Hyers-Ulam Stability of Euler-Lagrange-Jensen *k*-Cubic Functional Equations on Generalized Non-Archimedean Normed Spaces

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Abstract: One of the essential researches in mathematics is the study of functional equations, which originated in problems related to applied mathematics. Among those was the question concerning a general solution and a stability result of the functional equation, exceptionally fundamental equation in the theory of functional equations related to the Cauchy functional equation or additive functional equation. Nowadays, the number of mathematics papers concerning the stability results for many functional equations on normed spaces and non-Archimedean normed spaces via the direct method is still increasing. The fixed point theorem is an innovative tool to prove the stability result of the functional equation. Moreover, the stability result of the Euler-Lagrange-Jensen *k*-cubic functional equation was investigated for a mapping from a normed space to a Banach space. In this work, we prove the Hyers-Ulam stability for the Euler-Lagrange-Jensen *k*-cubic functional equation on generalized non-Archimedean normed spaces by using the fixed point method. Also, we can obtain the stability results of the Euler-Lagrange-Jensen *k*-cubic functional equation in the sense of Banach space from our main results.

Key words: Cubic functional equation, Euler-Lagrange-Jensen functional equation, Hyers-Ulam stability, non-Archimedean normed spaces.

1. Introduction

An equation in which the unknown variable as a function is called a functional equation. The most famous functional equation is Cauchy's functional equation or the additive functional equation [1] which is of the form

$$f(x+y)=f(x)+f(y) \tag{1}$$

for all $x,y\in\mathbb{R}$, where f maps from $\mathbb{R}$ into $\mathbb{R}$, and any solutions of (1) are called Cauchy's function or additive function. It is easy to know that the general solution of (1) is given by $f(x)=ax$ for all $x\in\mathbb{R}$, where a is an arbitrary real constant. The problem of functional equations consists of finding a general solution, and investigating its stability. In 1940, Ulam [2] gave a stability problem of homomorphisms as follows:

Let f be a mapping from a group $(G_1,\bullet)$ into a metric group $(G_2,\circ)$ with the metric d such that

$$d(f(x\bullet y), f(x)\circ f(y))\le \varepsilon$$

for all $x,y\in G_1$, where $\varepsilon>0$. Do there exist $\delta>0$ and a unique homomorphism $h:G_1\to G_2$ such that

$$d(f(x),h(x))\le \delta$$

for all $x\in G_1$?

If there is an answer of the above question, the Cauchy's functional equation is said to be stable. In 1941, Hyers [3] had shown answer the question of Ulam [2] by considering the stability problem of the Cauchy's functional equation on Banach spaces.

The result of Hyers [3] is extensively elevation the research in this way, which is called Hyers-Ulam stability of functional equations. A generalization of the result of Hyers [3] was proved by Aoki [4] by replacing the unbounded additive difference with the sum of powers of norms. Moreover, the stability result of Aoki [4] was extended by Gajda [5] and Găvruta [6], respectively. Furthermore, the stability result of Cauchy's functional equation was investigated in various spaces.

On the other hand, Rassias [7] introduced the cubic functional equation which is of the form

$$f(x+2y)=3f(x+y)+f(x-y)-3f(x)+6f(y), \tag{2}$$

where f is a mapping between two real vector spaces, and its stability was also proved. In 2016, Mohiuddine, Rassias, and Alotaibi [8] introduced a new cubic functional equation named Euler-Lagrange-Jensen *k*-cubic functional equation

$$k\left[f(kx+y)+f(x+ky)\right]+(k-1)^3\left[f\left(\frac{kx-y}{k-1}\right)+f\left(\frac{x-ky}{1-k}\right)\right]=\left(k^4-1\right)\left[f(x)+f(y)\right]+8k(k^2+1)f\left(\frac{x+y}{2}\right), \tag{3}$$

where $k\in\mathbb{R}$ with $m:=k+1\ne 0,\pm 1$ and f maps between two normed spaces. They also proved Hyers-Ulam stability of the Euler-Lagrange-Jensen *k*-cubic functional equation (3).

In this paper, we prove the generalized Hyers-Ulam stability of the Euler-Lagrange-Jensen *k*-cubic functional equation (3), where f maps from a normed space into a non-Archimedean quasi-β-Banach space by using the fixed point method.

2. Preliminaries

In this section, we recall some important definitions and the well-known fixed point theorem which is the main tool for proving our main results.

Definition 2.1 ([9]). Let F be a field. A non-Archimedean valuation on F is a function $|\cdot|:F\to[0,\infty)$ satisfying the following conditions for all $x,y\in F$:

1) $|x|=0$ if and only if $x=0$;
2) $|xy|=|x||y|$;
3) $|x+y|\le\max\{|x|,|y|\}$.

Also, a field F with a non-Archimedean valuation on F is called a non-Archimedean field.

Definition 2.2 ([9]). Let X be a vector space over a scalar filed F with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|: X \to [0,\infty)$ is called a non-Archimedean norm if it satisfies the following conditions:

1) $\|x\| = 0$ if and only if $x = 0$;
2) $\|rx\| = |r|\|x\|$ for all $r \in F$ and all $x \in X$;
3) $\|x + y\| \le \max\{\|x\|, \|y\|\}$ for all $x, y \in X$.

Also, $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Next, we give the definition of a non-Archimedean quasi-β-normed space.

Definition 2.3 ([10]). Let X be a vector space over a non-Archimedean field F with a non-Archimedean non-trivial valuation $|\cdot|$ and β be a real number with $0 < \beta \le 1$. A function $\|\cdot\|: X \to [0,\infty)$ is called a non-Archimedean quasi-β-norm if it satisfies the following conditions for all $x, y \in X$:

1) $\|x\| = 0$ if and only if $x = 0$;
2) $\|rx\| = |r|^{\beta}\|x\|$ for all $r \in F$;
3) $\|x + y\| \le \max\{\|x\|, \|y\|\}$.

Also, $(X, \|\cdot\|)$ is called a non-Archimedean quasi-β-normed space.

Definition 2.4 ([10]). The sequence $\{x_n\}$ in a non-Archimedean quasi-β-norm $(X, \|\cdot\|)$ is called convergent if for a given $\varepsilon > 0$ there are a positive integer N and $x \in X$ such that $\|x_n - x\| \le \varepsilon$ for all $n \ge N$ and x is called a limit of the sequence $\{x_n\}$, denote by $\lim_{n\to\infty} x_n = x$.

Definition 2.5 ([10]). The sequence $\{x_n\}$ in a non-Archimedean quasi-β-norm $(X, \|\cdot\|)$ is called Cauchy if for a given $\varepsilon > 0$ there is a positive integer N such that $\|x_n - x_m\| \le \varepsilon$ for all $m, n \ge N$.

Definition 2.6 ([10]). A non-Archimedean quasi-β-normed space is complete if every Cauchy sequence is convergent. A non-Archimedean quasi-β-Banach space is a complete non-Archimedean quasi-β-normed space.

In the proof the stability problem of the functional equation (3), we apply the following fixed point theorem.

Theorem 2.7 ([11]). Let (X, d) be a complete generalized metric space and let $J: X \to X$ be a strictly contractive mapping with some Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that the following assertions hold:

1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \ge n_0$;
2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

3. Main Results

In this section, we will investigate the stability results of the functional equation (3). First, we give the following lemma to guarantee some consequence in our main theorem.

Lemma 3.1 ([8]). Let X and Y be two vector spaces and $f: X \to Y$ be a mapping satisfying (3). Then we have

$$f(x)=m^3 f\left(\frac{x}{m}\right)$$

for all $x\in X$, where $m:=k+1\neq 0,\pm 1.$

For a normed space X and a non-Archimedean quasi-β-Banach space Y a real number k with $m:=k+1\neq 0,\pm 1,\quad x,y\in X$, we will use the following symbol:

$$Df(x,y):=k\left[f(kx+y)+f(x+ky)\right]+(k-1)^3\left[f\left(\frac{kx-y}{k-1}\right)+f\left(\frac{x-ky}{1-k}\right)\right]-\left(k^4-1\right)\left[f(x)+f(y)\right]-8k(k^2+1)f\left(\frac{x+y}{2}\right).$$

Next, we will consider the stability problem of the Euler-Lagrange-Jensen k-cubic functional equation (3) in non-Archimedean quasi-β-normed spaces by using the fixed point method.

Theorem 3.2. Let X be a normed space, Y be a non-Archimedean quasi-β-Banach space, $k\in\mathbb{R}$ with $m:=k+1\neq 0,\pm 1,$ and $\phi: X^2\to[0,\infty)$ be a function such that

$$\phi(mx,my)\leq L\phi(x,y) \tag{4}$$

for some real number L with $0\leq L<|m|^{3\beta}$ and for all $x,y\in X$. Suppose that $f:X\to Y$ is a mapping such that

$$\|Df(x,y)\|\leq\phi(x,y) \tag{5}$$

for all $x,y\in X$. Then there exists a unique cubic mapping $C:X\to Y$ satisfying (3) such that

$$\|f(x)-C(x)\|\leq\frac{\phi(x,x)}{2^\beta|m-1|^\beta\left(|m|^{3\beta}-L\right)} \tag{6}$$

for all $x\in X.$

Proof. Let $\Omega:=\{f:X\to Y\}$. Define a generalized metric d on Ω by

$$d(f,g)=\inf\left\{c>0:\|f(x)-g(x)\|\leq c\phi(x,x)\text{ for all } x\in X\right\}$$

for all $f,g\in\Omega$. Since Y is a non-Archimedean quasi-β-Banach space, we obtain (Ω,d) is a generalized complete metric space. Substituting y by x in (5), we obtain

$$2^\beta|k|^\beta\left\|f((k+1)x)-(k^3+3k^2+3k+1)f(x)\right\|\leq\phi(x,x) \tag{7}$$

and so

$$\left\|\frac{f(mx)}{m^3}-f(x)\right\|\leq\frac{\phi(x,x)}{2^\beta|m|^{3\beta}|m-1|^\beta} \tag{8}$$

for all $x\in X$. Define a map $T:\Omega\to\Omega$ by

$$(Tf)(x)=\frac{f(mx)}{m^3}$$

for all $x\in X$ and for all $f,g\in\Omega$. Next, we will show that

$$d(Tf,Tg)\leq\frac{L}{|m|^{3\beta}}d(f,g) \tag{9}$$

for all $f,g\in\Omega$. Let $f,g\in\Omega$. If $d(f,g)=\infty,$ then the inequality (9) is true. So we may assume that $d(f,g)<\infty.$ Assume that

$$A:=\left\{c>0\,\|f(x)-g(x)\|\leq c\phi(x,x)\text{ for all }x\in X\right\}.$$

Since $d(f,g)<\infty,$ we obtain $A\neq\varnothing.$ Suppose that $c\in A.$ For each $x\in X,$ we get

$$\begin{aligned}\|(Tf)(x)-(Tg)(x)\|&=\left\|\frac{f(mx)}{m^3}-\frac{g(mx)}{m^3}\right\|\\&=\frac{1}{|m|^{3b}}\|f(mx)-g(mx)\|\\&\leq\frac{cL}{|m|^{3\beta}}\phi(mx,mx)\\&\leq\frac{cL}{|m|^{3\beta}}\phi(x,x)\end{aligned}$$

and thus

$$d(Tf,Tg)\leq\frac{cL}{|m|^{3\beta}}.$$

By taking the infimum on $c\in A,$ we have

$$d(Tf,Tg)\leq\frac{L}{|m|^{3\beta}}d(f,g)$$

for all $f,g\in\Omega$. From (8), it yields that

$$d(Tf,f)\leq\frac{1}{2^{\beta}|m|^{3\beta}|m-1|^{\beta}}. \tag{10}$$

Hence, by Theorem 2.7, there is the unique fixed point C of T in Ω such that $\{T^n f\}$ converges to C in (Ω,d) and

$$d(C,f)\leq\left(\frac{1}{1-|m|^{-3\beta}L}\right)d(Tf,f).$$

From (10), we obtain

$$d(C,f) \le \left(\frac{1}{1-|m|^{-3\beta} L} \right) \frac{1}{2^{\beta}|m|^{3\beta}|m-1|^{\beta}}$$
$$\le \frac{1}{2^{\beta}|m-1|^{\beta}\left(|m|^{3\beta}-L\right)}. \tag{11}$$

By (5), we get

$$\left\| \frac{Df(m^n x, m^n y)}{m^{3n}} \right\| = \frac{1}{|m|^{3n\beta}} \left\| Df(m^n x, m^n y) \right\|$$
$$\le \frac{1}{|m|^{3n\beta}} \phi(m^n x, m^n y)$$
$$\le \frac{L^n}{|m|^{3n\beta}} \phi(x,y)$$

for all $x, y \in X$. Taking the limit as $n \to \infty$ in the above inequality, we obtain

$$DC(x,y) = 0$$

for all $x, y \in X$ and so C is satisfying (3). By Lemma 3.1, we get C is a cubic mapping. From (11), we obtain

$$\| f(x) - C(x) \| \le \frac{\phi(x,x)}{2^{\beta}|m-1|^{\beta}\left(|m|^{3\beta}-L\right)} \tag{12}$$

for all $x \in X$.

To prove the uniqueness of a mapping C, assume that there is a cubic mapping $C' : X \to Y$ satisfying (3) and (6). Since $C(mx) = m^3 C(x)$ and $C'(mx) = m^3 C'(x)$ for all $x \in X$. By using (4) and (5), we obtain

$$\|C(x) - C'(x)\| = \frac{1}{|m|^{3n\beta}} \left\| C(m^n x) - f(m^n x) - C'(m^n x) + f(m^n x) \right\|$$
$$\le \frac{1}{|m|^{3n\beta}} \max\left\{ \left\| C(m^n x) - f(m^n x) \right\|, \left\| C'(m^n x) - f(m^n x) \right\| \right\}$$
$$\le \frac{1}{|m|^{3n\beta}} \cdot \frac{\phi(m^n x, m^n x)}{2^{\beta}|m-1|^{\beta}\left(|m|^{3\beta}-L\right)}$$
$$\le \left(\frac{L}{|m|^{3\beta}} \right)^n \frac{\phi(x,x)}{2^{\beta}|m-1|^{\beta}\left(|m|^{3\beta}-L\right)}$$

for all $x \in X$. By taking the limit as $n \to \infty$ in the above inequality, the right-hand side converges to zero and so $C(x) = C'(x)$ for all $x \in X$. It yields that $C = C'$ verifying the uniqueness of C. This completes the proof.

Next, we give the following stability result of the Euler-Lagrange-Jensen *k*-cubic functional equation (3), which is proved by using the similar technique in Theorem 3.2.

Theorem 3.3. Let X be a normed space, Y be a non-Archimedean quasi-β- Banach space, $k \in \mathbb{R}$ with $m = k+1 \neq 0, \pm 1$ and $\phi : X^2 \to [0,\infty)$ be a function such that

$$\phi\left(\frac{x}{m}, \frac{y}{m}\right) \leq \frac{L}{|m|^{3\beta}} \phi(x,y) \tag{13}$$

for some real number L with $0 \leq L < |m|^{3\beta}$ and for all $x, y \in X$. Suppose that $f : X \to Y$ is a mapping such that

$$\|Df(x,y)\| \leq \phi(x,y) \tag{14}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying (3) such that

$$\| f(x) - C(x)\| \leq \frac{L\phi(x,x)}{2^{\beta} |m|^{3\beta} |m-1|^{\beta} (1-L)} \tag{15}$$

for all $x \in X$.

From Theorems 3.2-3.3, we can choose a suitable control function ϕ to apply in many results.

4. Conclusion

In this work, we proved the Hyers-Ulam stability for the Euler-Lagrange-Jensen *k*-cubic functional equation on generalized non-Archimedean normed spaces by using the fixed point method. Moreover, we can choose a suitable control function ϕ from the main stability theorems to apply in many results.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

The full manuscript was written by AT. WS had approved to the final version of the manuscript.

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