Fixed Point Results for Orthogonal Z-Contraction Mappings in O-Complete Metric Spaces

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Abstract: In this paper, we introduce the notion of orthogonal Z-contraction mappings and prove fixed point theorems for such contraction mappings in orthogonally metric spaces, which are generalizations of fixed point results for Z-contraction mappings in metric spaces. As an application, we apply our main results to show the existence of a unique positive definite solution of a nonlinear matrix equation.

Key words: Fixed point, orthogonally metric spaces, Z-contraction mappings.

1. Introduction

The most well-known fixed point theorem is the Banach contraction principle (briefly, BCP) due to Banach [1]. After that, Ran and Reuring [2] established fixed point results on partially ordered metric spaces and also applied to the existence and uniqueness results of a solution for a nonlinear matrix equation. Especially, Gordji et al. [3] extended the BCP to the setting of an orthogonal set (briefly, O-set). They applied obtained results to prove the existence of a solution for a differential equation, which can not be applied by the BCP [1] and the results of Ran and Reurings [2].

In 2012, Khojasteha et al. [4] introduced a new control function namely a simulation function and defined a new contraction namely Z-contraction as follows:

Definition 1.1 ([4]). A function \( \zeta : (0, \infty) \times (0, \infty) \to \mathbb{R} \) is called a simulation function if it satisfies the following conditions:

1. \( \zeta(0,0) = 0 \);
2. \( \zeta(t, s) < s - t \) for all \( t, s > 0 \);
3. \( \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0 \) such that \( \lim_{n \to \infty} \zeta(t_n, s_n) < 0 \).

We denote the set of all simulation functions by \( Z \).

Definition 1.2 ([4]). Let \( (X, d) \) be a metric space and \( \zeta \in Z \). A mapping \( T : X \to X \) is called a Z-contraction mapping with respect to \( Z \) if
They showed that the class of $Z$-contraction mappings can be expressed in various contractive classes in a simple and unified way and also established fixed point results for $Z$-contraction mappings in complete metric spaces.

The aims of this work are to introduce the concept of orthogonal $Z$-contraction mappings with respect to simulation functions and establish fixed point theorems for such contraction mappings in orthogonal metric spaces. As an application, we apply our main results to consider the existence of a unique positive definite solution of a nonlinear matrix equation.

2. Preliminaries

Throughout this paper, we denote by $X$, $\mathbb{N}$, and $\mathbb{N}_0$ the nonempty set, the set of positive integers and the set of nonnegative integers, respectively.

Now, we recall the concept of an orthogonal set (or O-set), some examples and some properties of the orthogonal sets as follows:

**Definition 2.1 ([3]).** Let $X$ be a nonempty set and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies

$$\exists x_0 \left( (\forall y \in X, y \perp x_0) \text{ or } (\forall y \in X, x_0 \perp y) \right),$$

then it is called an orthogonal set (briefly, O-set) and $x_0$ is called an orthogonal element. We denote this O-set by $(X, \perp)$.

**Example 2.2 ([3]).** Let $X$ be the set of all peoples in the world. Define the binary relation $\perp$ on $X$ by $x \perp y$ if $x$ can give blood to $y$. If $x_0$ is a person such that his (her) blood type is O-, then we have $x_0 \perp y$ for all $y \in X$. This means that $(X, \perp)$ is an O-set. In this O-set, $x_0$ (in definition) is not unique. Note that $x_0$ may be a person with blood type AB+. In this case, we have $y \perp x_0$ for all $y \in X$.

**Definition 2.3 ([3]).** Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n).$$

**Definition 2.4 ([3]).** The triplet $(X, \perp, d)$ is called an orthogonal metric space if $(X, \perp)$ is an O-set and $(X, d)$ is a metric space.

**Definition 2.5 ([3]).** Let $(X, \perp, d)$ be an orthogonal metric space. Then a mapping $T : X \to X$ is said to be orthogonally continuous (or $\perp$-continuous) in $x \in X$ if for each O-sequence $\{x_n\}$ in $X$ with $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$. Also, $T$ is said to be $\perp$-continuous on $X$ if $T$ is $\perp$-continuous in each $x \in X$.

**Definition 2.6 ([3]).** Let $(X, \perp, d)$ be an orthogonal metric space. Then $X$ is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

**Definition 2.7 ([3]).** Let $(X, \perp)$ be an O-set. A mapping $T : X \to X$ is said to be $\perp$-preserving if
\( Tx \perp Ty \) whenever \( x \perp y \).

Moreover, we define some new properties of the orthogonal sets as follows:

**Definition 2.8.** We say that an O-set \((X, \perp)\) is a transitive orthogonal set if \( \perp \) is transitive.

**Definition 2.9.** Let \((X, \perp)\) be an O-set. A path of length \( k \) in \( \perp \) from \( x \) to \( y \) is a finite sequence
\[ \{z_0, z_1, \ldots, z_k\} \subseteq X \] such that
\[ z_0 = x^*, z_k = y^*, z_i \perp z_{i+1} \] or \[ z_{i+1} \perp z_i \]
for all \( i = 0, 1, 2, \ldots, k-1 \).

Let \( Y(x, y, \perp) \) be denoted as all path of length \( k \) in \( \perp \) from \( x \) to \( y \).

The following lemma will be useful later.

**Lemma 2.10 ([5]).** Let \((X, d)\) be a metric space and \( \{x_n\} \) a sequence in \( X \) such that
\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \] If \( \{x_n\} \) is not a Cauchy, then there exists \( \varepsilon > 0 \) and two subsequence \( \{x_{n(k)}\} \) and \( \{x_{m(k)}\} \) of \( \{x_n\} \) where \( n(k) > m(k) > k \) such that
\[ \lim_{n \to \infty} d(x_{n(k)-1}, x_{m(k)+1}) = \lim_{n \to \infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon \]
and
\[ \lim_{n \to \infty} d(x_{n(k)}, x_{m(k)}) = \lim_{n \to \infty} d(x_{n(k)-1}, x_{m(k)+1}) = \varepsilon. \]

3. Main Results

In this section, we introduce a new \( Z \)-contraction mapping and prove some fixed point theorems for \( Z \)-contraction mappings in orthogonally metric spaces.

**Definition 3.1.** Let \((X, \perp, d)\) be an orthogonally metric space. A mapping \( T : X \to X \) is called an orthogonal \( Z \)-contraction mapping with respect to \( \zeta \) (briefly, \( Z \perp \)-contraction) if there is \( \zeta \in Z \) such that
\[ \zeta(d(Tx, Ty), d(x, y)) \geq 0 \] (1)

for all \( x, y \in X \) with \( x \perp y \).

**Theorem 3.2.** Let \((X, \perp, d)\) be an O-complete metric space with an orthogonal element \( x_0 \) and \( T \) be a self-mapping on \( X \) satisfying the following conditions:
1) \((X, \perp)\) is a transitive orthogonal set;
2) \( T \) is \( \perp \)-preserving;
3) \( T \) is a \( Z \perp \)-contraction mapping;
4) \( T \) is \( \perp \)-continuous.

Then \( T \) has a fixed point \( x^* \in X \). Also, the Picard sequence \( \{T^n x_0\} \) converges to the fixed point of \( T \).

**Proof.** By the definition of orthogonality, there exists \( x_0 \in X \) such that
\[ (\forall y \in X, x_0 \perp y) \] or \[ (\forall y \in X, y \perp x_0) \).

It follows that
\[ x_0 \perp T x_0 \text{ or } T x_0 \perp x_0. \]

Let \( x_{n+1} = T x_n = T^n x_0 \) for all \( n \in \mathbb{N}_0 \). If \( x_n = x_{n+1} \) for some \( n^* \in \mathbb{N}_0 \), then \( x_n \) is a fixed point of \( T \) and so the proof is completed. So we may assume that \( x_n \neq x_{n+1} \) for all \( n \in \mathbb{N}_0 \). Thus we have \( d(x_{n+1}, x_n) > 0 \) for all \( n \in \mathbb{N}_0 \). Since \( T \) is \( \perp \)-preserving, we have \[
(\forall n, x_n \perp x_{n+1}) \text{ or } (\forall n, x_{n+1} \perp x_n).
\]

Since \( T \) is a \( Z_\perp \)-contraction mapping, we have
\[
0 \leq \zeta(d(T^{n+2} x_0, T^{n+1} x_0), d(T^n x_0, T^n x_0)) < d(T^{n+1} x_0, T^n x_0) - d(T^{n+2} x_0, T^{n+1} x_0)
\]
for all \( n \in \mathbb{N} \). It yields that \( \{d(T^{n+1} x_0, T^n x_0)\} \) is a monotonically decreasing sequence of positive reals and then there exists \( c \geq 0 \) such that
\[
\lim_{n \to \infty} d(T^{n+1} x_0, T^n x_0) = c.
\]

Suppose that \( c > 0 \). Using (1) and \( (\zeta_3) \) of Definition 1.1, we have
\[
0 \leq \limsup_{n \to \infty} \zeta(d(T^{n+2} x_0, T^{n+1} x_0), d(T^{n+1} x_0, T^n x_0)) < 0,
\]
which is a contradiction. Therefore, \( c = 0 \) and so
\[
\lim_{n \to \infty} d(T^{n+1} x_0, T^n x_0) = 0.
\]

Now, we show that \( \{x_n\} \) is a Cauchy \( 0 \)-sequence. Suppose by contradiction that \( \{x_n\} \) is not a Cauchy \( 0 \)-sequence. By Lemma 2.10, there exists \( \varepsilon > 0 \) and two subsequence \( \{x_{n(k)}\} \) and \( \{x_{m(k)}\} \) of \( \{x_n\} \) where \( n(k) > m(k) > k \) such that
\[
\lim_{n \to \infty} d(x_{n(k)+1}, x_{m(k)+1}) = \lim_{n \to \infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2}
\]

Since \( (X, \perp) \) is a transitive orthogonal set, we have
\[
(\forall k, x_{n(k)} \perp x_{m(k)}) \text{ or } (\forall k, x_{m(k)} \perp x_{n(k)}).
\]

From (b), (1), (2), and \( (\zeta_3) \), we have
\[
0 \leq \limsup_{n \to \infty} \zeta(d(x_{n(k)+1}, x_{m(k)+1}), d(x_{n(k)}, x_{m(k)})) < 0,
\]
which is a contradiction. Thus, \( \{ x_n \} \) is a Cauchy O-sequence in \( X \). Since \( X \) is O-complete, there exists \( x^* \in X \) such that \( x_n \to x^* \) as \( n \to \infty \). Since \( T \) is \( \bot \)-continuous, we have \( Tx_n \to Tx^* \) as \( n \to \infty \). Thus,

\[
Tx^* = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = x^*.
\]

Hence, \( x^* \in X \) is a fixed point of \( T \).

**Theorem 3.3.** In addition to the hypothesis of Theorem 3.2, suppose that \( \Upsilon(x, y, \bot) \) is nonempty for all \( x, y \in X \). Then \( T \) has a unique fixed point.

**Proof:** Suppose that \( x^*, y^* \) are two fixed points of \( T \) such that \( x^* \neq y^* \). Since \( \Upsilon(x, y, \bot) \) is nonempty, for all \( x, y \in X \), there exists a path \( \{ z_0, z_1, \ldots, z_k \} \) of some finite length \( k \) in \( \bot \) from \( x \) to \( y \) such that

\[
z_0 = x^*, \quad z_k = y^*, \quad z_i \bot z_{i+1} \quad \text{or} \quad z_{i+1} \bot z_i \quad \text{for all} \quad i = 0, 1, 2, \ldots, k-1.
\]

Since \( (X, \bot) \) is a transitive orthogonal set, we get \( x^* \bot y^* \) or \( y^* \bot x^* \). From (1) and \((\zeta_1)\), we have

\[
0 \leq \zeta(d(Tx^*, Ty^*), d(x^*, y^*)) = \zeta(d(x^*, y^*), d(x^*, y^*)) < 0,
\]

which is a contradiction. Therefore, \( T \) has a unique fixed point. This completes the proof.

**Corollary 3.4.** Let \( (X, d) \) be a complete metric space and \( T : X \to X \) be a \( \mathcal{Z} \)-contraction mapping. Then \( T \) has a unique fixed point in \( X \). Moreover, for each \( x_0 \in X \), the Picard sequence \( \{ T^n x_0 \} \) converges to the fixed point of \( T \).

### 4. Application

In this section, we use the following matrix notations: \( M(n) \) denotes the set of all \( n \times n \) complex matrices, \( P(n) \) denotes the set of all \( n \times n \) positive definite matrices, \( H^+(n) \) denotes the set of all \( n \times n \) positive semidefinite matrices. We write \( A \succeq B \) (\( A \succ B \)) if \( A - B \in H^+(n) \) (or \( A - B \in P(n) \)). In particular, \( A \succeq 0 \) (\( A \succ 0 \)) implies \( A \in H^+(n) \) (or \( A \in P(n) \)). We also write \( A^* \) denotes the conjugate transpose of an \( n \times n \) matrix \( A \). In addition, we let \( d_T \) denotes the Thompson metric on \( P(n) \), which is defined by

\[
d_T(A, B) = \log \left\{ \max \{ \alpha, \beta \} \right\}
\]

where \( \alpha = \inf \{ \delta : A \leq \delta B \} = \lambda^+ \left( B^{-\frac{1}{2}} AB^{-\frac{1}{2}} \right) \) that is, the maximum eigenvalue of \( B^{-\frac{1}{2}} AB^{-\frac{1}{2}} \) and \( \beta = \inf \{ \delta : B \leq \delta A \} = \lambda^+ \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right) \), that is, the maximum eigenvalue of \( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \).
The goal of this section is to apply fixed point results for $Z_\perp$-contraction mappings via Thompson metrics to solve the nonlinear matrix equation

$$X^r = Q + \sum_{i=1}^{m} A_i^* K_i(X) A_i$$

(3)

where $r \geq 1$, $A_i$ is an $n \times n$ nonsingular matrix, $Q$ is a Hermitian positive definite matrix and $K_i$ is a continuous order preserving self-mapping on $P(n)$.

We recall properties of the Thompson metric for Hermitian positive definite matrices as follows:

**Lemma 4.1** ([6]). Let $d_T$ be a Thompson metric on $P(n)$.

(i) $d_T(A,B) = d_T(A^*,B^*) = d_T(MA^*,MB^*)$ for all $A,B \in P(n)$ and a nonsingular matrix $M$;

(ii) $d_T(A,B) \leq r d_T(A,B)$ for all $A,B \in P(n)$ and $r \in [-1,1]$;

(iii) $d_T(A + B,C + D) \leq \max\{d_T(A,C),d_T(B,D)\}$ for all $A,B,C,D \in P(n)$.

**Theorem 4.2.** Consider the matrix equation (3). Let $Q \in P(n)$ and for each $i=1,2,\ldots,m$, $K_i : P(n) \to P(n)$ be a continuous order-preserving mapping. Suppose that there are positive number $r \geq 1$ with

$$d_T(K_i(X),K_i(Y)) \leq r[d_T(X,Y) - \varphi(d_T(X,Y))]$$

for all $X,Y \in P(n)$, $i=1,2,\ldots,m$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous functions such that $\varphi(t) = 0$ if and only if $t = 0$. Then the equation (3) has a unique positive solution.

**Proof.** Define the relation $\perp$ on $P(n)$ by

$$A \perp B \iff A \preceq B.$$

Then, by setting $X_0 = 0$, it follows that $(P(n),\perp)$ is an $O$-set. Also, $\perp$ is transitive on $P(n)$. Since $(P(n),d_T)$ is a complete metric space, this implies that $(P(n),\perp, d_T)$ is an $O$-complete metric space. Next, we define a mapping $T : P(n) \to P(n)$ by

$$T(X) = \left(Q + \sum_{i=1}^{m} A_i^* K_i(X) A_i\right)^{\dagger}$$

for all $X \in P(n)$. Then $T$ is well-defined, $\perp$-continuous, and $\perp$-preserving.

Now, we show that $T$ is a $Z_\perp$-contraction mapping with a function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ which is defined by

$$\zeta(t,s) = s - \varphi(s) - t$$

for all $t,s \in [0,\infty)$. Let $X,Y \in P(n)$ such that $X \perp Y$. Then
\[ d_r(T(X), T(Y)) = d_r \left( (Q + \sum_{i=1}^{m} A_i^* k_i(X) A_i)^\frac{r}{2}, (Q + \sum_{i=1}^{m} A_i^* k_i(Y) A_i)^\frac{r}{2} \right) \]

\[ \leq \frac{1}{r} d_r \left( (Q + \sum_{i=1}^{m} A_i^* k_i(X) A_i), (Q + \sum_{i=1}^{m} A_i^* k_i(Y) A_i) \right) \]

\[ \leq \frac{1}{r} \max_{i \in \{1, 2, \ldots, m\}} d_r \left( k_i(X), k_i(Y) \right) \]

\[ \leq d_r(X, Y) - \varphi(d_r(X, Y)) \]

Thus,

\[ 0 \leq d_r(X, Y) - \varphi(d_r(X, Y)) - d_r(T(X), T(Y)) \]

and so

\[ 0 \leq \zeta (d_r(T(X), T(Y)), d_r(X, Y)). \]

Therefore, \( T \) is a \( Z_\perp \)-contraction mapping.

By Theorem 3.2, there exists \( X^* \in P(n) \) such that \( T(X^*) = X^* \). That is, \( X^* \) is a positive definite solution of the Equation (3). Since there is a greatest lower bound and a least upper bound, we have \( Y(x, y, \perp) \) is nonempty for all \( X, Y \in P(n) \). By Theorem 3.3, it follows that \( T \) has a unique fixed point in \( P(n) \). This implies that Equation (3) has a unique solution in \( P(n) \).

5. Conclusion

Our main theorem is a real generalization of the Khojasteha’s fixed point result and the Gordji’s fixed point result. Moreover, our new fixed point theorem for \( Z_\perp \)-contraction mappings can be applied to show that the matrix equation (3) always has a unique positive definite solution.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

KS conceived of the presented idea and developed the fixed point theorem and performed the applications to a nonlinear matrix equation via Thompson metrics. WS verified the analytical methods and encouraged KS to investigate and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

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