Generating Functions for the Extended Multivariable Fourth Type Horn Functions

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Abstract: Recently, there are many interested studies including various families of multiple hypergeometric functions and extended hypergeometric functions. Here, we will study on extended multivariable fourth type Horn functions which are both extended and multiple hypergeometric functions. In this article, we establish some generating functions for the extended multivariable fourth type Horn functions and then obtain a family of multilinear and multilateral generating functions for each of them. We also give special cases of the results presented in this paper.

Key words: Multiple hypergeometric functions, extended multivariable fourth type Horn functions, horn functions, generating functions, multilinear and multilateral generating functions.

1. Introduction

The extended multivariable hypergeometric functions (extensions of Appell and Lauricella functions) are generalized with two extra parameters and the extended beta function with $\Theta(\{\kappa_l\}_{l\in \mathrm{N}_0};z)$ (see [1]). Using the same method, the extended fourth type Horn, the extended multivariable fourth type Horn and the extended multivariable hypergeometric functions were defined in our recent papers [2], [3].

In this study, the extended fourth type Horn and multivariable fourth type Horn functions, which introduced below, has been used for obtaining new generating functions.

Definition 1.1. *Let a function* $\Theta(\{\kappa_l\}_{l\in \mathrm{N}_0};z)$ *be analytic within the disk* $|z|<R$ $(0<R<\infty)$ *and let its Taylor-Maclaurin coefficients be explicitly denoted by sequence* $\{\kappa_l\}_{l\in \mathrm{N}_0}$ *. Suppose also that the function* $\Theta(\{\kappa_l\}_{l\in \mathrm{N}_0};z)$ *can be continued analytically in the right half-plane* $\mathrm{Re}(z)>0$ *with the asymptotic property given as follows [1]:*

$$\Theta(\kappa_l;z)\equiv\Theta(\{\kappa_l\}_{l\in \mathrm{N}_0};z)=\begin{cases}\sum\kappa_l \frac{z^l}{l!} & (|z|<R;\ 0<R<\infty;\ \kappa_0=1)\\ M_0 z^w \exp(z)\left[1+O\left(\frac{1}{z}\right)\right] & (\mathrm{Re}(z)\to\infty;\ M_0>0;\ w\in \mathrm{C})\end{cases} \tag{1}$$

for some suitable constants M_0 *and* w *depending essentially on the sequence* $\{\kappa_l\}_{l\in\mathbb{N}_0}$.

The extended multivariable fourth type Horn functions are defined as follows (see [2]):

$$ {}^{(k)}H^{(r)}_{4,\left(\{\kappa_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\alpha,\beta_{k+1},...,\beta_r;\gamma_1,...,\gamma_r;x_1,...,x_k,x_{k+1},...,x_r\right) $$

$$:= \sum_{m_1,...,m_r=0}^{\infty} \frac{(\alpha)_{2(m_1+...+m_k)+m_{k+1}+...+m_r}}{(\gamma_1)_{m_1}...(\gamma_k)_{m_k}} \prod_{j=k+1}^{r} \frac{B_{p,q}^{\left(\{\kappa_l\}_{l\in\mathbb{N}_0}\right)}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \tag{2} $$

$$ \left(\left(2(\sqrt{|x_1|}+...+\sqrt{|x_k|})+...+|x_r|<1\right),\ \min\{\operatorname{Re}(p),\operatorname{Re}(q)\}\geq 0\right), $$

where the extended beta function $B_{p,q}^{\left(\{\kappa_l\}_{l\in\mathbb{N}_0}\right)}(\alpha,\beta)$ is given by (see [1])

$$ B_{p,q}^{\left(\{\kappa_l\}_{l\in\mathbb{N}_0}\right)}(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\ \Theta\left(\kappa_l;-\tfrac{p}{t}-\tfrac{q}{(t-1)}\right)dt, $$
$$ (\min\{\operatorname{Re}(\alpha),\operatorname{Re}(\beta)\}>0,\ \ \min\{\operatorname{Re}(p),\operatorname{Re}(q)\}\geq 0). $$

When $\kappa_l=\frac{(\rho)_l}{(\sigma)_l}, \rho=\sigma$ and $p=q=0$ in (2), then (2) reduces to multivariable Horn functions [4].

If we take $k=1,\ r=2$, in (2), then function (2) reduces to extended fourth type Horn functions defined by [2]:

$$ H_4^{\left(\{\kappa_l\}_{l\in\mathbb{N}_0};p,q\right)}(\alpha,\beta;\gamma_1,\gamma_2;x_1,x_2):=\sum_{m,r=0}^{\infty}\frac{(\alpha)_{2m+r}}{(\gamma_1)_m}\frac{B_{p,q}^{\left(\{\kappa_l\}_{l\in\mathbb{N}_0}\right)}(\beta+r,\gamma_2-\beta)}{B(\beta,\gamma_2-\beta)}\frac{x_1^m}{m!}\frac{x_2^r}{r!}, \tag{3} $$

$$ \left(\left(2\sqrt{|x_1|}+|x_2|<1\right),\ \min\{\operatorname{Re}(p),\operatorname{Re}(q)\}\geq 0\right). $$

When $\kappa_l=\frac{(\rho)_l}{(\sigma)_l}, \rho=\sigma$ and $p=q=0$ in (3), then (3) reduces to Horn functions [5], [6].

The aim of this article is to obtain generating functions for the extended multivariable fourth type Horn functions. Then, we derive various families of multilinear and multilateral generating functions for these functions and their special cases are also given.

2. Generating Functions

In this section, we give two generating functions for the extended multivariable fourth type Horn functions.

Theorem 2.1. *We have the following generating function for the extended multivariable fourth type Horn functions given by (2):*

$$ \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}^{(k)}H^{(r)}_{4,\left(\{\kappa_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\alpha,\beta_{k+1},...,\beta_r;1-\lambda-n,\gamma_2,...,\gamma_r;x_1,...,x_r\right)t^n $$
$$ =(1-t)^{-\lambda}\,{}^{(k)}H^{(r)}_{4,\left(\{\kappa_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\alpha,\beta_{k+1},...,\beta_r;1-\lambda,\gamma_2,...,\gamma_r;x_1(1-t),x_2...,x_r\right), \tag{4} $$

where $\lambda \in C$ *and* $|t|<1.$

Proof. Let T denote the first member of assertion (4). Then,

$$T=\sum_{n=0}^{\infty}\sum_{m_1,\dots,m_r=0}^{\infty}(\lambda)_n\frac{(\alpha)_{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r}}{(1-\lambda-n)_{m_1}(\gamma_2)_{m_2}\dots(\gamma_k)_{m_k}}\prod_{j=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathrm{N}_0}\right)}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{(x_1)^{m_1}}{m_1!}\dots\frac{(x_r)^{m_r}}{m_r!}\frac{t^n}{n!}$$

$$=\sum_{m_1,\dots,m_r=0}^{\infty}\frac{(\alpha)_{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r}}{(1-\lambda)_{m_1}(\gamma_2)_{m_2}\dots(\gamma_r)_{m_r}}\prod_{j=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathrm{N}_0}\right)}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{(x_1)^{m_1}}{m_1!}\dots\frac{(x_r)^{m_r}}{m_r!}\sum_{n=0}^{\infty}\frac{(\lambda-m_1)_n}{n!}t^n$$

$$=(1-t)^{-\lambda}\sum_{m_1,\dots,m_r=0}^{\infty}\frac{(\alpha)_{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r}}{(1-\lambda)_{m_1}(\gamma_2)_{m_2}\dots(\gamma_r)_{m_r}}\prod_{j=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathrm{N}_0}\right)}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{\left(x_1(1-t)\right)^{m_1}}{m_1!}\dots\frac{(x_r)^{m_r}}{m_r!}$$

$$=(1-t)^{-\lambda}\,{}^{(k)}H_{4,\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}^{(r)}\left(\alpha,\beta_{k+1},\dots,\beta_r;1-\lambda,\gamma_2,\dots,\gamma_r;x_1(1-t),x_2\dots,x_r\right),$$

which completes the proof.

Corollary 2.1. *If we choose* $k=1$, $r=2$ *in Theorem 2.1, we have the following relation for the extended fourth kind Horn functions:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}H_4^{\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}\left(\alpha,\beta;1-\lambda-n,\gamma_2;x_1,x_2\right)t^n=(1-t)^{-\lambda}\,H_4^{\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}\left(\alpha,\beta;1-\lambda,\gamma_2;x_1(1-t),x_2\right)$$

where $\lambda\in\mathrm{C}$ *and* $|t|<1.$

In the next theorem, let $\Psi_m^{\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}$ denote, the following special functions:

$$\Psi_m^{\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}(x_1,\dots,x_r)={}^{(k)}H_{4,\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}^{(r)}\left(\alpha,\beta_{k+1},\dots,\beta_r;1-\lambda-m,\gamma_2,\dots,\gamma_r;x_1,\dots,x_r\right).$$

Theorem 2.2. *We have the following generating function for the extended multivariable fourth type Horn functions given by (2):*

$$\sum_{n=0}^{\infty}\binom{\lambda+m+n-1}{n}\Psi_{m+n}^{\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}\left(x_1,\dots,x_r\right)t^n=(1-t)^{-\lambda-m}\,\Psi_m^{\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}\left(x_1(1-t),x_2,\dots,x_r\right).\qquad(5)$$

Proof. Let T denote the first member of assertion (5). Then,

$$T=\sum_{n=0}^{\infty}\binom{\lambda+m+n-1}{n}\Psi_{m+n}^{\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}\left(x_1,\dots,x_r\right)t^n$$

$$=\sum_{n=0}^{\infty}\binom{\lambda+m+n-1}{n}{}^{(k)}H_{4,\left(\{K_l\}_{l\in\mathrm{N}_0};p,q\right)}^{(r)}\left(\alpha,\beta_{k+1},\dots,\beta_r;1-\lambda-m-n,\gamma_2,\dots,\gamma_r;x_1,\dots,x_r\right)t^n.$$

By using Theorem 2.1, we observe

$$T=(1-t)^{-\lambda-m}\ {}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in\mathbb{N}_0};\,p,q\right)}\left(\alpha,\beta_{k+1},\dots,\beta_r;1-\lambda-m,\gamma_2,\dots,\gamma_r;x_1\left(1-t\right),x_2,\dots,x_r\right)$$
$$=(1-t)^{-\lambda-m}\ \Psi_m^{\left(\{K_l\}_{l\in\mathbb{N}_0};\,p,q\right)}\left(x_1\left(1-t\right),x_2,\dots,x_r\right),$$

which completes the proof.

3. Multilinear and Multilateral Generating Functions

In this section, we derive several families of bilinear and bilateral generating functions for the extended multivariable fourth type Horn functions by using the similar method considered in [2], [3], [7].

Theorem 3.1. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,\dots,y_r)$ *of* r *complex variables* $y_1,\dots,y_r$ $(r\in N)$ *and of complex order* μ, *let*

$$\Lambda_{m,d}(x_1,\dots,x_r;y_1,\dots,y_r;t):=\sum_{n=0}^{\infty}a_n\ \Psi_{m+nd}^{\left(\{K_l\}_{l\in\mathbb{N}_0};\,p,q\right)}(x_1,\dots,x_r)\Omega_{\mu+bn}(y_1,\dots,y_r)t^n,$$

where $(a_n\neq 0,\ \mu\in C)$ and $\Psi_{m+nd}^{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}$ is defined by (5) and

$$N_{n,m,d}^{b,\mu}(y_1,\dots,y_r;z):=\sum_{k=0}^{[n/d]}a_k\Omega_{\mu+bk}(y_1,\dots,y_r)z^k.$$

Then, for every nonnegative integer m, we have

$$\sum_{n=0}^{\infty}\binom{\lambda+m+n-1}{n}\Psi_{m+n}^{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}(x_1,\dots,x_r)N_{n,m,d}^{b,\mu}(y_1,\dots,y_r;z)t^n$$
$$=(1-t)^{-\lambda-m}\Lambda_{m,d}\left(x_1\left(1-t\right),x_2,\dots,x_r;y_1,\dots,y_r;\frac{zt^d}{\left(1-t\right)^d}\right), \tag{6}$$

provided that each member of (6) exists.

Proof. For convenience, let S denote the first member of the assertion (6). Then, we may write that

$$T=\sum_{n=0}^{\infty}\binom{\lambda+m+n-1}{n}\ \Psi_{m+n}^{\left(\{K_l\}_{l\in\mathbb{N}_0};\,p,q\right)}(x_1,\dots,x_r)\sum_{k=0}^{[n/d]}a_k\Omega_{\mu+bk}(y_1,\dots,y_r)z^kt^n$$

$$=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\binom{\lambda+m+n+dk-1}{n}\ \Psi_{m+n+dk}^{\left(\{K_l\}_{l\in\mathbb{N}_0};\,p,q\right)}(x_1,\dots,x_r)t^n\right)a_k\Omega_{\mu+bk}(y_1,\dots,y_r)\left(zt^d\right)^k$$

$$=\sum_{k=0}^{\infty}(1-t)^{-\lambda-m-dk}\,\Psi_{m+dk}^{\left(\{K_l\}_{l\in N_0};p,q\right)}\left(x_1(1-t),x_2,\dots,x_r\right)a_k\Omega_{\mu+bk}(y_1,\dots,y_r)\left(zt^d\right)^k$$

$$=(1-t)^{-\lambda-m}\sum_{k=0}^{\infty}a_k\Psi_{m+dk}^{\left(\{K_l\}_{l\in N_0};p,q\right)}\left(x_1(1-t),x_2,\dots,x_r\right)\Omega_{\mu+bk}(y_1,\dots,y_r)\left(\frac{zt^d}{(1-t)^d}\right)^k$$

$$=(1-t)^{-\lambda-m}\,\Lambda_{m,d}\left(x_1(1-t),x_2,\dots,x_r;y_1,\dots,y_r;\frac{zt^d}{(1-t)^d}\right),$$

which completes the proof.

In a similar manner, we also get the next result, immediately.

Theorem 3.2. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,\dots,y_s)$ *of* s *complex variables* $y_1,\dots,y_s$ $(s\in N)$ *and of complex order* μ, *let*

$$\Lambda_{\mu,\psi}(y_1,\dots,y_s;\zeta):=\sum_{l=0}^{\infty}a_l\,\Omega_{\mu+\psi l}(y_1,\dots,y_s)\zeta^l,$$

where $a_l\neq 0$, $\mu,\psi\in C$ *and*

$$\Theta_{n,b}^{\mu,\psi}\left(x_1,\dots,x_r;y_1,\dots,y_s;\xi\right)$$

$$:=\sum_{l=0}^{[n/b]}a_l\,(\lambda)_{n-bl}\;{}^{(k)}H_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\alpha,\beta_{k+1},\dots,\beta_r;1-\lambda-n+bl,\gamma_2,\dots,\gamma_r;x_1,\dots,x_r\right)$$
$$\times\Omega_{\mu+\psi k}(y_1,\dots,y_s)\tfrac{\xi^l}{(n-bl)!}.$$

Then, for $b\in N$, *we have*

$$\sum_{n=0}^{\infty}\Theta_{n,b}^{\mu,\psi}\left(x_1,\dots,x_r;y_1,\dots,y_s;\tfrac{\eta}{t^b}\right)t^n=\Lambda_{\mu,\psi}(y_1,\dots,y_s;\eta)(1-t)^{-\lambda}$$
$$\times{}^{(k)}H_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\alpha,\beta_{k+1},\dots,\beta_r;1-\lambda,\gamma_2,\dots,\gamma_r;x_1(1-t),x_2,\dots,x_r\right)\tag{7}$$

provided that each member of (7) exists.

4. Special Cases

In this section, we will show the applications for the theorems given above. When the multivariable function $\Omega_\mu(y_1,\dots,y_s)$, $l\in N_0$, $s\in N$ is expressed in terms of several simpler functions of one and more variables, we shall be led to an interesting class of multilateral generating functions for the extended multivariable fourth type polynomials considered and, of course, for the extended fourth type Horn functions when $k=1$, $r=2$. We first set

$$s=r,\quad \Omega_{\mu+\psi l}(y_1,\ldots,y_r)={}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0};p,q\right)}(\lambda+\mu+\psi l,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;y_1,\ldots,y_r)$$

in Theorem 3.2, where the extended multivariable hypergeometric functions ${}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0};p,q\right)}$ generated by [3]:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\lambda+n,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;x_1,\ldots,x_r\right)t^n$$
$$=(1-t)^{-\lambda}\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\lambda,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;\frac{x_1}{(1-t)^{\rho}},\ldots,\frac{x_k}{(1-t)^{\rho}},\frac{x_{k+1}}{(1-t)},\ldots,\frac{x_r}{(1-t)}\right). \tag{8}$$

We are thus led to the following result which provides a class of bilateral generating functions for the extended multivariable fourth type Horn functions and the extended multivariable hypergeometric functions.

Corollary 4.1. *If*

$$\Lambda_{\mu,\psi}(y_1,\ldots,y_r;\xi):=\sum_{l=0}^{\infty}a_l\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0};p,q\right)}(\lambda+\mu+\psi l,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;y_1,\ldots,y_r)\xi^l$$
$$(a_l\neq 0,\ \mu,\psi\in C)$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{l=0}^{[n/b]}a_l\,(\lambda)_{n-bl}\;{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\alpha,\beta_{k+1},\ldots,\beta_r\\ 1-\lambda-n+bl,\gamma_2,\ldots,\gamma_r\end{matrix};x_1,\ldots,x_r\right)$$
$$\times\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+\mu+\psi l,\beta_{k+1},\ldots,\beta_r\\ \gamma_1,\ldots,\gamma_r\end{matrix};y_1,\ldots,y_r\right)\frac{\eta^l\,t^{n-bl}}{(n-bl)!}$$
$$=\Lambda_{\mu,\psi}(y_1,\ldots,y_r;\eta)(1-t)^{-\lambda}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\alpha,\beta_{k+1},\ldots,\beta_r;1-\lambda,\gamma_2,\ldots,\gamma_r;x_1(1-t),x_2,\ldots,x_r\right) \tag{9}$$

provided that each member of (9) exists.

Remark 4.1. *Using the generating relation (8) for the extended multivariable hypergeometric functions and getting* $a_l=\dfrac{(\lambda)_l}{l!},\ \mu=0,\ \psi=1$ *in Corollary 4.1, we find that*

$$\sum_{n=0}^{\infty}\sum_{l=0}^{[n/b]}(\lambda)_l\,(\lambda)_{n-bl}\;{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\alpha,\beta_{k+1},\ldots,\beta_r\\ 1-\lambda-n+bl,\gamma_2,\ldots,\gamma_r\end{matrix};x_1,\ldots,x_r\right)$$
$$\times\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+l,\beta_{k+1},\ldots,\beta_r\\ \gamma_1,\ldots,\gamma_r\end{matrix};y_1,\ldots,y_r\right)\frac{\eta^l\,t^{n-bl}}{l!(n-bl)!}$$

$$=(1-t)^{-\lambda}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\alpha,\beta_{k+1},\ldots,\beta_r\\ 1-\lambda,\gamma_2,\ldots,\gamma_r\end{matrix};x_1(1-t),x_2,\ldots,x_r\right)$$
$$\times(1-\eta)^{-\lambda}\,{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda,\beta_{k+1},\ldots,\beta_r\\ \gamma_1,\ldots,\gamma_r\end{matrix};\frac{y_1}{(1-\eta)^{\rho}},\ldots,\frac{y_k}{(1-\eta)^{\rho}},\frac{y_{k+1}}{(1-\eta)},\ldots,\frac{y_r}{(1-\eta)}\right).$$

On the other hand, we set $\Omega_{\mu+\psi l}(y) = {}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\alpha,\beta_{k+1},\ldots,\beta_r;1-\lambda-(\mu+\psi l),\gamma_2,\ldots,\gamma_r;x_1,\ldots,x_r\right)$ in Theorem 3.2, we have the bilinear generating function relations for the extended multivariable fourth type Horn functions.

Furthermore, for every suitable choice of the coefficients a_i $(i \in N_0)$, if the multivariable function $\Omega_{\mu+\psi l}(y_1,\ldots,y_s)$ $s \in N$ is expressed as an appropriate product of several simpler functions, the assertions of Theorems 3.1 and 3.2 can be applied in order to derive various families of multilinear and multilateral generating functions for the extended fourth type Horn functions.

5. Conclusion

When we study on sequences of special functions, it is possible to analyze the following situations with the help of generating functions:

- finding an exact formula for the members of sequence,
- finding a recurrence formula,
- finding an asymptotic formula.

In this article, we established some generating functions for the extended multivariable fourth type Horn functions and obtained a family of multilinear and multilateral generating functions for each of them. The method used here enables us to obtain generating function relations for other sequences of special functions.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

Duriye Korkmaz-Duzgun and Esra Erkuş-Duman conducted the research together; Duriye Korkmaz-Duzgun wrote the paper; all authors had approved the final version.

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