skewed Exponential Power Distribution; AEPD= Asymmetric Exponential Power Distribution

In this paper, Method of Maximum Likelihood Estimation (MLE) is used to estimate. Markov Chain Monte Carlo method (MCMC) is used to generate random variables from AEPD for simulation. In empirical analysis, quarterly change rate of GDP and Cyclical unemployment rate are studied from 1999 Q1 to 2019 Q4. Akaike Information Criterion (AIC) is applied to measure fitness. Kolmogorov-Smirnov (KS) test is used for residual check. Hypothesis testing on parameter restrictions are based on the Likelihood Ratio (LR) test.

The organization of this paper is as follows. Section 2 explains Okun coefficient based on AEPD and methodology. Simulation analysis is in Section 3. Empirical analysis is conducted in Section 4. Section 5 is the conclusion.

2. Model and Methodology

2.1. The Okun-AEPD Model

The new Okun Law is proposed by introducing the Asymmetric Exponential Power Distribution (AEPD) as residual distribution. For convenience, this new model is denoted as Okun-AEPD and has following forms

$$Y_t - Y_{t-1} = \beta_1 + \beta_2(U_t - U_t^n) + u_t$$

$$u_t \sim AEPD(\alpha, p_1, p_2, \mu, \sigma) \tag{1}$$

where Y_t (t = 1,2, ..., T) is the GDP. T is the sample size. U_t is the unemployment rate. U_t^n is the long-term natural unemployment rate. β_1 and β_2 are the coefficient parameters. The probability density function (PDF) of u_t is:

$$f_{AEP}(x|\beta) = \begin{cases} \left(\frac{\alpha}{\alpha^*}\right)\frac{1}{\sigma}K_{EP}(p_1)\,exp\left(-\frac{1}{p_1}\left|\frac{x-\mu}{2\alpha^*\sigma}\right|^{p_1}\right) & \text{if } x \leq \mu \\ \left(\frac{1-\alpha}{1-\alpha^*}\right)\frac{1}{\sigma}K_{EP}(p_2)\,exp\left(-\frac{1}{p_2}\left|\frac{x-\mu}{2(1-\alpha^*)\sigma}\right|^{p_2}\right) & \text{if } x > \mu \end{cases} \tag{2}$$

$$K_{EP}(p) \equiv 1/[2p^{1/p}\Gamma(1+1/p)]$$

$$\alpha^* = \alpha K_{EP}(p_1)/[\alpha K_{EP}(p_1) + (1-\alpha)K_{EP}(p_2)]$$

$\mu \in R$ is the location parameter. σ > 0 is the scale parameter. $\alpha \in (0,1)$ is the skewness parameter. p_1 > 0 and p_2 > 0 are the left and the right tail parameters, respectively. $\Gamma(.)$ is the gamma function. AEPD includes a class of distributions. For example, if $p_1 = p_2$, the AEPD can be reduced to SEPD. If $p_1 = p_2$ and α = 0.5, the AEPD can be reduced to symmetric EPD. If α = 0.5 and $p_1 = p_2$ = 1, the AEPD will be the Laplace distribution. If α = 0.5 and $p_1 = p_2$= 2, the AEPD will be reduced to the Normal distribution. The skewness, fat tails, asymmetry and jumps in the short rate may be captured by the AEPD.

2.2. Okun-Normal Model

The Okun Law suggested by Okun (denoted as Okun-Normal) is

$$Y_t - Y_{t-1} = \beta_1 + \beta_2(U_t - U_t^n) + u_t$$

$$u_t \sim Normal(\mu, \sigma) \tag{3}$$

If $\alpha = 0.5$ and $p_1 = p_2 = 2$, the Okun-AEPD model in previous section will be the Okun-Normal. That means, Okun-Normal is nested in Okun-AEPD.

2.3. Method of Maximum Likelihood Estimation

Method of Maximum Likelihood Estimation (MLE) is used to estimate the parameters. The maximum likelihood function of the Okun-AEPD model is

$$L(Y_2 - Y_1, \dots, Y_t - Y_{t-1}, U_1 - U_1^n, \dots, U_t - U_t^n; \theta) = \prod_{t=1}^{T} f(Y_t - Y_{t-1})$$

$$= \prod_{i=1}^{n} \begin{cases} \left(\frac{\alpha}{\alpha^*}\right)\frac{1}{\sigma}K_{EP}(p_1) \, exp\left(-\frac{1}{p_1}\left|\frac{u_t-\mu}{2\alpha^*\sigma}\right|^{p_1}\right) & u_t \le \mu \\ \left(\frac{1-\alpha}{1-\alpha^*}\right)\frac{1}{\sigma}K_{EP}(p_2) \, exp\left(-\frac{1}{p_2}\left|\frac{u_t-\mu}{2(1-\alpha^*)\sigma}\right|^{p_2}\right) & u_t > \mu \end{cases} \tag{4}$$

$$u_t = Y_t - Y_{t-1} - \beta_1 - \beta_2(U_t - U_t^n)$$

where $\theta = (\beta_1, \beta_2, \alpha, p_1, p_2, \mu, \sigma)$ are the parameters to be estimated. We used the MatLab to estimate these parameters.

3. Simulation

3.1. Data Generation

In this section, simulation analysis for the Okun-AEPD model is done. The random variables from AEPD are drawn through Hastings (1970) algorithm. The reason we choose Hastings algorithm is that the marginal distribution of random variable u_t is known[1]. The data generation process is as follows.

1. Select a set of true parameter values for $\theta = (\beta_1, \beta_2, \alpha, p_1, p_2, \mu, \sigma)$ Assume values $u^{(i)}$, are known (If $i = 0$, set $u^{(0)} = 0$).

2. Generate a random variable from Standard Normal distribution $N(0,1)$ and denote it as $u^{(tem)}$[2].

3. Draw v from Uniform (0, 1) distribution. Then, $(i + 1)th$ draw of μ will be

$$u^{(i+1)} = \begin{cases} u^{(tem)}, if \ v \le min\left\{1, \frac{f(u^{(tem)})}{f(u^{(i)})}\right\}, \\ u^{(i)}, \ if \ v > min\left\{1, \frac{f\left(u_1^{(tem)}\right)}{f\left(u_1^{(i)}\right)}\right\}. \end{cases} \tag{5}$$

4. Set $i=i+1$ and repeat step 3-4 until $i=3001$.

5. Suppose u_t follow Uniform distribution (0,1) and generate Y_t by following formula

$$Y_t = \beta_1 + \beta_2 * X_t + u_t \tag{6}$$

3.2. Simulation

The simulation results are listed in Table 2. We find out the estimates are close to the true values. To exam

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[1]We use the marginal PDF of random variable u_t (see equation (4)) as the objective PDF $\pi(u_t)$.
[2]Tem means temporary.

the robustness, we also choose different set of true values. Hence, the MATLAB program is fine and can be used to analyze empirical data.

Table 2. Simulation Results

P	1		2		3		4		5	
	T	E	T	E	T	E	T	E	T	E
α	0.50	0.57	0.40	0.38	0.70	0.70	0.60	0.69	0.50	0.57
p_1	2.00	1.98	1.50	1.48	2.00	2.03	2.00	2.29	2.00	2.44
p_2	2.00	1.62	2.00	2.15	2.00	1.83	2.00	1.65	3.00	2.77
β_1	0.50	0.54	0.50	0.57	0.30	0.37	0.40	0.38	0.60	0.52
β_2	2.00	1.98	2.00	2.34	3.00	3.56	1.60	1.43	1.50	1.54
μ	1.00	0.80	1.50	1.35	1.70	1.93	2.00	1.62	2.00	1.83
σ	0.00	0.10	0.40	0.37	0.30	0.24	0.20	0.20	0.30	0.28

Notes: P=Parameter; T=True value; E=Estimate.

4. Empirical Analysis

4.1. Data

We choose and analyze the Okun Law for U.S. and Japan. Data is downloaded from Federal Reserve Bank database. The sample period is from 1999 Q1 to 2019 Q4.

The descriptive statistics of sample data are listed in Table 3. For each observation, the skewness of each GDP growth rate is negative, which means both GDP growth rate are skewed to the left. And the kurtosis of both GDP and cyclical unemployment rate is more than 3, which means both have fatter tails. The *P*-value of Jarque-Bera test for each data is zero, which means all data are not distributed as normal under 5% significance level. Hence, we conclude that both GDP growth rate and cyclical unemployment do not follow normal distribution.

Table 3. Descriptive Statistics

		Mean	St. Dev	Skewness	Kurtosis	JB	P
US	Unemployed	-0.0293	0.3548	0.1748	5.6507	45.2744	0
	GDP	0.0068	0.0065	-0.8162	5.7757	65.6718	0
Japan	Unemployed	-0.0188	0.1143	-7.6995	67.0670	215.1701	0
	GDP	0.0022	0.0099	-1.6913	10.1257	67.0670	0

4.2. Estimation Results

The estimates for the Okun-AEPD are listed in Panel A of Table 4. We find out the estimates for the skewness parameter α are not equal to .5, which captures the skewness in the data. All estimates for the tail parameters (p_1 and p_2) are smaller than 2, which documents the fat tailness. Also, both countries have smaller estimates for the left tail parameter p_2, which means both have fatter left tails than right tails. That is, Okun-AEPD can document the asymmetric effects.

The estimates for the Okun-Normal are listed in Panel B of Table 4. Compared with those in Okun-AEPD, the estimates for the location parameter μ are much further to 0. The estimates of the scale parameter σ are no smaller than those in Okun-AEPD. It is very interesting to find out the estimates of constant term β_1 are much closer to zero while the slope estimates of β_2 are lower if Normal error assumed.

One obvious shortcoming of the estimates of Okun-Normal is they do not show any hints about data skewness, fat tailness and asymmetric effects.

Table 4. Estimates for Parameters

	β_1	β_2	α	p_1	p_2	μ	σ
Panel A: Okun-AEPD							
U.S.	0.0092	-0.013	0.5207	1.0286	1.1841	-0.0018	0.0048
Japan	0.0032	-0.019	0.4556	0.6872	0.9979	0.0003	0.0058
Panel B: Okun-Normal							
U.S.	0.0031	-0.044	---	---	---	0.0036	0.0063
Japan	0.0010	-0.095	---	---	---	0.0010	0.0097

4.3. Model Diagnostics

4.3.1. Significance tests of coefficients

To test the significance of regressors in Okun-AEPD, we use Likelihood Ratio test (LR), which is calculated using equation (7). The null hypothesis is H0: β_i = 0 in Okun-AEPD, *i* = 1 or 2.

$$\text{LR} = -2\ln (likelihood\ for\ null) + 2\ln (likelihood\ for\ alternative). \tag{7}$$

The values of LR are listed in Panel A of Table 5. For example, the LR value of β_1 for US is 42.19, which is greater than the critical value 3.84. That means, under 5% significance level, we reject the null and conclude that the coefficient β_1 is statistically significant. Also, the LR value of β_2 for US is 70.31, which is greater than the critical value 3.84. That means, under 5% significance level, we reject the null and conclude that the coefficient β_2 is statistically significant. Same analysis can be applied to Japan. The results show Okun theory is partly supported by US and Japan if we change the error assumption from Normal to AEPD.

Table 5. Likelihood Ratio Test (LR)

	Panel A Significance Test		Panel B Parameter Restriction Tests					
	H1	**H2**	**H3**	**H4**	**H5**	**H6**	**H7**	**H8**
US	42.19	70.31	70.31	29.52	18.98	2.26	22.96	28.86
Japan	166.53	132.95	132.95	28.33	14.89	166.43	20.22	118.79
$\mathbf{X_{0.05}^2}$	3.84	3.84	5.99	7.84	3.84	3.84	5.99	3.84

Notes: H1= $H_0: \beta_1 = 0$. H2= $H_0: \beta_2 = 0$. H3= $H_0: \beta_1 = \beta_2 = 0$. H4= $H_0: \alpha = 0.5, p_1 = p_2 = 2$. H5= $H_0: p_1 = 2$. H6= $H_0: p_2 = 2$. H7= $H_0: p_1 = p_2 = 2$. H8= $H_0: \alpha = 0.5$. CV(5%)=Critical Value under 5% Significance Level.

4.3.2. Residual check

In this subsection, we check the residuals for models of Okun-AEPD and Okun-Normal with Kolmogorov-Smirnov (KS) test. The null hypothesis of KS test is H_0: Data follows a specified distribution. We set the significance level of all tests at 0.05 and put the *P*-value of KS test in Table 6. If the *P*-value of KS test is bigger than 0.05, then we do not reject the null hypothesis.

We first apply KS test for the Okun-AEPD residuals with the null hypothesis of H_0: Okun-AEPD residuals are distributed as AEPD $(\hat{\alpha}, \hat{p}_1, \hat{p}_2, \hat{\mu}, \hat{\sigma})$. For U.S., the *P*-value is 0.85, which means, under 5% significance level, we can't reject the null hypothesis and conclude that the residuals from Okun-AEPD do follow AEPD. In addition to Japan, we can see KS can't reject the null hypothesis. Hence, we conclude that the error terms of both countries do follow AEPD.

Table 6. P-values for Residual Checks

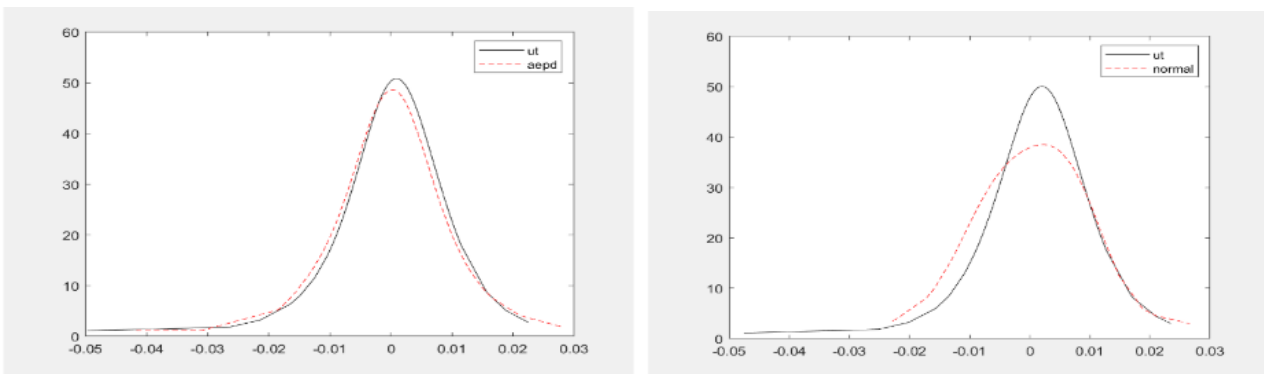|	KS(Okun-AEPD)	JB(Okun-Normal)
U.S.	0.85	0.02
Japan	0.85	0.02

For the residuals from Okun-Normal model, we calculate the *P*-values of the Jarque-Bera test (see Table 6). *P*-values of JB test for both countries are smaller than 5% significance level, which means we reject the null of normal distribution. Hence, we conclude that the Okun-Normal model is not adequate.

In Fig. 1a and Fig. 2a, we compare the probability density function (PDF) for the estimated residuals $\hat{u}_t$ with that of AEPD $(\hat{\alpha}, \hat{p}_1, \hat{p}_2, \hat{\mu}, \hat{\sigma}.)$. We find out these curves are very close to each other. Similarly, we compare the PDF of the residuals from Okun-Normal with that of Normal $(\hat{\mu}, \hat{\sigma})$. We find out there are big differences between these curves (see Fig. 1b and Fig. 2b), which means the residuals are not distributed as Normal. Hence, from the graphs, we may also conclude the Okun-AEPD fits the data better.
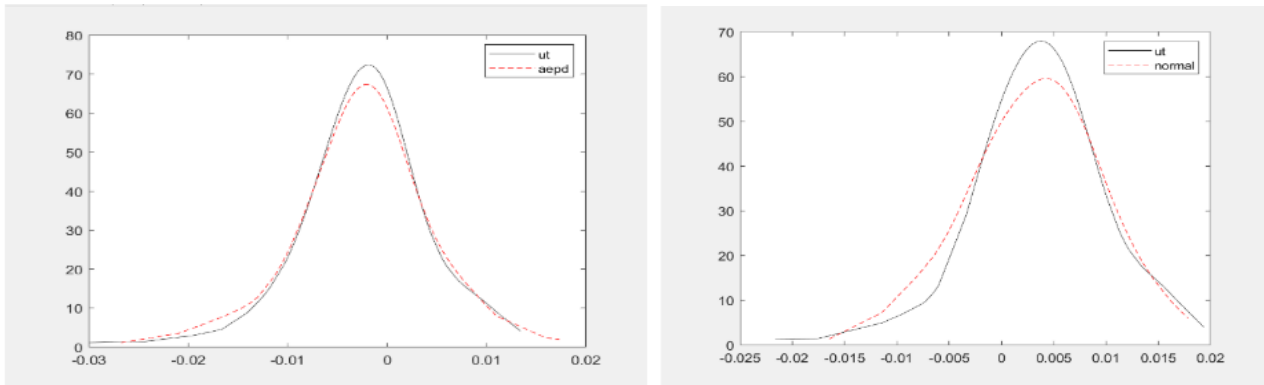
a) U.S. PDFs of Okun-AEPD residuals and AEPD; b) U.S. PDFs of Okun-Normal residuals and Normal

Fig. 1. PDFs of Residuals for U.S.

a) Japan PDFs of Okun-AEPD residuals and AEPD; b) Japan PDFs of Okun-Normal residuals and Normal

Fig. 2. PDFs of Residuals for Japan.

4.4. Tests for Parameter Restrictions

In this section, we test some restrictions on the parameters in Okun-AEPD and Likelihood Ratio test in equation (7) is used. And the results are listed in Panel B of Table 4. H3 column (i.e., Hypothesis 3 H_0: $\beta_1=\beta_2=0$), lists the joint significance test results for the coefficient parameter β_1 and β_2. The results of joint significance test (H3) show coefficients β_1 and β_2 in each country are statistically significant under 5% significance level.

Also, we test the Normality using parameter restrictions. The test results of parameter restrictions show strong non-Normality. For example, for column H4 (i.e., Hypothesis 4 H_0: $\alpha = 0.5$, $p_1 = p_2 = 2$), all values of LR are higher than the critical value 7.84 under 5% significance level. That means, AEPD error terms cannot be reduced to Normal errors.

To check robustness of this result, we also run Normality test on other hypotheses on the parameters of AEPD. Results listed in column H5, H6 and H7. 7 out of 8 results reject their null hypotheses under 5%

significance level. Hence, we conclude there exists strong non-Normality in the data.

Lastly, the skewness of data is tested. Column H8 (i.e., Hypothesis 8 H_0: α = 0.5) lists test results for skewness. Both reject the null, which documents significant skewness under 5% significance level.

4.5. Model Comparisons

In this subsection, for each country, we compare 9 models using Akaike Information Criterion. The AIC formula is

$$AIC = \frac{2k}{T} - \frac{2lnLikelihood}{T} \tag{8}$$

where k is the number of parameters to be estimated. T is the sample size. *lnLikelihood* is the value of log likelihood function. The AIC values are listed in Table 7. M1 in Table 7 is the Okun-AEPD and M5 is the Okun-Normal.

Table 7. Values of Akaike Information Criterion (AIC)

	M1	**M2**	**M3**	**M4**	**M5**
U.S.	-647.11	-572.80	-602.92	-574.80	-634.22
Japan	-757.07	-718.00	-755.59	-720.00	-747.71
	M6	**M7**	**M8**	**M9**	
U.S.	-642.45	-619.34	-645.02	-638.26	
Japan	-586.75	-752.96	-751.89	-749.86	

Notes: M1=Okun-AEPD. M2=Okun-AEPD with $\beta_1 = \beta_2 = 0$. M3=Okun-AEPD with $\beta_1 = 0$. M4=Okun-AEPD with $\beta_2 = 0$. M5=Okun-Normal. M6=Okun-AEPD with $\alpha = 0.5$. M7=Okun-AEPD with $p_1 = 2$. M8=Okun-AEPD with $p_2 = 2$. M9=Okun-AEPD with $p_1 = p_2 = 2$.

For U.S., the value of AIC in M1 is -647.11, the smallest among 9 models, which means the Okun-AEPD is better than others. Same conclusion can be drawn from Japan. Hence, we conclude the Okun-AEPD model has better in-sample fit.

5. Conclusions and Future Extensions

Asymmetric Exponential Power Distribution (AEPD) is first suggested in Zhu and Zinde-Walsh (2009) and it is one extension of Normal distribution to capture fat tails and asymmetric effects in financial data. In this paper, we suggest a new Okun coefficient by adding Asymmetric Exponential Power Distribution. Maximum Likelihood Estimation method is used to estimate this model. U.S. and Japan from 1999 Q1 to 2019 Q4 are analyzed. Likelihood Ratio test (LR) is used to test both significance of and restrictions on parameters. Model comparisons are used Akaike Information Criterion (AIC).

Empirical results show Okun theory is partly supported by US and Japan if the error assumption is changed from Normal to AEPD. Likelihood Ratio (LR) test proves the existence of fat tailness and skewness in residuals and based on Kolmogorov-Smirnov (KS) test, it is accepted that residuals follow AEPD under 5% significance level. Finally, Okun-AEPD has better in-sample fit than Okun-Normal by Akaike Information Criterion (AIC).

Future extensions will include but not limited to follows. First, we can add GARCH type volatility into the Okun-AEPD model. Second, we can exam our results with different data.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

The author conducted all this work ,including data collection, data analysis and wrote the paper. The author had approved the final version.

Acknowledgment

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