A Note on Integrating Factors of a Conformable Fractional Differential Equation

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Abstract: Recently exact fractional differential equations have been introduced, using the conformable fractional derivative. In this paper, we propose and prove some new results on the integrating factor. We introduce a conformable version of several classical special cases for which the integrating factor can be determined. Specifically, the cases we will consider are where there is an integrating factor that is a function of only *x*, or a function of only *y*, or a simple formula of x and y. In addition, using the Conformable Euler's Theorem on homogeneous functions, an integration factor for the conformable homogeneous differential equations is established. Finally, the above results apply in some interesting examples.

Key words: A conformable Euler´s theorem, conformable fractional derivative, exact fractional differential equation, integrating factor.

1. Introduction

For the many years, many definitions of fractional derivative have been introduced by various researchers. One of them is the Riemann-Liouville fractional derivative and the second one is the so-called Caputo derivative. These definitions are mostly used for mathematical models in many applications and are defined, respectively,

1) *Riemann-Liouville definition.* For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_{t_0}^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

2) *Caputo definition.* For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_{t_0}^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx$$

Now, all definitions attempt to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of fractional derivative is the linearity property. However, the following are the setbacks of one definition or another:

1) The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$ ($D_a^{\alpha}(1) = 0$ for the Caputo derivative.), if α is not natural number.
2) All fractional derivatives lost some of the basic properties that usual derivatives have such as the product rule, the quotient rule and the chain rule.

3) All fractional derivatives do not satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$ in general.

4) The Caputo definition assumes that function f is differentiable.

Recently, Khalil et al. introduced a new definition of fractional derivative called the conformable fractional derivative, [1]. Unlike other definitions, this new definition satisfies formulas of derivative of product and quotient of two functions and has a simpler chain rule. In addition, these authors introduce the conformable fractional derivative definition, the conformable integral definition, Rolle theorem and Mean value theorem for conformable fractional differentiable functions. In [2], Abdeljawad improves this new theory, establishing important elements of fractional calculus, such as: definitions of left and right conformable fractional derivatives and fractional integrals of higher order (i.e. of order $\alpha > 1$), the fractional power series expansion, the fractional transform Laplace definition, fractional integration by parts formulas, chain rule and Gronwall inequality.

In the field of multivariate calculus, several works propose the extension of important concepts and results in the conformable sense. In [3], [4], the conformable partial derivative of the order $\alpha \in (0,1]$ of the real value of several variables and conformable gradient vector are defined; and a conformable version of Clairaut´s Theorem for partial derivatives of conformable fractional orders is proven. In [5], two new results on homogeneous functions involving their conformable partial derivatives are introduced, specifically, homogeneity of the conformable partial derivatives of a homogeneous function and the conformable version of Euler´s Theorem.

Finally, it is also a remarkable fact the large number of studies in the theory and application of fractional differential equations based on this new definition of derivative, which have been developed in a short time, [6]-[17].

The paper is organized as follows. In Section 2, the main concepts of conformable fractional calculus are presented. In Section 3, some results on exact fractional differential equations are recalled first, and then two new results on the integrating factor are proposed and proven. Specifically, the cases we will consider are where there is an integrating factor that is a function of only x, or a function of only y, or a simple formula of x and y. Finally, an integrating factor is determined for homogeneous fractional differential equations.

2. Basic Definitions and Tools

Definition 2.1. [1]. Given a function $f: [0, \infty) \to R$. Then the conformable fractional derivative of f of order α, is defined by

$$(T_\alpha f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{1}$$

For all $t > 0$, $0 < \alpha \leq 1$. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} (T_\alpha f)(t)$ exist, then it is defined as

$$(T_\alpha f)(0) = \lim_{t \to 0^+} (T_\alpha f)(t) \tag{2}$$

As a consequence of the above definition, the following useful theorem is obtained, [1].

Theorem 2.1. If a function $f: [0, \infty) \to R$ is α-differentiable at $t_0 > 0$, $0 < \alpha \leq 1$, then f is continuous at t_0.

It is easily shown that T_α satisfies the following properties, [1].

Theorem 2.2. Let $0 < \alpha \leq 1$ and f, g be α-differentiable at a point $t > 0$. Then

1) $T_\alpha(af + bg) = a\,(T_\alpha f) + b\,(T_\alpha g), \forall a, b \in R$.

2) $T_\alpha(t^p) = pt^{p-\alpha}, \forall p \in R$.

3) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

4) $T_\alpha(fg) = f(T_\alpha g) + g(T_\alpha f)$.

5) $T_\alpha\left(\frac{f}{g}\right) = \frac{g(T_\alpha f) - f(T_\alpha g)}{g^2}$.

6) If, in addition, f is differentiable, then $(T_\alpha f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

The conformable fractional derivative of certain functions for above definition is given as follows:

1) $T_\alpha(1) = 0$.

2) $T_\alpha(sin(at)) = at^{1-\alpha}cos(at),\ a \in R$.

3) $T_\alpha(cos(at)) = -at^{1-\alpha}sin(at),\ a \in R$.

4) $T_\alpha(e^{at}) = ae^{at},\ a \in R$.

Further, many functions behave as in the usual derivative. Here are some formulas:

1) $T_\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1$

2) $T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$

3) $T_\alpha\left(\sin\left(\frac{1}{\alpha}t^\alpha\right)\right) = cos\left(\frac{1}{\alpha}t^\alpha\right)$

4) $T_\alpha\left(\cos\left(\frac{1}{\alpha}t^\alpha\right)\right) = -sin\left(\frac{1}{\alpha}t^\alpha\right)$

Remark 2.1. One should notice that a function could be α-differentiable at a point but not differentiable. For example, take $f(t) = 3\sqrt[3]{t}$. Then $\left(T_{\frac{1}{3}}f\right)(0) = \lim_{t\to 0^+}\left(T_{\frac{1}{3}}f\right)(t) = 1$, where $\left(T_{\frac{1}{3}}f\right)(t) = 1$, for $t > 0$. But $\frac{df}{dt}(0)$ does not exist.

Theorem 2.3 (Mean Value Theorem). [6]. Let $a > 0$ and $f: [a, b] \to R$ be a function that satisfies,

1) f is continuous in $[a, b]$,

2) f is α-differentiable on (a, b), for some $\alpha \in (0,1]$.

Then, there exists $c \in (a, b)$ such that

$$(T_\alpha f)(c) = \left[\frac{f(b) - f(a)}{b - a}\right]c^{1-\alpha} \tag{3}$$

Definition 2.2. [2]. The (left) conformable derivative starting from a of a function $f: [a, \infty) \to R$ of f of order $0 < \alpha \le 1$, is defined by

$$(T_\alpha^a f)(t) = \lim_{\epsilon\to 0}\frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon} \tag{4}$$

When $a = 0$, it is written as $(T_\alpha f)(t)$. If f is α-differentiable in some (a, b), then define

$$(T_\alpha^a f)(a) = \lim_{t\to a^+}(T_\alpha^a f)(t) \tag{5}$$

Note that if f is differentiable, then $(T_\alpha^a f)(a) = (t - a)^{1-\alpha}\frac{df}{dt}(t)$. Theorem 2.2 holds for Definition 2.2 when changing by $(t - a)$.

Theorem 2.4. *(Chain Rule).*[2]. Assume $f, g: (a, \infty) \to R$ be (left) α-differentiable functions, where $0 < \alpha \le 1$. Let $h(t) = f(g(t))$. The $h(t)$ is α-differentiable for all $t \ne a$ and $g(t) \ne 0$, therefore

$$(T_\alpha^a h)(t) = (T_\alpha^a f)\big(g(t)\big) \cdot (T_\alpha^a g)(t).(g(t))^{\alpha-1} \tag{6}$$

If $t = a$, then

$$(T_\alpha^a h)(a) = \lim_{t\to a^+}(T_\alpha^a f)\big(g(t)\big) \cdot (T_\alpha^a g)(t).(g(t))^{\alpha-1} \tag{7}$$

Now, there is the following definition for the α-fractional integral of a function f starting from $a \ge 0$.

Definition 2.3. [1]. $I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} \cdot dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

With the above definition, it was shown that

Theorem 2.5. [1]. $T_\alpha^a I_\alpha^a(f)(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of I_α.

Lemma 2.1. [2]. Let $f:(a,b) \to R$ be differentiable and $\alpha \in (0,1]$. Then, for all $a > 0$ we have,

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - f(a) \tag{8}$$

Finally, [3], [4], the conformable partial derivative of a real valued function with several variables is defined as follows:

Definition 2.4. Let f be a real valued function with n variables and $\boldsymbol{a} = (a_1, \dots, a_n) \in R^n$ be a point whose *ith* component is positive. Then the limit

$$\lim_{\epsilon \to 0} \frac{f(a_1,..,a_i+\varepsilon a_i{}^{1-\alpha},\dots,a_n) - f(a_1,\dots,a_n)}{\varepsilon} \tag{9}$$

If it exists, is denoted $\frac{\partial^\alpha}{\partial x_i^\alpha} f(\boldsymbol{a})$, and called the *ith* conformable partial derivative of f of the order $\alpha \in (0,1]$ at a.

Remark 2.2. If a real valued function f with n variables has all conformable partial derivatives of the order $\alpha \in (0,1]$ at $\boldsymbol{a} = (a_1, \dots, a_n) \in R^n$, each $a_i > 0$, then the conformable gradient of f of the order α at **a** is

$$\nabla^\alpha f(\boldsymbol{a}) = \left(\frac{\partial^\alpha}{\partial x_1^\alpha} f(\boldsymbol{a}), \dots, \frac{\partial^\alpha}{\partial x_n^\alpha} f(\boldsymbol{a}) \right) \tag{10}$$

Remark 2.3. Let $\alpha \in (0,1]$ and f be a real valued function with n variables defined on an open set $D \subset R^n$, such that, for all $(x_1, \dots, x_n) \in D$, each $x_i > 0$. Function f is said to be in $C_\alpha(D,R)$ if all its conformable fractional partial derivatives of order α exists and are continuous on D, [5].

In [3], Clairaut's Theorem for conformable partial derivatives fractional orders presented as follows:

Theorem 2.6. Let α, β be positive constants such that $0 < \alpha, \beta < 1$. Assume That $f(x,y)$ is function for which $\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\beta f(x,y)}{\partial y^\beta} f(x,y)\right)$ and $\frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\alpha f(x,y)}{\partial x^\alpha} f(x,y)\right)$ exists and are continuous over a domain $X \subset R^n$ such that for all $(x,y) \in X$, $x, y > 0$, then

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\beta f(x,y)}{\partial y^\beta} f(x,y)\right) = \frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\alpha f(x,y)}{\partial x^\alpha} f(x,y)\right) \tag{11}$$

For all $(x,y) \in X$.

3. Conformable Fractional Differential Equation Reducible to Exact: Integrating Factor

In this section, some results on exact fractional differential equations are recalled, [6].

Definition 3.1. Let $0 < \alpha \leq 1$. A first order differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is called $\alpha - exact$ if there exists a function $\Phi(x,y)$ such that

$$\frac{\partial^\alpha \Phi}{\partial y^\alpha} = M \text{ and } \frac{\partial^\alpha \Phi}{\partial x^\alpha} = N$$

Consequently,

$$d^\alpha \Phi(x,y) = M(x,y)dx + N(x,y)dy = 0$$

From the properties of the conformable fractional derivative, we get Φ is a constant function.

Theorem 3.1. Let $0 < \alpha \leq 1$. Let M, N be a real valued function with two variables defined on a set D and class C_α on D. Then $M(x,y)dx + N(x,y)dy = 0$ *is* $\alpha - exact$ *if and only if*

$$\frac{\partial^\alpha N}{\partial x^\alpha} = \frac{\partial^\alpha M}{\partial y^\alpha}, \forall (x,y) \in D \tag{12}$$

Definition 3.2. Let $0 < \alpha \leq 1$. Let M, N, μ be a real valued function with two variables defined on a set D and class C_α on D, with $x, y > 0\ \forall(x, y) \in D$. The function $\mu(x, y)$ is an integrating factor to the fractional differential equation $M(x, y)dx + N(x, y)dy = 0$, if the fractional differential equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \tag{13}$$

is $\alpha - exact$.

Remark 3.1. To find an integrating factor $\mu(x, y)$, apply the $\alpha - exactness$ condition on the equation (13)

$$\frac{\partial^\alpha(\mu N)}{\partial x^\alpha} = \frac{\partial^\alpha(\mu M)}{\partial y^\alpha} \tag{14}$$

That is,

$$\mu\left(\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right) = N(x,y)\frac{\partial^\alpha \mu}{\partial x^\alpha} - M(x,y)\frac{\partial^\alpha \mu}{\partial y^\alpha} \tag{15}$$

This is a fractional partial differential equation for the unknown function $\mu(x, y)$, which is more difficult to solve than the original fractional ordinary differential equation. However, for some special cases, equation (15) can be solved for an integrating factor.

3.1. Special Cases

3.1.1. μ is a function of x or y

If μ is a function of x only, that is, $\mu = \mu(x)$, [2], then

$$\frac{\partial^\alpha \mu}{\partial x^\alpha} = T_\alpha \mu\,, \frac{\partial^\alpha \mu}{\partial y^\alpha} = 0$$

And equation (15) becomes

$$NT_\alpha \mu = \mu\left(\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right)$$

This implies that

$$\frac{1}{\mu}T_\alpha \mu = \frac{1}{N}\left(\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right)$$

Since $\mu(x)$ is a function of x only, left hand side is a function of x only. Hence, if an integrating factor of the form $\mu(x)$ is to exit, right hand side must be is a function of x only. Then the integrating factor is, [6],

$$\mu(x) = exp\left[\int \frac{1}{N}\left(\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right)x^{\alpha-1}dx\right] \tag{16}$$

Interchanging M and N, and x and y in equation (15), one obtains an integrating factor for another special case, [6],

$$\mu(y) = exp\left[\int \underbrace{\frac{1}{M}\left(\frac{\partial^\alpha N}{\partial x^\alpha} - \frac{\partial^\alpha M}{\partial y^\alpha}\right)}_{function\ of\ y\ only} y^{\alpha-1}dy\right] \tag{17}$$

Example 3.1. Consider

$$2x^\alpha y^\alpha dx + (2x^{2\alpha} - 4y^{2+\alpha})dy = 0$$

For some $\alpha \in (0,1]$.

Solution. Here

$$\frac{\partial^\alpha(2x^\alpha y^\alpha)}{\partial y^\alpha} = 2\alpha x^\alpha \text{ and } \frac{\partial^\alpha(2x^{2\alpha} - 4y^{2+\alpha})}{\partial y^\alpha} = 4\alpha x^\alpha$$

Thus the equation is not conformable $\alpha - exact$.
So,

$$\frac{\partial^\alpha(2x^{2\alpha} - 4y^{2+\alpha})}{\partial y^\alpha} - \frac{\partial^\alpha(2x^\alpha y^\alpha)}{\partial y^\alpha} = 2\alpha x^\alpha$$

Thus

$$\frac{1}{\mu}T_\alpha\mu = \frac{2\alpha x^\alpha}{2x^\alpha y^\alpha} = \frac{\alpha}{y^\alpha}$$

So now it is matter routine to solve the equation noticing that

$$\mu(y) = exp\left[\alpha\int\frac{1}{y^\alpha}y^{\alpha-1}dy\right] = y^\alpha$$

3.1.2. μ is a function of x and y

If μ is a function of z only, where $z = z(x, y)$, [2], then

$$\frac{\partial^\alpha\mu}{\partial x^\alpha} = (T_\alpha\mu)(z)\cdot z^{\alpha-1}\cdot\frac{\partial^\alpha z}{\partial x^\alpha}, \frac{\partial^\alpha\mu}{\partial y^\alpha} = (T_\alpha\mu)(z)\cdot z^{\alpha-1}\cdot\frac{\partial^\alpha z}{\partial y^\alpha}$$

And equation (15) becomes

$$(T_\alpha\mu)(z)\cdot z^{\alpha-1}\cdot\left(N\cdot\frac{\partial^\alpha z}{\partial x^\alpha} - M\cdot\frac{\partial^\alpha z}{\partial y^\alpha}\right) = \mu\left(\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right)$$

This implies that

$$\frac{1}{\mu(z)}(T_\alpha\mu)(z) = \frac{\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}}{z^{\alpha-1}\cdot\left(N\cdot\frac{\partial^\alpha z}{\partial x^\alpha} - M\cdot\frac{\partial^\alpha z}{\partial y^\alpha}\right)} \tag{18}$$

Since $\mu(z)$ is a function of z only, left hand side is a function of z only. Hence, if an integrating factor of the form $\mu(z)$ is to exit, right hand side must be is a function of z only. Then the integrating factor is,

$$\mu(z) = exp\left[\int\frac{\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}}{z^{\alpha-1}\cdot\left(N\cdot\frac{\partial^\alpha z}{\partial x^\alpha} - M\cdot\frac{\partial^\alpha z}{\partial y^\alpha}\right)}\cdot z^{\alpha-1}\cdot dz\right] \tag{19}$$

Example 3.2. Find an integrating factor of the form $\mu(z)$, where $z = xy$, of the following equation

$$(4y^{1+\alpha} - 6xy^\alpha)dx + (6x^\alpha y - 4x^{1+\alpha})dy = 0$$

for Some $\alpha \in (0,1]$.
Solution. Here

$$\frac{\partial^\alpha(4y^{\alpha+1} - 6xy^\alpha)}{\partial y^\alpha} = 4(1+\alpha)y - 6\alpha x$$

And

$$\frac{\partial^\alpha(6x^\alpha y - 4x^{\alpha+1})}{\partial x^\alpha} = 6\alpha y - 4(1+\alpha)x$$

Thus, the equation is not $\alpha - exact$.
Now, computing the conformable fractional partial derivative of function $\mu(z)$ with respect to x and y, [3], then

$$\frac{\partial^\alpha\mu}{\partial x^\alpha} = (T_\alpha\mu)(z)\cdot z^{\alpha-1}\cdot\frac{\partial^\alpha z}{\partial x^\alpha} = y^\alpha\cdot(T_\alpha\mu)(z)$$

And

$$\frac{\partial^\alpha \mu}{\partial y^\alpha} = (T_\alpha \mu)(z) \cdot z^{\alpha-1} \cdot \frac{\partial^\alpha z}{\partial y^\alpha} = x^\alpha \cdot (T_\alpha \mu)(z)$$

Substituting these derivatives in equation (19) so that

$$\frac{1}{\mu(z)} (T_\alpha \mu)(z) = \frac{(2-\alpha)}{z^\alpha}$$

Finally, applying the fractional integral with respect to z on both sides of above equation, an integrating factor is obtained

$$\mu(x,y) = x^{2-\alpha} y^{2-\alpha}$$

Example 3.3. Find an integrating factor of the form $\mu(z)$, where $z = x + y$, of the following equation

$$\left(3x^{\frac{1}{2}} - y^{\frac{1}{2}}\right) dx + \left(3y^{\frac{1}{2}} + x^{\frac{1}{2}}\right) dy = 0$$

Solution. Here

$$\frac{\partial^{\frac{1}{2}}\left(3x^{\frac{1}{2}} - y^{\frac{1}{2}}\right)}{\partial y^{\frac{1}{2}}} = -\frac{1}{2} \quad \text{and} \quad \frac{\partial^{\frac{1}{2}}\left(3y^{\frac{1}{2}} + x^{\frac{1}{2}}\right)}{\partial x^{\frac{1}{2}}} = \frac{1}{2}$$

Now, computing the conformable fractional partial derivative of function $\mu(z)$ with respect to x and y, [2], then

$$\frac{\partial^{\frac{1}{2}} \mu}{\partial x^{\frac{1}{2}}} = \left(T_{\frac{1}{2}} \mu\right)(z) \cdot z^{-\frac{1}{2}} \cdot \frac{\partial^{\frac{1}{2}} z}{\partial x^{\frac{1}{2}}} = x^{\frac{1}{2}} \cdot z^{-\frac{1}{2}} \cdot \left(T_{\frac{1}{2}} \mu\right)(z)$$

And

$$\frac{\partial^{\frac{1}{2}} \mu}{\partial y^{\frac{1}{2}}} = \left(T_{\frac{1}{2}} \mu\right)(z) \cdot z^{-\frac{1}{2}} \cdot \frac{\partial^{\frac{1}{2}} z}{\partial y^{\frac{1}{2}}} = y^{\frac{1}{2}} \cdot z^{-\frac{1}{2}} \cdot \left(T_{\frac{1}{2}} \mu\right)(z)$$

Substituting these derivatives in equation (19) so that

$$\frac{1}{\mu(z)} \left(T_{\frac{1}{2}} \mu\right)(z) = -\frac{1}{\sqrt{z}}$$

Finally, applying the fractional integral with respect to z on both sides of above equation, an integrating factor is obtained

$$\mu(x,y) = \frac{1}{x+y}$$

3.1.3. Application to homogeneous fractional differential equation

Let $\alpha \in (0,1]$ and M, N are a real valued functions with two variables defined on an open set D for witch $(tx, ty) \in D$ whenever $t > 0$ and $(x,y) \in D$, with $x, y > 0$, that satisfies

1) M, N are homogeneous function of degree r

2) $M, N \in C_\alpha\ (D, R)$Then an integrating factor of the homogeneous differential equation $M(x,y)dx + N(x,y)dy = 0$, is given by

$$(x,y) = \frac{1}{x^\alpha \cdot M(x,y) + y^\alpha \cdot N(x,y)} \tag{20}$$

providing $x^{\alpha} \cdot M(x,y) + y^{\alpha} \cdot N(x,y) \neq 0, \forall (x,y) \in D$

In effect, computing the conformable fractional partial derivatives of functions μM and μN with respect to x and y, respectively, then

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}}\left(\frac{M(x,y)}{x^{\alpha} \cdot M(x,y) + y^{\alpha} \cdot N(x,y)}\right) = \frac{y^{\alpha} \cdot N(x,y) \cdot \frac{\partial^{\alpha} M(x,y)}{\partial y^{\alpha}} - \alpha M(x,y)N(x,y) - y^{\alpha} \cdot M(x,y) \cdot \frac{\partial^{\alpha} N(x,y)}{\partial y^{\alpha}}}{(x^{\alpha} \cdot M(x,y) + y^{\alpha} \cdot N(x,y))^2}$$

And

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{N(x,y)}{x^{\alpha} \cdot M(x,y) + y^{\alpha} \cdot N(x,y)}\right) = \frac{x^{\alpha} \cdot M(x,y) \cdot \frac{\partial^{\alpha} N(x,y)}{\partial x^{\alpha}} - \alpha M(x,y)N(x,y) - x^{\alpha} \cdot N(x,y) \cdot \frac{\partial^{\alpha} M(x,y)}{\partial x^{\alpha}}}{(x^{\alpha} \cdot M(x,y) + y^{\alpha} \cdot N(x,y))^2}$$

Finally, using the Conformable Euler´s Theorem on homogeneous functions, [5], the result is followed.

Example 3.4. Consider

$$(x^{2+\alpha} + yx^{1+\alpha})dx + (2x^{2+\alpha} + 3y^{2+\alpha})dy = 0$$

For some $\alpha \in (0,1]$.

Since $M(x,y) = x^{2+\alpha} + yx^{1+\alpha}$ and $N(x,y) = 2x^{2+\alpha} + 3y^{2+\alpha}$are homogeneous function of degree $2+\alpha$ and class C_{α} on open set D, with $x, y > 0$, then above equation is an homogeneous differential equation and

$$\mu(x,y) = \frac{1}{x^{\alpha}(x^{2+\alpha} + yx^{1+\alpha}) + y^{\alpha}(2x^{2+\alpha} + 3y^{2+\alpha})} = \frac{1}{x^{2+2\alpha} + yx^{1+2\alpha} + 2x^{2+\alpha}y^{\alpha} + 3y^{2+2\alpha}}$$

is an integrating factor of it.

4. Conclusions

The main objective of this work has been to generalize in the field of fractional calculus, some important results about the integrating factor for ordinary differential equations. The objective has been successfully achieved, so the definition of a partial derivative has been used to construct some results, such as: an integrating factor that is a function of only x, or a function of only y, an integrating factor that is a simple function of the variables x and y, or an integrating factor for a differential equation homogeneous. It seems that the results obtained in this work correspond to the results obtained in the classic case.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

F. Martínez and I. Martínez conceived the discussed idea and proved the results. S. Paredes wrote the paper and supervised the results of the work. All of authors had approved the final results and contributed to the final manuscript.

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