Fixed Point Results for Orthogonal Z-Contraction Mappings in O-Complete Metric Spaces

Kanokwan Sawangsup*, Wutiphol Sintunavarat
Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathumthani 12120, Thailand.

* Corresponding author. Tel.: +6682-4939779; email: juneks.ks@gmail.com
Manuscript submitted June 10, 2019; accepted September 23, 2019.
doi: 10.17706/ijapm.2020.10.1.33-40

Abstract: In this paper, we introduce the notion of orthogonal Z-contraction mappings and prove fixed point theorems for such contraction mappings in orthogonally metric spaces, which are generalizations of fixed point results for Z-contraction mappings in metric spaces. As an application, we apply our main results to show the existence of a unique positive definite solution of a nonlinear matrix equation.

Key words: Fixed point, orthogonally metric spaces, Z-contraction mappings.

1. Introduction

The most well-known fixed point theorem is the Banach contraction principle (briefly, BCP) due to Banach [1]. After that, Ran and Reuring [2] established fixed point results on partially ordered metric spaces and also applied to the existence and uniqueness results of a solution for a nonlinear matrix equation. Especially, Gordji et al. [3] extended the BCP to the setting of an orthogonal set (briefly, O-set). They applied obtained results to prove the existence of a solution for a differential equation, which can not be applied by the BCP [1] and the results of Ran and Reurings [2].

In 2012, Khojasteha et al. [4] introduced a new control function namely a simulation function and defined a new contraction namely Z-contraction as follows:

Definition 1.1 ([4]). A function $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0,0)=0$;

(ζ_2) $\zeta(t,s)<s-t$ for all $t,s>0$;

(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n>0$, then

$$\limsup_{n\to\infty}\zeta(t_n,s_n)<0.$$

We denote the set of all simulation functions by Z.

Definition 1.2 ([4]). Let (X,d) be a metric space and $\zeta\in Z$. A mapping $T:X\to X$ is called a Z-contraction mapping with respect to Z if

$$\zeta(d(Tx,Ty),d(x,y)) \geq 0$$

for all $x,y \in X$.

They showed that the class of $\mathcal{Z}$-contraction mappings can be expressed in various contractive classes in a simple and unified way and also established fixed point results for $\mathcal{Z}$-contraction mappings in complete metric spaces.

The aims of this work are to introduce the concept of orthogonal Z-contraction mappings with respect to simulation functions and establish fixed point theorems for such contraction mappings in orthogonal metric spaces. As an application, we apply our main results to consider the existence of a unique positive definite solution of a nonlinear matrix equation.

2. Preliminaries

Throughout this paper, we denote by X, $\mathbb{N}$, and $\mathbb{N}_0$ the nonempty set, the set of positive integers and the set of nonnegative integers, respectively.

Now, we recall the concept of an orthogonal set (or O-set), some examples and some properties of the orthogonal sets as follows:

Definition 2.1 ([3])**.** Let X be a nonempty set and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition:

$$\exists x_0[(\forall y \in X, y \perp x_0) \text{ or } (\forall y \in X, x_0 \perp y)],$$

then it is called an orthogonal set (briefly, O-*set*) and x_0 is called an orthogonal element. We denote this O-set by $(X,\perp)$.

Example 2.2 ([3])**.** Let X be the set of all peoples in the word. Define the binary relation $\perp$ on X by $x \perp y$ if x can give blood to y. If x_0 is a person such that his (her) blood type is O-, then we have $x_0 \perp y$ for all $y \in X$. This means that $(X,\perp)$ is an O-set. In this O-set, x_0 (in definition) is not unique. Note that x_0 may be a person with blood type AB+. In this case, we have $y \perp x_0$ for all $y \in X$.

Definition 2.3 ([3])**.** Let $(X,\perp)$ be an O-set. A sequence $\{x_n\}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n).$$

Definition 2.4 ([3])**.** The triplet $(X,\perp,d)$ is called an orthogonal metric space if $(X,\perp)$ is an O-set and (X,d) is a metric space.

Definition 2.5 ([3])**.** Let $(X,\perp,d)$ be an orthogonal metric space. Then a mapping $T: X \to X$ is said to be orthogonally continuous (or $\perp$-continuous) in $x \in X$ if for each O-sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$. Also, T is said to be $\perp$-continuous on X if T is $\perp$-continuous in each $x \in X$.

Definition 2.6 ([3])**.** Let $(X,\perp,d)$ be an orthogonal metric space. Then X is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

Definition 2.7 ([3])**.** Let $(X,\perp)$ be an O-set. A mapping $T: X \to X$ is said to be $\perp$-preserving if

$Tx \perp Ty$ whenever $x \perp y$.

Moreover, we define some new properties of the orthogonal sets as follows:

Definition 2.8. We say that an O-set $(X, \perp)$ is a transitive orthogonal set if $\perp$ is transitive.

Definition 2.9. Let $(X, \perp)$ be an O-set. A path of length k in $\perp$ from x to y is a finite sequence

$\{z_0, z_1, \ldots, z_k\} \subset X$ such that

$$z_0 = x^*, z_k = y^*, z_i \perp z_{i+1} \text{ or } z_{i+1} \perp z_i$$

for all $i = 0, 1, 2, \ldots, k-1$.

Let $\Upsilon(x, y, \perp)$ be denoted as all path of length k in $\perp$ from x to y.

The following lemma will be useful later.

Lemma 2.10 ([5]). Let (X, d) be a metric space and $\{x_n\}$ a sequence in X such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy, then there exists $\varepsilon > 0$ and two subsequence $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ where $n(k) > m(k) > k$ such that $\lim_{n\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \lim_{n\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon$ and $\lim_{n\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{n\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \lim_{n\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \varepsilon$.

3. Main Results

In this section, we introduce a new Z-contraction mapping and prove some fixed point theorems for Z-contraction mappings in orthogonally metric spaces.

Definition 3.1. Let $(X, \perp, d)$ be an orthogonally metric space. A mapping $T: X \to X$ is called an orthogonal Z-contraction mapping with respect to ζ (briefly, $Z_\perp$-contraction) if there is $\zeta \in Z$ such that the following condition holds:

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \tag{1}$$

for all $x, y \in X$ with $x \perp y$.

Theorem 3.2. Let $(X, \perp, d)$ *be an O-complete metric space with an orthogonal element* x_0 *and* T *be a self-mapping on* X *satisfying the following conditions:*

1) $(X, \perp)$ *is a transitive orthogonal set;*
2) T *is* $\perp$*-preserving;*
3) T *is a* $Z_\perp$*-contraction mapping;*
4) T *is* $\perp$*-continuous.*

Then T *has a fixed point* $x^* \in X$ *Also, the Picard sequence* $\{T^n x_0\}$ *converges to the fixed point of* T.

Proof. By the definition of orthogonality, there exists $x_0 \in X$ such that

$$(\forall y \in X, x_0 \perp y) \text{ or } (\forall y \in X, y \perp x_0).$$

It follows that

$$x_0 \perp Tx_0 \text{ or } Tx_0 \perp x_0.$$

Let $x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N}_0$. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N}_0$, then x_{n^*} is a fixed point of T and so the proof is completed. So we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Thus we have $d(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}_0$. Since T is $\perp$-preserving, we have

$$(\forall n, x_n \perp x_{n+1}) \text{ or } (\forall n, x_{n+1} \perp x_n).$$

Since T *is a* $Z_\perp$-contraction mapping, we have

$$0 \leq \zeta(d(T^{n+2}x_0, T^{n+1}x_0), d(T^{n+1}x_0, T^n x_0))$$

$$< d(T^{n+1}x_0, T^n x_0) - d(T^{n+2}x_0, T^{n+1}x_0)$$

for all $n \in \mathbb{N}$. It yield that $\{d(T^{n+1}x_0, T^n x_0)\}$ is a monotonically decreasing sequence of positive reals and then there exists $c \geq 0$ such that

$$\lim_{n\to\infty} d(T^{n+1}x_0, T^n x_0) = c.$$

Suppose that $c > 0$. Using (1) and (ζ_3) of Definition 1.1, we have

$$0 \leq \limsup_{n\to\infty} \zeta(d(T^{n+2}x_0, T^{n+1}x_0), d(T^{n+1}x_0, T^n x_0)) < 0,$$

which is a contradiction. Therefore, $c = 0$ and so

$$\lim_{n\to\infty} d(T^{n+1}x_0, T^n x_0) = 0.$$

Now, we show that $\{x_n\}$ is a Cauchy O-sequence. Suppose by contradiction that $\{x_n\}$ is not a Cauchy O-sequence. By Lemma 2.10, there exists $\varepsilon > 0$ and two subsequence $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ where $n(k) > m(k) > k$ such that

$$\lim_{n\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \lim_{n\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2}$$

Since $(X, \perp)$ is a transitive orthogonal set, we have

$$(\forall k, x_{n(k)} \perp x_{m(k)}) \text{ or } (\forall k, x_{m(k)} \perp x_{n(k)}).$$

From *(b)*, (1), (2), and (ζ_3), we have

$$0 \leq \limsup_{n\to\infty} \zeta(d(x_{n(k)+1}, x_{m(k)+1}), d(x_{n(k)}, x_{m(k)})) < 0,$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy O-sequence in X. Since X is O-complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since T is $\perp$-continuous, we have $Tx_n \to Tx^*$ as $n \to \infty$. Thus,

$$Tx^* = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Hence, $x^* \in X$ is a fixed point of T.

Theorem 3.3. *In addition to the hypothesis of Theorem 3.2, suppose that* $\Upsilon(x, y, \perp)$ *is nonempty for all* $x, y \in X$. *Then* T *has a unique fixed point.*

Proof. Suppose that x^*, y^* are two fixed points of T such that $x^* \neq y^*$. Since $\Upsilon(x, y, \perp)$ is nonempty, for all $x, y \in X$, there exists a path $\{z_0, z_1, \ldots, z_k\}$ of some finite length k in $\perp$ from x to y such that

$$z_0 = x^*, z_k = y^*, z_i \perp z_{i+1} \text{ or } z_{i+1} \perp z_i \text{ for all } i = 0,1,2,\ldots,k-1.$$

Since $(X, \perp)$ is a transitive orthogonal set, we get $x^* \perp y^*$ or $y^* \perp x^*$. From (1) and (ζ_3), we have
$0 \le \zeta(d(Tx^*, Ty^*), d(x^*, y^*))$

$$\begin{aligned} &= \zeta(d(x^*, y^*), d(x^*, y^*)) \\ &< 0, \end{aligned}$$

which is a contradiction. Therefore, T has a unique fixed point. This completes the proof.

Corollary 3.4. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a* Z*-contraction mapping. Then* T *has a unique fixed point in* X. *Moreover, for each* $x_0 \in X$, *the Picard sequence* $\{T^n x_0\}$ *converges to the fixed point of* T.

4. Application

In this section, we use the following matrix notations: $M(n)$ denotes the set of all $n \times n$ complex matrices, $P(n)$ denotes the set of all $n \times n$ positive definite matrices, $H^+(n)$ denotes the set of all $n \times n$ positive semidefinite matrices. We write $A \ge B$ ($A > B$) if $A - B \in H^+(n)$ (or $A - B \in P(n)$). In particular, $A \ge 0$ ($A > 0$) implies $A \in H^+(n)$ (or $A \in P(n)$). We also write A^* denotes the conjugate transpose of an $n \times n$ matrix A. In addition, we let d_T denotes the Thompson metric on $P(n)$, which is defined by

$$d_T(A, B) = \log\{\max\{\alpha, \beta\}\}$$

where $\alpha = \inf\{\delta : A \le \delta B\} = \lambda^+\left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\right)$ that is, the maximum eigenvalue of $B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$ and $\beta = \inf\{\delta : B \le \delta A\} = \lambda^+\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)$, that is, the maximum eigenvalue of $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$.

The goal of this section is to apply fixed point results for $Z_\perp$ -contraction mappings via Thompson metrics to solve the nonlinear matrix equation

$$X^r = Q + \sum_{i=1}^{m} A_i^* \mathrm{K}_i(X) A_i \tag{3}$$

where $r \geq 1$, A_i is an $n \times n$ nonsingular matrix, Q is a Hermitian positive definite matrix and K_i is a continuous order preserving self-mapping on $P(n)$.

We recall properties of the Thompson metric for Hermitian positive definite matrices as follows:

Lemma 4.1 ([6]). Let d_T be a Thompson metric on $P(n)$.

(i) $d_T(A,B) = d_T(A^{-1},B^{-1}) = d_T(MAM^*, MBM^*)$ for all $A, B \in P(n)$ and a nonsingular matrix M;

(ii) $d_T(A^r, B^r) \leq |r| d_T(A,B)$ for all $A, B \in P(n)$ and $r \in [-1,1]$;

(iii) $d_T(A+B, C+D) \leq \max\{d_T(A,C), d_T(B,D)\}$ for all $A, B, C, D \in P(n)$.

Theorem 4.2. *Consider the matrix equation (3). Let* $Q \in P(n)$ *and for each* $i = 1, 2, \ldots, m$, $\mathrm{K}_i : P(n) \to P(n)$ *be a continuous order-preserving mapping. Suppose that there are positive number* $r \geq 1$ *with*

$$d_T(\mathrm{K}_i(X), \mathrm{K}_i(Y)) \leq r[d_T(X,Y) - \varphi(d_T(X,Y))]$$

for all $X, Y \in P(n)$, $i = 1, 2, \ldots, m$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous functions such that $\varphi(t) = 0$ if and only if $t = 0$. *Then the equation* (3) *has a unique positive solution.*

Proof. Define the relation $\perp$ on $P(n)$ by

$$A \perp B \Leftrightarrow A \preceq B.$$

Then, by setting $X_0 = 0$, it follows that $(P(n), \perp)$ is an O-set. Also, $\perp$ is transitive on $P(n)$. Since $(P(n), d_T)$ is a complete metric space, this implies that $(P(n), \perp, d_T)$ is an O-complete metric space. Next, we define a mapping $T : P(n) \to P(n)$ by

$$T(X) = \left(Q + \sum_{i=1}^{m} A_i^* \mathrm{K}_i(X) A_i \right)^{\frac{1}{r}}$$

for all $X \in P(n)$. Then T is well-defined, $\perp$-continuous, and $\perp$-preserving.

Now, we show that T is a $Z_\perp$ -contraction mapping with a function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ which is defined by

$$\zeta(t,s) = s - \varphi(s) - t$$

for all $t, s \in [0,\infty)$. Let $X, Y \in P(n)$ such that $X \perp Y$. Then

$$
\begin{aligned}
d_T(T(X),T(Y)) &= d_T\left((Q+\sum_{i=1}^{m} A_i^*\mathrm{K}_i(X)A_i)^{\frac{1}{r}}, d_T(Q+\sum_{i=1}^{m} A_i^*\mathrm{K}_i(Y)A_i)^{\frac{1}{r}}\right)\\
&\le \frac{1}{r} d_T\left(Q+\sum_{i=1}^{m} A_i^*\mathrm{K}_i(X)A_i, Q+\sum_{i=1}^{m} A_i^*\mathrm{K}_i(Y)A_i\right)\\
&\le \frac{1}{r} d_T\left(\sum_{i=1}^{m} A_i^*\mathrm{K}_i(X)A_i, \sum_{i=1}^{m} A_i^*\mathrm{K}_i(Y)A_i\right)\\
&\le \frac{1}{r} \max_{i\in\{1,2,\ldots,m\}} d_T\left(\mathrm{K}_i(X),\mathrm{K}_i(Y)\right)\\
&\le d_T(X,Y)-\varphi(d_T(X,Y))
\end{aligned}
$$

Thus,

$$0 \le d_T(X,Y)-\varphi(d_T(X,Y))-d_T(T(X),T(Y))$$

and so

$$0 \le \zeta(d_T(T(X),T(Y)), d_T(X,Y)).$$

Therefore, T is a $Z_\perp$ -contraction mapping.

By Theorem 3.2, there exists $X^* \in P(n)$ such that $T(X^*)=X^*$. That is, X^* is a positive definite solution of the Equation (3). Since there is a greatest lower bound and a least upper bound, we have $\Upsilon(x,y,\perp)$ is nonempty for all $X,Y \in P(n)$. By Theorem 3.3, it follows that T has a unique fixed point in $P(n)$. This implies that Equation (3) has a unique solution in $P(n)$.

5. Conclusion

Our main theorem is a real generalization of the Khojasteha's fixed point result and the Gordji's fixed point result. Moreover, our new fixed point theorem for $Z_\perp$ -contraction mappings can be applied to show that the matrix equation (3) always has a unique positive definite solution.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

KS conceived of the presented idea and developed the fixed point theorem and performed the applications to a nonlinear matrix equation via Thompson metrics. WS verified the analytical methods and encouraged KS to investigate and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

Acknowledgment

The first author would like to thank the Thammasat University Doctoral Student Scholarship for financial support during the preparation of this manuscript. This study was supported by Thammasat University Research Fund, Contract No. TUGG 37/2562.

References

[1] Banach, S. (1922). Sur les oprations dans les ensembles abstraits et leurs applications. *Fundamenta Mathematicae*, 133–181.

[2] Ran, A. C. M., & Reurings, M. C. B. (2003). A fixed point theorem in partially ordered sets and some applications to matrix equations*. Proceedings of the American Mathematical Society*, *132*, 1435–1443.

[3] Gordji, M. E., Rameani, M., Sen, M., & Cho, Y. J. (2017). On orthogonal sets and Banach fixed point theorem. *Fixed Point Theory*, *18*, 569–578.

[4] Khojasteha, F., Shuklab, S., & Radenovi´c, S. (2015). A new approach to the study of fixed point theory for simulation functions. *Filomat*, *29(6)*, 1189–1194.

[5] Radenovic, S., & Chandok, S. (2018). Simulation type functions and coincidence point results. *Filomat*, *32(1)*, 141-147.

[6] Hussain, N., Ahmad, J., & Azam, A. (2015). On Suzuki-Wardowski type fixed point theorems. *Journal of Nonlinear Sciences and Applications*, *8(6)*, 1095–1111.

Copyright © 2020 by the authors. This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited (CC BY 4.0).

K. Sawangsup was born on June 1, 1992 in Lopburi, Thailand. She graduated from Thammasat University, Thailand, the bachelor degree and master degree of science in mathematics in 2014 and 2016, respectively. Her research interests focus on fixed point theory and its applications.

She have 4 published papers. Now, she work on fixed point theory and applications.

W. Sintunavarat was born on January 7, 1985 in Bangkok, Thailand. He received the bachelor and master of science degree in mathematics at Thammasat University, Thailand and the doctor of philosophy degree in applied mathematics at King Mongkut's University of Technology Thonburi (KMUTT), Thailand under the Research Professional Development Project Under the Science Achievement Scholarship of Thailand (SAST).

He is the lecturer at Thammasat University, Thailand. His research interests focus on fixed point theory and its application in nonlinear functional analysis.

W. Sintunavarat got the distinction award for Graduation (Ph.D Level) with the highest score from Professor Tab Nilanidhi Foundation in Thailand, the good thesis 2014 award from National Research Council of Thailand (NRCT), the World's Most Influential Scientific Minds 2015 from Thomson Reuters. He has published more than 140 research articles in reputed international journals of mathematics.