Strong Transcendental Numbers and Linear Independence

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Abstract: Notions of strong and weak transcendental numbers are introduced. Consequently, proofs of several longstanding conjectures about the transcendence of the numbers such as $e \pm \pi, \frac{\pi}{e}, e^e, \pi^e, \pi^{\sqrt{2}}, e^{\pi^2}$ are obtained. Direct proof of the algebraic independence of the numbers e and π is derived.

Key words: Algebraic, irrational, transcendental numbers, Gel'fond-Schneider theorem, Hermite-Lindemann theorem, Baker's theorem, algebraic independence.

1. Introduction

A complex number α is called *algebraic* if there is a nonzero polynomial $p(z)$ with rational coefficients such that $p(\alpha) = 0$. A number that is not algebraic is called *transcendental*. Springing from such diverse sources as the ancient Greek question concerning the squaring of the circle, the researches of Liouville [1] and Cantor [2], Hermite's investigations on the exponential function [3] and the seventh of Hilbert's famous list of 23 problems [4], the study of transcendental numbers has developed into a fertile and extensive theory, enriching widespread branches of mathematics. Baker [5] gave a comprehensive account of the major discoveries in the field. Notwithstanding its long history and its major advances in recent years, the theory of transcendental numbers is far from being complete and several famous long-standing problems remain open.

The first constructed transcendental number is the Liouville's constant $\sum_{n=0}^{\infty} 10^{-n!}$ [1]. Hermite (1873) and Lindemann (1882) were able to prove to respectively show that e and π are transcendental [6]. The proof of the transcendence of a specific number is usually beset by considerable difficulties, and such results are still few and far between. Despite many efforts, the transcendence of numbers such as $e \pm \pi$, $e\pi, \pi/e, e^e, \pi^e, \pi^{\sqrt{2}}, e^{\pi^2}$, $\ln \pi$, conjectured for long time to be true, remains widely open problem.

Another major problem in transcendence theory is showing that a specific set of numbers is algebraically independent. The first result on algebraic independence of transcendental numbers was proved in late 19th century by Lindemann and Weierstrass [7]. It states that whenever $a_1, \dots, a_n$ are algebraic numbers that are linearly independent over the field of rational numbers $\mathbb{Q}$, then $e^{a_1}, \dots, e^{a_n}$ are algebraically independent over $\mathbb{Q}$. It is often used to prove that some sets are algebraically independent over the rationals. It is known for example that π and e^{π} are algebraically independent (see for example [8], [9]). However, it remains an open problem to prove that π and e are algebraically independent over the field of rational numbers.

In Section 2 of the present paper, we introduce a very natural way of classifying complex numbers. In

Section 3, we generalize the main ideas in the Baker's proof of the Lindemann-Weierstrass Theorem and derive new criteria for establishing the transcendence of numbers. In Section 4, we further classify the complex numbers as weak or strong. In Section 5, we discuss some linear independence results. Along the way, we obtain proofs of several longstanding conjectures in transcendence of numbers.

2. A Classification of Transcendental Numbers

As usual, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote respectively the sets of, respectively all natural numbers, all integers, all rational numbers, all real numbers and all complex numbers. We also denote by $\overline{\mathbb{Q}}$ the set of algebraic number over the field $\mathbb{Q}$. The definition of transcendental number can be restated as follows.

Definition 1. A complex number c is transcendental if every finite family of distinct non-negative integer powers of c is $\mathbb{Q}$-linearly independent.

The non-negative condition in the above definition can be removed.

Theorem 2. *A complex number c is transcendental if and only if every finite family of distinct integer powers of c is $\mathbb{Q}$-linearly* independent.

Proof. Clearly, if c satisfies the condition, then c is transcendental. Conversely, assume that c is transcendental and that there exist $n_0, \ldots, n_p$ distinct integers and rational numbers $b_0, \ldots, b_p$ not all zeros such that $\sum_{k=0}^{p} b_k c^k = 0$. Rearranging if necessary, we assume that $n_0, \ldots, n_{p-q}$ are the negative integers where $q \leq p$. Then

$$\sum_{k=0}^{p-q} b_k c^{n_0 \cdots n_{p-q} n_k^{-1}} c^{n_k} + \sum_{k=p-q+1}^{p} b_k c^{n_0 \cdots n_{p-q}} c^{n_k} = 0.$$

The transcendence of c implies that $b_k c^{n_0 \cdots n_{p-q} n_k^{-1}} = 0$ for $k = 0,1, \ldots, p-q$ and $b_k c^{n_0 \cdots n_{p-q}} = 0$ for $k = p-q+1, \ldots, p$. Thus $b_k = 0$ for all k 's. Contradiction. □

We further notice that if c is a zero of $\sum_{k=0}^{p} b_k x^{1/n_k}$. where $n_0, \ldots, n_p$ are distinct positive integers, then $c^{1/(n_1 n_2 \ldots n_p)}$ is a zero of the polynomial $\sum_{k=0}^{p} b_k x^{n_1 n_2 \ldots n_p / n_k}$. That is, $c^{1/(n_1 n_2 \ldots n_p)}$, and hence $c = b^{n_1 n_2 \ldots n_p}$, is an algebraic number. We obtain the following characterization of transcendental numbers.

Theorem 3. *A complex number c is a transcendental number if and only if every finite family of distinct rational powers of c is $\mathbb{Q}$-linearly independent.*

Baker's formulation (see e.g. [5], [10]) of the Lindemann-Weierstrass Theorem states that:

Theorem 4. (Lindemann-Weierstrass) *Every finite family of distinct algebraic powers of the number e is $\overline{\mathbb{Q}}$-linearly independent.*

For a proof, see [5]. In the transcendental number theory, such a theorem proves to be very useful in establishing the transcendence of numbers. It particularly implies that the number e has a property that is stronger than the transcendence. Such a fact naturally prompts us to introduce the following classification.

Definition 5. We say that a complex number x is *a strong transcendental number* if every finite family of distinct algebraic powers of x is $\overline{\mathbb{Q}}$-linearly independent. Otherwise, we say that the number x is a *weak transcendental number*.

It is clear that strong transcendental numbers are transcendental, and that Theorem 4. establishes the fact that the number e is a strong transcendental number. The following theorem immediately follows.

Theorem 6. *Let x be a complex number. Then the following statements are equivalent:*

1. *x is a strong transcendental number;*
2. *Every nonzero algebraic power of x is a strong transcendental number;*
3. *Every $\overline{\mathbb{Q}}$-linear combination of distinct algebraic powers of x is a transcendental number.*

Proof. Clearly, we have *3.* ⇒ *2.* ⇒ *1.* . To see *1.* ⇒ *3.* , let $y = \sum_{k=0}^{p} b_k x^{\beta_k}$ be a $\overline{\mathbb{Q}}$ -linear combination of distinct algebraic powers of x. Then for any distinct non-negative integers $n_0, \dots, n_m$ and any rational numbers not all zero $r_0, \dots, r_m$, it is quickly seen that the expression

$$\sum_{l=0}^{m} r_l \, y^{n_l} = \sum_{l=0}^{m} r_l \left(\sum_{k=0}^{p} b_k x^{\beta_k} \right)^{n_l}$$

is a $\overline{\mathbb{Q}}$-linear combination of distinct algebraic powers of x and therefore cannot be equal to 0 since x is a strong transcendental number. Thus y is transcendental. The proof is complete. □

Some immediate consequences are the strong transcendence of numbers like $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sinh\alpha$, $\cosh\alpha$, $\tanh\alpha$, $\log\alpha$ whenever α is a nonzero algebraic number. And generally, the inverse functions of all the above listed functions are strong transcendental for all $\alpha \in \overline{\mathbb{Q}} \setminus \{0,1\}$.

3. Strong Transcendental Numbers

By a transcendental function, we mean an analytic function that does not satisfy a polynomial equation. The proofs of the transcendence of e and π are quite similar. They both revolve around an analytic part and an algebraic part. The analytic part relies on the so-called *Hermite's identity* that we can generalize as follows.

Lemma 7. *Let φ be a transcendental function satisfying $\varphi(0) = 1$. Let f be a complex polynomial with degree m. For $u \in \mathbb{C}$, define*

$$I(u, f) := \int_0^u \varphi'(u-t) f(t) dt$$

where the integral is along the line segment from 0 to u. Then

$$I(u, f) = \varphi(u) \textstyle\sum_{j\geq 0} f^{(j)}(0) - \sum_{j\geq 0} f^{(j)}(u). \qquad (1)$$

Proof. Using integrating by parts, one has

$$I(u, f) = \int_0^u \varphi'(u-t) f(t) dt = \int_0^u f(t) d\big(-\varphi(u-t)\big) = -f(u) + \varphi(u) f(0) + I(u, f').$$

The identity (1) is obtained by repeating this process $m-1$ times. □

We need the following simple estimate.

Lemma 8. *Let φ be a transcendental function. Let f be a complex polynomial and let $I(u,f)$ be given by* (1)*. Then* $|I(u,f)| \leq |u| \max_{t \in [0,u]} |\varphi'(u-t)| \max_{t \in [0,u]} |f(t)|$.

Our next result generalizes the Lindemann-Weierstrass Theorem 4.

Theorem 9. *Let φ be a transcendental function satisfying $\varphi(0) = 1$. Then for distinct $a_1, \dots, a_n \in \overline{\mathbb{Q}} \setminus \{0\}$, and for $b_1, \dots, b_n \in \overline{\mathbb{Q}}$ not all zero, $b_1 \varphi(a_1) + \cdots + b_n \varphi(a_n) \neq 0$.*

Proof. We leave out many of the technical details as they are the same exactly as in the Baker's proof of the Lindemann-Weierstrass Theorem [5].

We assume that for some distinct $a_1, \dots, a_n \in \overline{\mathbb{Q}} \setminus \{0\}$, and $b_1, \dots, b_n \in \overline{\mathbb{Q}}$ not all zero, we have $b_1 \varphi(a_1) + \cdots + b_n \varphi(a_n) = 0$. We shall derive a contradiction. It is enough to consider the b_i rational integers. We can then choose a positive integer A such that $Aa_1, \dots, Aa_n$ and $Ab_1, \dots, Ab_n$ are algebraic integers. For $i = 1,2, \dots, n$ and for p large prime number, consider the polynomial $f_i(x) = A^{np}(x - a_i)^{-p}(x - a_1)^p \cdots (x - a_n)^p$. Then $|f_i(x)| \le |A|^{np}\,(n^{n+1})^p$ and by the Hermite's identity (1)

$$I(u, f_i) = \varphi(u) \sum_{j \ge 0} f_i^{(j)}(0) - \sum_{j \ge 0} f_i^{(j)}(u).$$

The sums in the above expression are finite since f_i is a polynomial. Let $J_i := \sum_{k=1}^{n} b_k I(a_k, f_i)$. Then

$$J_i = \sum_{k=1}^{n} b_k \left(\varphi(a_k) \sum_{j \ge 0} f_i^{(j)}(0) - \sum_{j \ge 0} f_i^{(j)}(a_k) \right) = -\sum_{k=1}^{n} b_k \sum_{j \ge 0} f_i^{(j)}(a_k)$$

where the last equality follows from our assumption that $\sum_{k=1}^{n} b_k \varphi(a_k) = 0$.

One then notices that $|J_1 \cdots J_n|$ is an integer satisfying $\left((p-1)!\right)^n \le |J_1 \cdots J_n| \le cd^{np}$ for some constants c and d independent of p. The estimates are inconsistent for sufficiently large p. Such a contradiction proves the theorem.

Example 10. Theorem 9 promptly implies the following strong transcendence results:

1) e^a is strong transcendental for every $a \in \overline{\mathbb{Q}} \setminus \{0\}$: in Theorem 9, take $\varphi(z) = e^{az}$.
2) π^a is strong transcendental for every $a \in \overline{\mathbb{Q}} \setminus \{0\}$: in Theorem 9, take $\varphi(z) = \pi^{az}$.
3) log a is strong transcendental for every $a \in \overline{\mathbb{Q}} \setminus \{0\}$: and for any determination the logarithm: in Theorem 9, take $\varphi(z) = (\log a)^z$.

In particular $\pi^{\sqrt{2}}$ is a strong transcendental number. It is plain that (1.) in the above example is a restatement of the Hermite-Lindemann Theorem [4], [11]. The *Six Exponentials Theorem* [12] implies that at least one of the numbers $e^e, e^{e^2}, e^{e^3}, e^{e^4}$ is transcendental. By considering $\varphi(z) = e^{ze^a}$, it follows that:

Corollary 11. *The numbers e^{e^a} are strong transcendental numbers for all $a \in \overline{\mathbb{Q}}$.*

Likewise, by considering $\varphi(z) = e^{z\pi^a}$, we have

Corollary 12. *The numbers e^{π^a} are strong transcendental numbers for all $a \in \overline{\mathbb{Q}} \setminus \{1\}$.*

In particular, e^{π^2} is a strong transcendental number. We are also now able to confirm the transcendence of the number π^e. Indeed, taking $\varphi(z) = \pi^{(a+e^b)z}$, where $a, b \in \overline{\mathbb{Q}}$ in Theorem 9, we have

Corollary 13. *The numbers $\pi^{(a+e^b)}$ where $a, b \in \overline{\mathbb{Q}}$, are strong transcendental numbers.*

4. Weak and Strong Complex Numbers

In what follows, $a^b := e^{b \log a}$, where $e^z = \sum_{n=0}^{\infty} z^n/n!$ and $\log a = \log|a| + i \arg a$. The argument of a is determined only up to a multiple of 2π. Let us denote by $\mathcal{G}$ the set of all Gel'fond-Schneider transcendental numbers, that is to say, $\mathcal{G} = \{a^b : a \in \overline{\mathbb{Q}} \setminus \{0,1\}, b \in \overline{\mathbb{Q}} \setminus \mathbb{Q}\}$. The Gel'fond-Schneider Theorem ([13], [14]) states that every element of $\mathcal{G}$ is a weak transcendental number. In fact, our next result shows that the class of weak transcendental numbers coincides exactly to the class of Gel'fond-Schneider transcendental number.

Theorem 14. *A transcendental number τ is strong if and only if $\tau \notin \mathcal{G}$.*

Proof. Let $\tau \in \mathcal{G}$. Since $\tau^{\frac{1}{b}} - a\tau^0 = 0$ is a non-trivial 0 linear combination, we see that τ is not strong. Conversely, if $\tau \notin \mathcal{G}$ and is a transcendental number, then the function $\varphi(z) = \tau^z = e^{z \log \tau}$ is a transcendental function with $\varphi(0) = 1$. By Theorem 9 for distinct $a_1, \dots, a_n \in \overline{\mathbb{Q}} \setminus \{0\}$, and for $b_1, \dots, b_n \in \overline{\mathbb{Q}}$ not all zero, we have $b_1 \tau^{a_1} + \dots + b_n \tau^{a_n} \neq 0$ hence τ is a strong transcendental number. □

We introduce the following definition.

Definition 15. We say that a complex number is *weak* if it is either an algebraic number or a Gel'fond-Schneider transcendental number. A complex number is said to be *strong* if it is not weak. We denote by $\mathcal{W}$ the set of all weak complex numbers and by $\mathcal{S}$ its complement.

By our definition $\mathcal{W} = \overline{\mathbb{Q}} \cup \mathcal{G}$. It is easy to see that $\mathcal{W} = \{a^b : a, b \in \overline{\mathbb{Q}}\}$. It follows from Theorem 14 that the class of strong complex numbers coincides exactly to the class of strong transcendental numbers.

Theorem 16. *A complex number is strong if and only if it is a strong transcendental number.*

Thus, a complex number is either a weak complex number or a strong transcendental number. It is also worth noticing that since the set $\mathcal{G}$ of weak transcendental numbers is countable, so is the set $\mathcal{W}$. Therefore most (uncountably many) complex numbers are strong complex numbers.

Consider the set $\mathcal{L} := \{b \log a : a \in \overline{\mathbb{Q}} \setminus \{0,1\}, b \in \overline{\mathbb{Q}} \setminus \mathbb{Q}\}$. If $\lambda \in \mathcal{L}$, then $e^{\lambda} \in \mathcal{G}$, and therefore e^{λ} is weak. It turns out that the converse of such a statement holds.

Theorem 17. *If* $\notin \mathcal{L}$ *, then the number* $e^{\tau} \in \mathcal{S}$.

Proof. The result follows from Theorem 9 because if $\tau \notin \mathcal{L}$, then $\varphi(z) = e^{\tau z}$ is a transcendental function such that $\varphi(0) = 1$.

We have already noticed that if $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, then $e^{\alpha} \in \mathcal{S}$. An immediate consequence of the Theorem 17 establishes the following even stronger result:

Corollary 18. Let $\alpha, \beta \in \overline{\mathbb{Q}} \setminus \{0\}$. Then $e^{\alpha^{\beta}} \in \mathcal{S}$.

In particular, $e^{e^{\pi}} = e^{i^{-2i}}$ is a strong transcendental number.

Proof. We notice that for every $\alpha', \beta' \in \overline{\mathbb{Q}} \setminus \{0\}$, $\alpha^{\beta} \neq \beta' \log \alpha'$ because the left-hand side is a weak complex number while the right-hand side is a strong transcendental number (Example 10). It follows from Theorem 17 that $e^{\alpha^{\beta}}$ is a strong complex number.

5. Linear Independence Results

Many linear independence results can promptly be derived from our classification of transcendental numbers for certain pair of numbers.

Theorem 19. *If* $a, b \in \overline{\mathbb{Q}} \setminus \{0\}$*, then the set* $\{e^b, \log a\}$ *is* $\overline{\mathbb{Q}}$*-linearly independent.*

Proof. Assume that $b_1 e^b = b_2 \log a = 0$ for $b_1, b_2 \in \overline{\mathbb{Q}} \setminus \{0\}$. Then we have $a^{b_2/b_1} = e^{e^b}$. This is a contradiction because a^{b_2/b_1} is a weak complex number while according to Corollary 11, $e^{e^b} \in \mathcal{S}$. □

As a corollary, we obtain a proof of yet another important conjecture, namely:

Theorem 20. *The numbers* πe, π/e *are transcendental.*

Proof. Assume that $\pi e^{\pm 1} = a$ is algebraic. Then $\pi = a e^{\mp 1}$ and $e^{\pi} = \left(e^{e^{\mp 1}}\right)^a$. Theorem 17 implies that $e^{\mp 1} \notin \mathcal{L}$. Thus $e^{e^{\mp 1}}$, and hence $\left(e^{e^{\mp 1}}\right)^a$ is a strong transcendental number. This is a contradiction because e^{π} is a weak complex number. □

Theorem 21. *If* $a, b \in \overline{\mathbb{Q}} \setminus \{0\}$*, then the set* $\{\pi^b, \log a\}$ *is* $\overline{\mathbb{Q}}$*-linearly independent.*

Proof. Assume that $b_1 \pi^b = b_2 \log a = 0$ for $b_1, b_2 \in \overline{\mathbb{Q}} \setminus \{0\}$. Then we have $a^{b_2/b_1} = e^{\pi^b}$. This is a contradiction because a^{b_2/b_1} is a weak complex number while according to Corollary 12, $e^{\pi^b} \in \mathcal{S}$. □

Theorem 22. *If* $b \in \overline{\mathbb{Q}} \setminus \{0\}$*, then the set* $\{\pi, \log \pi^b\}$ *is* $\overline{\mathbb{Q}}$*-linearly independent.*

Proof. Assume that $b_1 \pi^b = b_2 \log \pi = 0$ for $b_1, b_2 \in \overline{\mathbb{Q}} \setminus \{0\}$. Then we have $\pi^{b b_2/b_1} = e^{\pi}$. This is a contradiction since $\pi^{b b_2/b_1}$ is strong transcendental number (Example 10) while $e^{\pi} \in \mathcal{W}$. □

The statement of Theorem 4 implies in particular that finitely many distinct algebraic powers of the number e form a finite family of distinct strong transcendental numbers. Our next result shows that similar conclusion can be stated if in the theorem, one replaces the distinct powers of e with just distinct strong transcendental numbers. Its proof uses the same line of ideas as the proof of the transcendence of either e.

Theorem 23. *If $\tau_1, \dots, \tau_n$ is a finite family of distinct strong transcendental numbers, then the set $\{1, \tau_1, \dots, \tau_n\}$ is $\overline{\mathbb{Q}}$-linearly independent.*

Proof. Assume to the contrary that

$$b_0 + b_1\,\tau_1 + \cdots + b_n\,\tau_n = 0 \tag{2}$$

where $b_0, \dots, b_n$ are in $\overline{\mathbb{Q}}$ and not all 0. By removing all i for which $b_i = 0$, we may assume that all coefficients are different from zero. We also notice that if $b_i \neq 0$, then one can multiply by τ_i^{-1} to obtain equation with the same form as in (2). Hence, we may and do assume that $b_0 \neq 0$.

For a large prime number $p >$ max $\{|b_0|, n\}$, consider the polynomial $f(x) = x^{p-1}(x-1)^p \cdots (x-n)^p$. Then for $0 < x < n$, we have $|f(x)| \leq n^{3p}$. For $i = 1, \dots, n$, let $I_i(f) \coloneqq \int_0^i (a \log \tau_i)\tau_i^{a(i-t)} f(t)dt$. Integrating by parts, one has

$$I_i(f) = \int_0^i f(t)d\left(-\,\tau_i^{a(\,i-t\,)}\right) \;= -f(i) + \tau_i^a f(0) + I_i(f').$$

Iterating the above procedure, we have $I_i(f) = \tau_i^a \sum_{j\geq 0} f_i^{(j)}(0) - \sum_{j\geq 0} f_i^{(j)}(i)$. The sums in the above expressions are finite since f is a polynomial and we further have

$$|I_i(f)| \leq \int_0^i \left|(a \log \tau_i)\tau_i^{a(i-t)} f(t)\right| dt \;\leq\; |\,a \log \tau_i| e^{|a \log \tau_i|}\, n^{3p}. \tag{3}$$

Define $J \coloneqq b_1 I_1(f) + \cdots + b_n I_n(f)$. Then we have

$$J = \sum_{i=1}^{n} b_i \left(\tau_i^a \sum_{j\geq 0} f_i^{(j)}(0) - \sum_{j\geq 0} f_i^{(j)}(i) \right) = -\, b_0 \sum_{j\geq 0} f_i^{(j)}(0) - \sum_{i=1}^{n} b_i \sum_{j\geq 0} f_i^{(j)}(i))$$

where the last equality follows from our assumption that $\sum_{i=1}^n b_i \tau_i^a = -b_0$. Note that the polynomial f can be written as $f(x) = \pm\,(n!)^p x^{p-1} + c_p x^p + c_{p+1}x^{p+1} + \cdots$ where the c_p are integers and thus

$f^{(j)}(0) = 0$ for $j < p-1$; $f^{(p-1)}(0) = \pm(p-1)!\,(n!)^p$ is not divisible by $p!$ for $p > n$ and $f^{(j)}(0)$ is an integer divisible by $p!$ for $j \geq p$.

For $k = 1, \dots, n$, $f(x+k)$ is a polynomial of the form $f(x+k) = d_p x^p + d_{p+1}x^{p+1} + \cdots$ where d_i are integers. Thus $f^{(j)}(k) = 0$ for $j < p$ and $f^{(j)}(k)$ is an integer divisible by $p!$ for $j \geq p$. It follows that if $p > |b_0|$, J is a nozero integer divisible by $(p-1)!$ and $(p-1)! \leq |J|$. The inequalities in (3) imply $|J| \leq \sum_{i=1}^n |b_i||I_i(f)|| \leq n^{3p} \sum_{i=1}^n |b_i||\,a \log \tau_i| e^{|a \log \tau_i|}$. Such estimates are inconsistent for sufficiently large p and the contradiction proves the theorem. □

Corollary 24 *If $\tau_1, \dots, \tau_n$ is a finite family of distinct strong transcendental numbers, then for any $a \in \overline{\mathbb{Q}} \setminus \{0\}$, any $\overline{\mathbb{Q}}$-linear combination of the numbers $\tau_1^a, \dots, \tau_n^a$ a is a transcendental number.*

Proof. We first notice that $\tau_1^a, \dots, \tau_n^a$ are distinct strong transcendental numbers and by Theorem 23, any $\overline{\mathbb{Q}}$-non trivial linear combination $b_1\,\tau_1^a + \cdots + b_n\,\tau_n^a \neq b_0$ for any $b_0 \in \overline{\mathbb{Q}}$ and must be transcendental.

In particular, since both e and π are strong transcendental numbers, Corollary 24 proves the conjecture about the transcendence of the numbers $e+\pi$ and $e-\pi$.

Corollary 25. *The numbers* $e^a+\pi^a$ *and* $e^a-\pi^a$ *for* $a \in \overline{\mathbb{Q}} \setminus \{0\}$ *are all transcendental numbers.*

Corollary 26. *For any* $a, b \in \overline{\mathbb{Q}} \setminus \{0\}$, *the set* $\{1, e^a, \pi^b\}$ *is* $\overline{\mathbb{Q}}$*-linearly independent. In particular, any* $\overline{\mathbb{Q}}$*-linear combination of the numbers* e^a, π^b *is a transcendental number.*

Proof. We notice that e^a, π^b strong transcendental numbers and that $e^a \neq \pi^b$ because if not we have $e^{\frac{a}{b}} = \pi$ and $e^{e^{a/b}} = e^\pi$. This is a contradiction because $e^\pi \in \mathcal{W}$ while $e^{e^{a/b}} \in \mathcal{S}$. We can apply then Theorem 23 to $\{e^a, \pi^b\}$.

Corollary 26 provides a proof of the long-standing conjecture of the algebraic independence of the numbers e and π.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

The author contributed to 100% of this work,

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