Flett Potential Spaces and a Weighted Wavelet-Like Transform

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Abstract: In this study, the Flett potential spaces are defined and a characterization of these potential spaces is given. Most of the known characterizations of classical potential spaces such as Riesz, Bessel potentials spaces and their generalizations are given in terms of finite differences. Here, by taking wavelet measure instead of finite differences, a weighted wavelet-like transform associated with Poisson semigroup is defined. And, by making use of this weighted wavelet-like transform, a new "truncated" integrals $\mathbb{D}_\varepsilon^\alpha \varphi$ are defined, then using these integrals a characterization of the Flett potential spaces is given.

Key words: Fractional integrals, weighted wavelet-like transform, flett potential spaces.

1. Introduction

The importance of weak-singular integral operators such as classical Riesz, Bessel and parabolic potentials and their various generalizations in harmonic analysis and its applications is well known. Producing inversion formulas for potentials is one of the important problems in potential theory. A number of approaches to this problem are known. The hypersingular integral technique, a very powerful tool for inversion of potentials, was introduced and studied by E. Stein [1], P. Lizorkin [2], S. Samko [3], [4], B. Rubin [5], [6] and many other. We refer the interested reader also to the papers [7]-[9] for various properties, generalizations and applications.

Continuous wavelet transforms are an alternative approach to find inversion formulas of potentials and this approach has been defined by B. Rubin ([5], [10]) and developed by I. A. Aliev and B. Rubin [11], [12], I. A. Aliev and M. Eryigit [13].

The Bessel potentials $J^\alpha \varphi\,, \varphi \in L_p(\mathbb{R}^n)$ and relevant function spaces $H_p^\alpha(\mathbb{R}^n) = \{f \ : \ f = J^\alpha \varphi\,, \varphi \in L_p(\mathbb{R}^n)\}$ were introduced by N. Aronszajn and K. Smith [14] and A. P. Calderón [15]. Parabolic Bessel potential spaces were defined by C. H. Sampson [16] and generalized by V. A. Nogin and B. Rubin [17]. Similarly, Riesz potential spaces were introduced by S. G. Samko [18]. Recently the wavelet type characterization of these spaces are introduced and studied by I. A. Aliev [19], S. Sezer and I. A. Aliev [20], [21].

The Flett potentials $\mathcal{T}^\alpha f$ of a function f are introduced by T. M. Flett in his fundamental paper [22]

$$(\mathcal{T}^\alpha f)^\wedge(x) = (1+ \mid x \mid)^{-\alpha} f^\wedge(x), (x \in \mathbb{R}^n, \alpha > 0). \tag{1}$$

The purpose of this paper is to attempt to study and define space of Flett potentials. The rest of the artical organized as follows: Notations and auxilary lemmas introduced in Section 2. Main results proved in Section 3.

In this work, we define the space of Flett potentials

$$\mathcal{T}^{\alpha}(L_p) = \ f \ : \ f = \mathcal{T}^{\alpha}\varphi \ , \ \ \varphi \in L_p \ \ , \alpha > 0 \ , \ 1 < p < \infty.$$

Then using the weighted wavelet-like transform

$$(w_{\mu} f)(x,t) = \int_{\mathbb{R}_+} e^{-t\eta} (\mathcal{P}_{t\eta} f)(x) d\mu(\eta)\,, \ (x \in \mathbb{R}^n, t > 0) \tag{2}$$

Which is generated by a finite Borel measure μ on $\mathbb{R}_+ = [0,\infty)$ and the Poisson integral $\mathcal{P}_t f$, we obtain a characterization of the Flett potential spaces $\mathcal{T}^{\alpha}(L_p)$.

2. Notation and Auxiliary Lemmas

$L_p \equiv L_p(\mathbb{R}^n)$ is the space of the measurable functions on $\mathbb{R}^n$ such that

$$\| f \|_p \equiv \left(\int_{\mathbb{R}^n} | f(x) |^p \ dx \right)^{1/p} < \infty\,, 1 \le p < \infty.$$

$C_0 \equiv C_0(\mathbb{R}^n)$ is the class of all continuous functions on $\mathbb{R}^n$ for which $\lim\limits_{|x| \to \infty} f(x) = 0$. The notation $S \equiv S(\mathbb{R}^n)$ is the class of Schwartz test functions. It is well known that this class is dense in $L_p\,, 1 \le p < \infty$ and C_0. The Fourier transform of a function $f \in L_1(\mathbb{R}^n)$ is defined by

$$f^{\wedge}(x) \equiv (Ff)(x) = \int_{\mathbb{R}^n} e^{-ix.\xi} f(\xi) d\xi, \quad x.\xi = x_1\xi_1 + \cdots + x_n\xi_n, \quad \xi, x \in \mathbb{R}^n. \tag{3}$$

The inverse Fourier transform is defined by

$$(F^{-1} f)(\xi) = (2\pi)^{-n} (Ff)(-\xi).$$

The action of a distribution f as a functional on the test function ω will be denoted by (f,ω). For a locally integrable function f , (f,ω) is defined by

$$(f,\omega) = \int_{\mathbb{R}^n} f(x)\omega(x) dx\,,$$

Provided that the last integral is finite for every $\omega \in \Phi$.

The Flett potential of order $\in$ is defined by (1), has the following convolution type representation:

$$(\mathcal{T}^{\alpha} f)(x) = (\Phi_{\alpha} * f)(x) \equiv \int_{\mathbb{R}^n} \Phi_{\alpha}(y) f(x-y) dy\,, \tag{4}$$

where $(F\Phi_{\alpha})(x) = (1+ | x |)^{-\alpha}$. The kernel $\Phi_{\alpha}(y)$ is as follows

$$\Phi_\alpha(y) = \frac{1}{\lambda_n(\alpha)} \mid y \mid^{\alpha-n} \int_0^\infty \frac{s^\alpha e^{-s|y|}}{(1+s^2)^{(n+1)/2}} ds, \tag{5}$$

With $\lambda_n(\alpha) = \pi^{(n+1)/2}\Gamma(\alpha) \,/\, \Gamma\big((n+1)/2\big)$.

It is not difficult to show that the kernel $\Phi_\alpha(y)$ has the following properties (see also: [22], [4]).

(a) If $0 < \alpha < n$, then

$$\Phi_\alpha(y) \sim c_n(\alpha) \mid y \mid^{\alpha-n} \text{ as } \mid y \mid \to 0$$

and

$$\Phi_n(y) \sim \frac{c_n}{(n-1)!} \log \frac{1}{\mid y \mid} \quad \text{as} \quad \mid y \mid \to 0 ,$$

where

$$c_n(\alpha) = \frac{\Gamma\ (\alpha+1)/2\ \ \Gamma\ (n-\alpha)/2}{2\Gamma(\alpha)\pi^{(n+1)/2}} \quad \text{and} \quad c_n = \frac{\Gamma\ (n+1)/2}{\pi^{(n+1)/2}} ;$$

(b) For all $\alpha > 0$

$$\Phi_\alpha(y) \sim \alpha c_n \mid y \mid^{-n-1} \quad as \quad \mid y \mid \to \infty ;$$

(c) $\Phi_\alpha \in L_1$ and $\| \Phi_\alpha \|_1 = 1$ for all $\alpha > 0$.

From (c) it follows that

$$\| \mathcal{T}^\alpha f \|_p \leq \| f \|_p , \quad \forall \alpha > 0 , \quad 1 \leq p \leq \infty . \tag{6}$$

If we use (5) and the Poisson integral $\mathcal{P}_t f$, we obtain the following equality for the Flett potential:

$$\mathcal{T}^\alpha f\ (x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t}\ \mathcal{P}_t f\ (x) dt , \tag{7}$$

where $f \in L_p$, $(1 \leq p \leq \infty)$ and

$$\mathcal{P}_t f\ (x) = \int_{\mathbb{R}^n} P(y,t) f(x-y) dy , \quad t > 0 , \ x \in \mathbb{R}^n \tag{8}$$

is the Poisson integral with the Poisson kernel $P(y,t)$, defined by

$$P(y,t) = \frac{c_n t}{(\mid y \mid^2 + t^2)^{(n+1)/2}} , \quad c_n = \Gamma\ (n+1)/2 \Big/ \pi^{(n+1)/2} . \tag{9}$$

The following lemma gives some properties of the Poisson integral $\mathcal{P}_t f$ which will be used later.

Lemma 2.1 [5] Let $f \in L_p$, $1 \leq p \leq \infty$ and $\mathcal{P}_t f$ be as in (8). Then

$$\text{(a)} \quad \int_{\mathrm{R}^n} P(y,t) dy = 1, \quad (P(.,t))^\wedge (y) = e^{-t|y|} , \forall t > 0 ; \tag{10}$$

$$\text{(b)} \quad \| \mathbb{P}_t f \|_p \leq \| f \|_p ; \tag{11}$$

$$\text{(c)} \quad \sup_x | (\mathbb{P}_t f)(x) | \leq c t^{-n/p} \| f \|_p , 1 \leq p < \infty, c = c(n,p) ; \tag{12}$$

$$\text{(d)} \quad \sup_{t>0} | (\mathbb{P}_t f)(x) | \leq (Mf)(x) \quad (Mf \text{ is the Hardy-Littlewood maximal function}) ; \tag{13}$$

$$\text{(e)} \quad (\mathbb{P}_\tau (\mathbb{P}_t f))(x) = (\mathbb{P}_{\tau+t} f)(x), \quad t > 0 , \ \tau > 0, \tag{14}$$

$$\text{(f)} \quad \lim_{t \to 0} \quad (\mathbb{P}_t f)(x) = f(x) , \tag{15}$$

where the limit is interpreted in L_p-norm and pointwise a.e.. For $f \in C_0$ the convergence is uniform on $\mathbb{R}^n$.

Definition 2.2 Let μ is a signed Borel measure on $\mathbb{R}_+ = [0,\infty)$ if

$$\| \mu \| \equiv | \mu | (\mathbb{R}_+) \equiv \int_{\mathbb{R}_+} d | \mu | (\eta) < \infty \qquad \text{and} \qquad \mu(\mathbb{R}_+) \equiv \int_0^\infty d\mu(\eta) = 0 \tag{16}$$

Then μ is called a wavelet measure.

Definition 2.3 The weighted wavelet-like transform of $f \in L_p$ is defined by

$$(w_\mu f)(x,t) = \int_{\mathbb{R}_+} e^{-t\eta} (\mathcal{P}_{t\eta} f)(x) d\mu(\eta), \quad (x \in \mathbb{R}^n, t > 0) \tag{17}$$

where μ is a finite Borel measure on $[0,\infty)$, $\mu([0,\infty)) = 0$ and $\mathcal{P}_{t\eta} f$ is the Poisson integral.

Owing to (15), it is assumed that $e^{-t\eta} (\mathcal{P}_{t\eta} f)(x) \Big|_{\eta=0} = f(x)$ and therefore

$$\int_{\mathbb{R}_+} e^{-t\eta} (\mathcal{P}_{t\eta} f)(x) d\mu(\eta) = \int_{(0,\infty)} e^{-t\eta} (\mathcal{P}_{t\eta} f)(x) d\mu(\eta) + \mu\{0\}. f(x) . \tag{18}$$

Furthermore, by (12)

$$\left\| (w_\mu f)(\cdot,t) \right\|_p \leq \ \left\| \mu \right\| \left\| f \right\|_p \ ,$$

where $\left\| \mu \right\| = \int_{[0,\infty)} d \,|\, \mu \,|\, (\eta) < \infty$.

Lemma 2.4 [23] Let $\{T_\varepsilon\}$, $\varepsilon > 0$, be a family of linear operators, mapping $L_p \equiv L_p(\mathbb{R}^n), 1 \leq p \leq \infty$ into the space of measurable functions on $\mathbb{R}^n$. Define $T^* f$ by setting

$$(T^* f)(x) = \sup_{\varepsilon > 0} | (T_\varepsilon f)(x) | , x \in \mathbb{R}^n.$$

Suppose that there exists a constant $c > 0$ and a real number $q \geq 1$ such that

$$meas \{x \ :| (T^* f)(x) |> t\} \leq \left(\frac{c \| f \|_p}{t} \right)^q$$

For all $t > 0$ and $f \in L_p$.

If there exists a dense subset $\mathcal{D}$ of L_p such that $\lim_{\varepsilon \to 0}(T_\varepsilon g)(x)$ exists and is finite a.e. whenever $g \in \mathcal{D}$ then for each $f \in L_p$, $\lim_{\varepsilon \to 0}(T_\varepsilon f)(x)$ exists and is finite a.e..

Lemma 2.5 [10] Let μ be a finite Borel measure on $\mathbb{R}_+$, and let $k_\alpha(s) = s^{-1}(I^{\alpha+1}\mu)(s)$, where

$$(I^{\alpha+1}\mu)(s) = \frac{1}{\Gamma(\alpha+1)}\int_0^s (s-t)^\alpha \, d\mu(t) \tag{19}$$

is the Riemann-Liouville fractional integral of order $\alpha+1$ of the measure μ. Let further, $\alpha' = \mathbf{Re}\,\alpha \geq 0$, and let μ satisfy the following conditions:

$$\text{(i)} \quad \int_0^\infty t^j d\mu(t) = 0 \ , \ j = 0,1,\ldots,[\alpha'] \quad (\text{the integer part of } \alpha');$$

$$\text{(ii)} \quad \int_1^\infty t^\gamma d\,|\,\mu\,|\,(t) < \infty \quad \text{for some} \quad \gamma > \alpha'.$$

Then

$$k_\alpha(s) = \left\{\begin{matrix} O(s^{\alpha'-1}) & \text{if } \ 0 < s < 1\,, \\ O(s^{-\delta-1}) & \text{if } \ s \geq 1\,,\, \delta = \min\{\gamma - \alpha', 1 + [\alpha'] - \alpha'\} > 0 \end{matrix}\right\}.$$

Furthermore, if $\tilde{\mu}(t) = \int_0^\infty e^{-t\eta}\, d\mu(\eta)$ is the Laplace transform of μ, then

$$\int_0^\infty k_\alpha(s)\, ds = \int_0^\infty \frac{\tilde{\mu}(t)}{t^{\alpha+1}} dt \equiv k_{\alpha,\mu} \tag{20}$$

where

$$k_{\alpha,\mu} = \left\{\begin{matrix} \Gamma(-\alpha)\int_0^\infty t^\alpha \ d\mu(t) \ if \quad \alpha \notin \mathbb{Z}_+ \ , \\ \frac{(-1)^{\alpha+1}}{\alpha!}\int_0^\infty t^\alpha \ln t \ d\mu(t) \quad \text{if} \quad \alpha \in \mathbb{Z}_+ \end{matrix}\right\}. \tag{21}$$

The following theorem gives an inversion formula for $w_\mu f$, defined in (17).

Theorem 2.6 [24] Let $\mathcal{T}^\alpha f$, $\alpha > 0$ be the Flett potentials of $f \in L_p$. Suppose that μ is a finite Borel measure on $\mathbb{R}_+$ satisfying

$$\text{(a)} \quad \int_1^\infty t^\beta d\,|\,\mu\,|\,(t) < \infty \quad \textit{for some} \quad \beta > \alpha \ ; \tag{22}$$

$$\text{(b)} \quad \int_0^\infty t^k d\mu(t) = 0 \ , \ k = 0,1,\ldots,m \quad (m = [\alpha] \ \text{ is the integer part of } \ \alpha). \tag{23}$$

Then

$$\int_0^\infty t^{-\alpha} \ \ w_\mu \mathcal{T}^\alpha f \ \ (x,t)\frac{dt}{t} \equiv \lim_{\varepsilon \to 0}\int_\varepsilon^\infty t^{-\alpha} \ \ w_\mu \mathcal{T}^\alpha f \ \ (x,t)\frac{dt}{t} = k_{\alpha,\mu} f(x) \ , \tag{24}$$

where the constant $k_{\alpha,\mu}$ is defined as (21).

The limit in (24) is understood in L_p-sense and pointwise a.e. for $1 \le p < \infty$. If $f \in C_0$, the convergence is uniform on $\mathbb{R}^n$.

Proof. Let $\varphi = \mathcal{T}^{\alpha} f,\ \ f \in L_p, 1 \le p < n / \alpha.$ Then

$$\begin{aligned} w_{\mu}\varphi\ \ (x,t) &= \int_0^{\infty} \mathcal{P}_{t\eta}\mathcal{T}^{\alpha}f\ \ (x)d\mu(\eta) = \int_0^{\infty} \mathcal{T}^{\alpha}\mathcal{P}_{t\eta}f\ \ (x)d\mu(\eta) \\ &\overset{(7)}{=} \frac{1}{\Gamma(\alpha)}\int_0^{\infty} d\mu(\eta)\int_0^{\infty}\tau^{\alpha-1}\ \ \mathcal{P}_{\tau}\mathcal{P}_{t\eta}f\ \ (x)d\tau \\ &\overset{(14)}{=} \frac{1}{\Gamma(\alpha)}\int_0^{\infty} d\mu(\eta)\int_0^{\infty}\tau^{\alpha-1}\ \ \mathcal{P}_{\tau+t\eta}f\ \ (x)d\tau \\ &= \frac{1}{\Gamma(\alpha)}\int_0^{\infty} d\mu(\eta)\int_0^{\infty}(\tau-\eta t)_+^{\alpha-1}\ \ \mathcal{P}_{\tau}f\ \ (x)d\tau, \end{aligned} \tag{25}$$

where $s_+ = \begin{cases} s, & if \quad s > 0 \\ 0, & if \quad s \le 0. \end{cases}$

We introduce ``truncated" integrals $\mathbb{D}_{\varepsilon}^{\alpha}\varphi$ as follows:

$$\mathbb{D}_{\varepsilon}^{\alpha}\varphi\ \ (x) \equiv \int_{\varepsilon}^{\infty} t^{-\alpha-1}\ \ w_{\mu}\varphi\ \ (x,t)dt. \tag{26}$$

By making use of (25), (26) and Fubini's theorem, we have

$$\begin{aligned} \mathbb{D}_{\varepsilon}^{\alpha}\varphi\ \ (x) &= \frac{1}{\Gamma(\alpha)}\int_{\varepsilon}^{\infty} t^{-\alpha-1}dt\int_0^{\infty}d\mu(\eta)\int_0^{\infty}(\tau-\eta t)_+^{\alpha-1}\ \ \mathcal{P}_{\tau}f\ \ (x)d\tau \\ &= \frac{1}{\Gamma(\alpha)}\int_0^{\infty}\ \mathcal{P}_{\tau}f\ \ (x)d\tau\int_0^{\tau/\varepsilon}\eta^{\alpha-1}d\mu(\eta)\int_{\varepsilon}^{\tau/\eta}t^{-\alpha-1}\left(\frac{\tau}{\eta}-t\right)^{\alpha-1}dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^{\infty}\ \mathcal{P}_{\varepsilon\tau}f\ \ (x)d\tau\int_0^{\tau}\eta^{\alpha-1}d\mu(\eta)\int_1^{\tau/\eta}t^{-\alpha-1}\left(\frac{\tau}{\eta}-t\right)^{\alpha-1}dt. \end{aligned}$$

If we apply the formula (cf. [25], formula no: 3.238(3))

$$\int_1^{s} t^{-\alpha-1}(s-t)^{\alpha-1}dt = \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)}\frac{1}{s}(s-1)^{\alpha},\ \ (s>1)$$

Then

$$\mathbb{D}_{\varepsilon}^{\alpha}\varphi\ \ (x) = \int_0^{\infty}(\mathcal{P}_{\varepsilon\tau}f)(x)\left(\frac{1}{\tau}\frac{1}{\Gamma(1+\alpha)}\int_0^{\tau}(\tau-\eta)^{\alpha}d\mu(\eta)\right)d\tau = \int_0^{\infty}\ \mathcal{P}_{\varepsilon\tau}f\ \ (x)K_{\alpha}(\tau)d\tau, \tag{27}$$

where $K_{\theta}(\tau) = \frac{1}{\tau}\ \ I_{0^+}^{\theta+1}\mu\ \ (\tau),$ and

$$I_{0^+}^{\theta+1}\mu\ \ (\tau) = \frac{1}{\Gamma(1+\theta)}\int_0^{\tau}(\tau-\eta)^{\theta}d\mu(\eta)$$

is the Riemann-Liouville integral of the measure μ. By Lemma 1 from [10], conditions (22) and (23) imply that $K_\theta(\tau)$ has a decreasing integrable majorant and $\int\limits_0^\infty K_\theta(\tau)d\tau = k_{\theta,\mu}$. Here $k_{\theta,\mu}$ is defined by (21).

By (27)

$$\left(\mathbb{D}_\varepsilon^\alpha \mathcal{T}^\alpha f\right)(x) - k_{\alpha,\mu} f(x) = \int\limits_0^\infty \left[\left(\mathcal{P}_{\varepsilon\tau} f\right)(x) - f(x)\right] K_\alpha(\tau) d\tau ,$$

and therefore,

$$\left\| \mathbb{D}_\varepsilon^\alpha \mathcal{T}^\alpha f - k_{\alpha,\mu} f \right\|_p \leq \int\limits_0^\infty \left\| \mathcal{P}_{\varepsilon\tau} f - f \right\|_p \mid K_\alpha(\tau) \mid d\tau .$$

By making use of the Lebesgue convergence theorem and formula (15) gives

$$\left\| \mathbb{D}_\varepsilon^\alpha \mathcal{T}^\alpha f - k_{\alpha,\mu} f \right\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0, \; 1 < p < \infty. \tag{28}$$

For $f \in L_p \cap C_0$ the proof is similar and based on Lemma 2.1.(f). The proof of the pointwise (a.e.) convergence is based on the maximal function technique. More precisely, from (27) and (13) we have

$$\left| \left(\mathbb{D}_\varepsilon^\alpha \mathcal{T}^\alpha f\right)(x) \right| \leq \sup_{t>0} \left| (\mathcal{P}_t f)(x) \right| \int\limits_0^\infty \mid K_\alpha(\tau) \mid d\tau \leq c(Mf)(x),$$

and therefore

$$\sup_{\varepsilon>0} \left| \left(\mathbb{D}_\varepsilon^\alpha \mathcal{T}^\alpha f\right)(x) \right| \leq c\,(Mf)(x), \quad c = c(\alpha) > 0. \tag{29}$$

Thus, the maximal operator

$$f(x) \mapsto \sup_{\varepsilon>0} \left| \left(\mathbb{D}_\varepsilon^\alpha \mathcal{T}^\alpha f\right)(x) \right|.$$

is weak (p,p), $1 \leq p < n / \alpha$. Then, by Theorem 3.12 from [23] it follows that $(\mathbb{D}_\varepsilon^\alpha T^\alpha f)(x) \to k_{\alpha,\mu} f(x)$ as $\varepsilon \to 0$ for almost all $x \in \mathbb{R}^n$.

3. Main Results

In this section we define Flett potential spaces $\mathcal{T}^\alpha(L_p)$ and give a characterization of these spaces by using the weighted wavelet-like transform $w_\mu f$.

Definition 3.1 Let $\alpha > 0$ and $1 < p < \infty$. The spaces $\mathcal{T}^\alpha(L_p)$ of Flett potential are defined as follows:

$$\mathcal{T}^\alpha(L_p) = \left\{ f \; : \; f = \mathcal{T}^\alpha \varphi \;,\; \varphi \in L_p \right\}.$$

The norm of $f \in \mathcal{T}^\alpha(L_p)$ is defined by $\|f\|_{\mathcal{T}^\alpha(L_p)} = \|\varphi\|_p$, which makes $\mathcal{T}^\alpha(L_p)$ a Banach space.

Theorem 3.2 Let $\alpha > 0$, $1 < p < \infty$ and $\beta > 0$. Then

$$f \in \mathcal{T}^\alpha(L_p) \Leftrightarrow f \in L_p \quad and \quad \sup_{\varepsilon>0} \left\| \mathbb{D}_\varepsilon^\alpha f \right\|_p < \infty.$$

Proof. We use some ideas from [5]. Suppose that $f \in \mathcal{T}^\alpha(L_p)$. Then for some $\varphi \in L_p$, $f = \mathcal{T}^\alpha \varphi$. Since

$\{\mathbb{D}_\varepsilon^\alpha \mathcal{T}^\alpha \varphi\}_{\varepsilon>0}$ converges in $L_p -$ norm to $k_{\alpha,\mu}\varphi$ (see (29)), there exists $c>0$ such that $\sup\limits_{\varepsilon>0}\left\|\mathbb{D}_\varepsilon^\alpha \mathcal{T}^\alpha \varphi\right\|_p \le c\left\|\varphi\right\|_p$.

Conversely, let $f\in L_p$ and $\sup\limits_{\varepsilon>0}\left\|\mathbb{D}_\varepsilon^\alpha f\right\|_p < \infty$. Firstly we show that $f=\mathcal{T}^\alpha\psi$ for some $\psi\in L_p$. Since the space $S=S(\mathbb{R}^n)$ is dense in L_p, it is sufficient to show that

$$f, w \;=\; \mathcal{T}^\alpha\psi, w \tag{30}$$

For some $\psi\in L_p$ and all $w\in S$. We use the following equality for the convolution-type operators:

$$h*\lambda, w \;=\; h, \lambda\_ * w \qquad , \qquad \lambda, w\in S,\; h\in S' \text{ and } \lambda\_(x)=\lambda(-x) \;. \tag{31}$$

Since the operators $\mathcal{T}^\alpha$ and $\mathbb{D}_\varepsilon^\alpha$ are the convolution-type operators with the radial kernels, it follows from (31) that

$$\text{a)}\quad \mathcal{T}^\alpha g, w \;=\; g, \mathcal{T}^\alpha w \qquad\qquad \text{b)}\quad \mathbb{D}_\varepsilon^\alpha g, w \;=\; g, \mathbb{D}_\varepsilon^\alpha w \tag{32}$$

For all $g\in L_p$ and $w\in S$. By the Banach-Alaoglu theorem, the condition $\sup\limits_{\varepsilon>0}\left\|\mathbb{D}_\varepsilon^\alpha f\right\|_p<\infty$ yields that there exist a function $\varphi\in L_p$ and a sequence $\varepsilon_k\to 0$, such that

$$\mathbb{D}_{\varepsilon_k}^\alpha f, w \;=\; \varphi, w \qquad as \quad \varepsilon_k\to 0, \tag{33}$$

For all $w\in L_q$, $1/q+1/p=1$ (in particular, for all $w\in S$). For this function $\varphi\in L_p$ and any Schwartz function w we have

$$\mathcal{T}^\alpha\varphi, w \;\overset{(32)-a)}{=}\; \varphi, \mathcal{T}^\alpha w \;\overset{(33)}{=}\; \lim_{\varepsilon_k\to 0} \mathbb{D}_{\varepsilon_k}^\alpha f, \mathcal{T}^\alpha w \;\overset{(32)-b)}{=}\; \lim_{\varepsilon_k\to 0} f, \mathcal{T}^\alpha w \;.$$

That is

$$\mathcal{T}^\alpha\varphi, w \;=\lim_{\varepsilon_k\to 0} f, \mathbb{D}_{\varepsilon_k}^\alpha \mathcal{T}^\alpha w \qquad , \quad \forall w\in S. \tag{34}$$

Let us show that

$$\lim_{\varepsilon_k\to 0} f, \mathbb{D}_{\varepsilon_k}^\alpha \mathcal{T}^\alpha w \;=\; f, k_{\alpha,\mu} w \tag{35}$$

where $k_{\alpha,\mu}$ is defined as (20). The Hölder inequality yields

$$\left|\; f, \mathbb{D}_{\varepsilon_k}^\alpha \mathcal{T}^\alpha w \;-\; f, k_{\alpha,\mu} w \;\right| = \left|\; f, \mathbb{D}_{\varepsilon_k}^\alpha \mathcal{T}^\alpha w - k_{\alpha,\mu} w \;\right| \le \left\|f\right\|_p \left\|\mathbb{D}_{\varepsilon_k}^\alpha \mathcal{T}^\alpha w - k_{\alpha,\mu} w\right\|_q, \quad \frac{1}{p}+\frac{1}{q}=1.$$

From (28), the right hand side of the last expression goes to zero as $\varepsilon_k\to 0$. Hence, (35) is true and therefore, by (34) we get

$$\mathcal{T}^\alpha\varphi, w \;=\; f, k_{\alpha,\mu} w \qquad , \quad \forall w\in S.$$

The latter equality shows that $f = \mathcal{T}^{\alpha}\psi$, where $\psi = k_{\alpha,\mu}^{-1}\varphi$, $\varphi \in L_p$, and therefore $f \in \mathcal{T}^{\alpha}(L_p)$. The proof is completed.

Remark 3.3 Examples of wavelet measures, which satisfy the conditions of Theorem 3.2 (with $k_{\alpha,\mu} \neq 0$), are the following:

1. Let $\mu = \sum_{j=0}^{l}(-1)^j \binom{l}{j}\delta_j$, where $\delta_j = \delta_j(s)$ denotes the unit Dirac mass at the point $s = j$, i.e. $\langle \delta_j, \varphi \rangle = \varphi(j)$, and $l > \alpha$ is a fixed integer. It is well known that (see [4])

$$\int_0^{\infty} s^k \, d\mu(s) \equiv \sum_{j=0}^{l}(-1)^j \binom{l}{j} j^k = 0 \ , \ \ \forall k = 0,1,\ldots,l-1.$$

On the other hand

$$\tilde{\mu}(t) \equiv \int_0^{\infty} e^{-ts}\, d\mu(s) = \sum_{j=0}^{l}(-1)^j \binom{l}{j} e^{-tj} = (1-e^{-t})^l,$$

and therefore, by (20),

$$k_{\alpha,\mu} \equiv \int_0^{\infty} t^{-\alpha-1}\tilde{\mu}(t)dt = \int_0^{\infty} t^{-\alpha-1}(1-e^{-t})^l dt \neq 0.$$

2. Let $l > \alpha$ be a fixed integer and $h(s)$, $s \in \mathbb{R}^1$ be a Schwartz test function such that $h^{(k)}(0) = 0, \forall k = 0,1,2,\ldots,l$ and $\int_0^{\infty} s^{\alpha-l}h(s)ds \neq 0$ (e.g. $h(s) = e^{-s^2-1/s^2}$, $h(0) = 0$; such functions are called the Lizorkin test functions). We set $d\mu(s) = h^{(l)}(s)ds$, $(s \geq 0)$. The integration by parts shows that

$$\int_0^{\infty} s^k d\mu(s) \equiv \int_0^{\infty} s^k h^{(l)}(s)ds = 0 \ , \ \ \forall k = 0,1,\ldots,m \ ; \ (m = [\alpha]),$$

And therefore, the conditions (22) and (23) are fulfilled. Further, integrating by parts, we obtain

$$\tilde{\mu}(t) \equiv \int_0^{\infty} e^{-ts} d\mu(s) = \int_0^{\infty} e^{-ts} h^{(l)}(s)ds = t^l \int_0^{\infty} e^{-ts}h(s)ds.$$

Now after simple calculations, we have

$$k_{\alpha,\mu} \equiv \int_0^{\infty} t^{-\alpha-1}\tilde{\mu}(t)dt = \int_0^{\infty} t^{l-\alpha-1}\left(\int_0^{\infty} e^{-ts}h(s)ds\right)dt$$
$$= \int_0^{\infty} h(s)\left(\int_0^{\infty} t^{l-\alpha-1}e^{-ts}dt\right)ds = \Gamma(l-\alpha)\int_0^{\infty} h(s)s^{\alpha-l}ds \neq 0.$$

4. Conclusion

Most of the known characterizations of potential spaces (for example Riesz, Bessel potentials space) and their generalizations are given in terms of finite differences ([3]-[5]). In the "wavelet language" finite differences are replaced by wavelet measures. So, in this study flett potentials space is defined and a characterization of this space is given by making use of wavelet measure.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

The authors contributed equally.

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References

[1] Stein, E. (1970). *Singular Integrals and Differentiability Properties of Functions*. Princeton, N.J.: Princeton Univ. Press.

[2] Lizorkin, P. I. (1970). Characterization of the spaces $L_p^r(R^n)$ in terms of difference singular integrals. *Mat. Sb. (N.S.), 81(1)*, 79-91.

[3] Samko, S. G. (2002). Hiypersingular integrals and their applications. *An International Series of Monographs in Mathematics, 5*.

[4] Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach Science Publishers.

[5] Rubin, B. (1996). Fractional integrals and potentials. *Pitman Monographs and Surveys in Pure and Applied Mathematics, 82*.

[6] Rubin, B. (1987). Inversion of potentials on $\mathbb{R}^n$ with the aid of Gauss-Weierstrass integrals. *Math. Notes, 41(1-2)*, 22-27.

[7] Almeida, A., & Samko, S. G. (2006). Characterization of Riesz and Bessel potentials on variable Lebesgue spaces. *Journal of Function Spaces and Applications, 4(2)*, 1-26.

[8] Hu, J., & Zahle, M. (2005). Potential spaces on fractals. *Stud. Math., 170*, 259-281.

[9] Nogin, V. A., & Samko, S. G. (1999). Some applications of potentials and approximative inverse operators in multidimensionel fractional calculus. *Fractional Calculus and Appl. Analysis, 2(2)*, 205-228.

[10] Rubin, B. (1999). Fractional integrals and wavelet transforms associated with Blaschke-Levy representations on the sphere. *Israel Journal of Math., 114*, 1-27.

[11] Aliev, I. A., & Rubin, B. (2001). Parabolic potentials and wavelet transforms with the generalized translations. *Stud. Math., 145*, 1-16.

[12] Aliev, I. A., & Rubin, B. (2005). Wavelet-like transforms for admissible semi-groups; Inversion formulas for potentials and Radon transforms. *Journal of Fourier Anal. and Appl., 11*, 333-352.

[13] Aliev, I. A., & Eryigit, M. (2002). Inversion of Bessel potentials with the aid of weighted wavelet transforms. *Math. Nachr.*, *242*, 27-37.

[14] Aronszajn, N., & Smith, K. T. (1961). Theory of Bessel potentials I. *Ann. Inst. Fourier, 11*, 385-475.

[15] Calderón, A. P. (1961). Lebesgue spaces of differentiable functions and distributions. *Proceedings of the Symp. in Pure Math., 5*, 33-49.

[16] Sampson, C. H. (1968). A characterization of parabolic Lebesque spaces. Dissertation, Rice Univ.

[17] Nogin, V. A., & Rubin, B. (1987). The spaces $L_{p,r}^{\alpha}(R^{n+1})$ of parabolic potentials. *Anal. Math., 2(13)*, 321-338.

[18] Samko, S. G. (1977). The spaces $L_{p,r}^{\alpha}(R^n)$ and hypersingular integrals (Russian). *Stud. Math. (PRL), 61(3)*, 193-230.

[19] Aliev, I. A. (2009). Bi-parametric potentials, relevant function spaces and wavelet-like transforms. *Integral Equations and Operator Theory, 65*, 151-167.
[20] Sezer, S. & Aliev, I. A. (2005). On space of parabolic potentials associated with the singular heat operator. *Turkish Journal of Mathematics, 29***,** 299-314.
[21] Sezer, S. & Aliev, I. A. (2010). A new characterization of the Riesz potential spaces with the aid of a composite wavelet transform. *Journal of Mathematical Analysis and Applications*, *372***,** 549-558.
[22] Flett, T. M. (1971). Temperatures, Bessel potentials and Lipschitz space. *London Math. Soc., 22(3)*, 385-451.
[23] Stein, E. M. & Weiss, G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton, N.J.: Princeton Univ. Press.
[24] Aliev, I. A., Sezer, S., & Eryigit, M. (2006). An integral transform associated to the Poisson integral and inversion of Flett potentials. *Journal of Mathematical Analysis and Applications, 321*, 691-704.
[25] Gradshteyn, I. S., & Ryzhik, I. M. (1994). *Table of Integrals, Series and Products* (5h ed.). Academic Press.

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