Sufficient and Necessary Conditions for Hölder's Inequality in Weighted Orlicz Spaces

Al Azhary Masta[1*], Ifronika[2], Siti Fatimah[3]

[1,3] Department of Mathematics Education, Universitas Pendidikan Indonesia, Bandung, Indonesia.
[2] Analysis and Geometry Research Group, Institut Teknologi Bandung, Bandung, West Java, Indonesia.

* Corresponding author. Tel.: +6285603118954; email: alazhari.masta@upi.edu
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Abstract: In this article, we present the sufficient and necessary conditions for Hölder's inequality in weighted Orlicz spaces and in their weak type. One of the keys to prove our results is to estimate the norms of characteristic function in $\mathbb{R}^n$.

Key words: Hölder's inequality, Orlicz spaces, weighted Orlicz spaces.

1. Introduction

The Orlicz spaces were first introduced by Orlicz in [1] are generalizations of Lebesgue spaces. Recently, Osançliol [2] also introduced weighted Orlicz spaces as generalization of Orlicz spaces and weighted Lebesgue spaces. Many researchers have been studying intensively about Orlicz spaces (see [3]-[9], etc.).

Hölder's inequality was first studied by L.C Rogers in 1888 and was reproved by O. Hölder in 1889. The sufficient and necessary conditions for generalized Hölder's inequality in Lebesgue spaces may be found in [3], [10], [11]. In 2018, Ifronika *et al.* [10] obtained the sufficient and necessary conditions for generalized Hölder's inequality in Morrey spaces, in generalized Morrey spaces, and in their weak type. Recently, Ifronika *et al.* [11] also obtained the sufficient and necessary conditions for generalized Hölder's inequality in Orlicz spaces. In 2019, Masta *et al.* [9] also discussed the sufficient condition for Hölder's inequality in weighted Orlicz spaces. Motivated by these results, we would like to discuss the Hölder's inequality in weighted Orlicz spaces and in weighted weak Orlicz spaces.

The novelty of this paper is a necessary condition of Hölder's inequality in weighted Orlicz spaces and in their weak type. From our results, we can also see what parameters are significant in the Hölder's inequality in weighted Orlicz spaces.

First we recall the definition of Young functions. A function $\Phi: [0,\infty) \to [0,\infty)$ is called a Young function if Φ is a convex, left-continuous, $\lim_{t\to 0}\Phi(t) = 0 = \Phi(0)$, and $\lim_{t\to\infty}\Phi(t) = \infty$. For Φ is a Young function, we define $\Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\}$ for every $s \geq 0$. For Φ is a Young function, *the Orlicz space* $L_\Phi(\mathbb{R}^n)$ is the set of functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\| f \|_{L_\Phi(\mathbb{R}^n)} := \inf\left\{b > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{b}\right) dx \leq 1\right\} < \infty. \tag{1}$$

Meanwhile, for Φ is a Young function, *the weak Orlicz space* $wL_\Phi(\mathbb{R}^n)$ is the set of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\| f \|_{wL_\Phi(\mathbb{R}^n)} := \inf\left\{ b > 0 : \sup_{t>0} \Phi(t)\mu\left(\{ x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t \} \right) \le 1 \right\} < \infty. \quad (2)$$

Now, we come to the definition of weighted Orlicz spaces and weighted weak Orlicz spaces. Let Φ be a Young function and u is a weight on $\mathbb{R}^n$ (i.e $u: \mathbb{R}^n \to (0,\infty)$ is a measurable function), *the weighted Orlicz space* $L_\Phi^u(\mathbb{R}^n)$ is the set of all functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\| f \|_{L_\Phi^u(\mathbb{R}^n)} := \| uf \|_{L_\Phi(\mathbb{R}^n)} = \inf\left\{ b > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|u(x)f(x)|}{b} \right) dx \le 1 \right\} < \infty . \quad (3)$$

Note that, if $u(x) = 1$ for every $x \in \mathbb{R}^n$, then $L_\Phi^u(\mathbb{R}^n) = L_\Phi(\mathbb{R}^n)$ is Orlicz space.

Analog with weighted Orlicz spaces, for a Young function Φ and a weight u on $\mathbb{R}^n$, *the weighted weak Orlicz space* $wL_\Phi^u(\mathbb{R}^n)$ is the set of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\| f \|_{wL_\Phi^u(\mathbb{R}^n)} := \| uf \|_{wL_\Phi(\mathbb{R}^n)} < \infty. \quad (4)$$

As well as the Orlicz space and the weak Orlicz space, the relation between weighted weak Orlicz spaces and (strong) weighted Orlicz spaces is

$$L_\Phi^u(\mathbb{R}^n) \subset wL_\Phi^u(\mathbb{R}^n)$$

with $\| f \|_{wL_\Phi^u(\mathbb{R}^n)} \le \| f \|_{L_\Phi^u(\mathbb{R}^n)}$ for every $f \in L_\Phi^u(\mathbb{R}^n)$.

The rest of this paper is organized as follows. In Section 2, we presented some lemmas which useful for obtain our results. The main results are presented in Section 3. In Section 3, we state the sufficient and necessary conditions for Hölder's inequality in weighted Orlicz spaces and in their weak type.

2. Methods

To obtain the sufficient and necessary conditions for Hölder's inequality in weighted Orlicz spaces, we use the norms of the characteristic function in $\mathbb{R}^n$ and some lemmas as in the following.

Lemma 2.1 [3], [4], [11], [12] *Suppose that* Φ *is a Young function and* $\Phi^{-1}(s) := \inf\{r \ge 0 : \Phi(r) > s\}$. *We have*

1) $\Phi^{-1}(0) = 0$.
2) $\Phi^{-1}(s_1) \le \Phi^{-1}(s_2)$ *for* $s_1 \le s_2$.
3) $\Phi(\Phi^{-1}(s)) \le s \le \Phi^{-1}(\Phi(s))$ *for* $0 \le s < \infty$.

Lemma 2.2 [2], [11], [12] *Let* $u: \mathbb{R}^n \to (0,\infty)$ *be a measurable function such that* $u(x+y) \le u(x) \cdot u(y)$ *for every* $x, y \in \mathbb{R}^n$. *If* Φ *is a Young function,* $T_x f(y) = f(y - x)$, *for* $f \in L_\Phi^u(\mathbb{R}^n)$ *and* $f \ne 0$, *then there exists a constant* $C > 0$ *(depends on* f*) such that*

$$\frac{u(x)}{C} \le \| L_x f \|_{wL_\Phi^u(\mathbb{R}^n)} \le \| T_x f \|_{L_\Phi^u(\mathbb{R}^n)} \le Cu(x). \quad (5)$$

Lemma 2.3 [12] *Let* Φ *be a Young function. If* $f \in wL_\Phi^u(\mathbb{R}^n)$, *then for arbitrary* $\epsilon > 0$ *we have*

$$\sup_{t>0} \Phi(t) \left| \left\{ x \in \mathbb{R}^n : \frac{|u(x)f(x)|}{\|f\|_{wL_\Phi^u(\mathbb{R}^n)} + \varepsilon} > t \right\} \right| \le 1. \quad (6)$$

3. Main Results

First, we present the sufficient and necessary conditions for Hölder's inequality in weighted Orlicz spaces in the following theorem.

Theorem 3.1. *Let* Φ_1, Φ_2, Φ_3 *be Young functions and* $u_1, u_2, u_3 : \mathbb{R}^n \to \mathbb{R}$ *be measurable functions such that* $\Phi_1^{-1}(t)\Phi_2^{-1}(t) \leq \Phi_3^{-1}(t)$ *for every* $t > 0$. *Then the following statements are equivalent:*

1) *There exists a constant* $C > 0$ *such that* $u_3(x) \leq C u_1(x) u_2(x)$ *for every* $x \in \mathbb{R}^n$.
2) *For* $f_1 \in L_{\Phi_1}^{u_1}(\mathbb{R}^n)$ *and* $f_2 \in L_{\Phi_2}^{u_2}(\mathbb{R}^n)$, *there exists a constant* $M > 0$ *such that*

$$\| f_1 f_2 \|_{L_{\Phi_3}^{u_3}(\mathbb{R}^n)} \leq M \| f_1 \|_{L_{\Phi_1}^{u_1}(\mathbb{R}^n)} \| f_2 \|_{L_{\Phi_2}^{u_2}(\mathbb{R}^n)}$$

for every $f_1 \in L_{\Phi_1}^{u_1}(\mathbb{R}^n)$ *and* $f_2 \in L_{\Phi_2}^{u_2}(\mathbb{R}^n)$.

Proof.

((1) $\Longrightarrow$ (2)). The proof of (1) implies (2) can be found in [11] and it goes as follows. Suppose that (1) holds. Since Φ is a convex function, we have

$$\int_{\mathbb{R}^n} \Phi_3\left(\frac{|u_3(x) f_1(x) f_2(x)|}{2\|f_1\|_{L_{\Phi_1}^{u_1}(\mathbb{R}^n)} \|f_2\|_{L_{\Phi_2}^{u_2}(\mathbb{R}^n)}}\right) dx \leq \frac{1}{2} \int_{\mathbb{R}^n} \Phi_3\left(\frac{|u_3(x) f_1(x) f_2(x)|}{\|f_1\|_{L_{\Phi_1}^{u_1}(\mathbb{R}^n)} \|f_2\|_{L_{\Phi_2}^{u_2}(\mathbb{R}^n)}}\right) dx$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^n} \Phi_3\left(\frac{|u_1(x) u_2(x) f_1(x) f_2(x)|}{\|f_1\|_{L_{\Phi_1}^{u_1}(\mathbb{R}^n)} \|f_2\|_{L_{\Phi_2}^{u_2}(\mathbb{R}^n)}}\right) dx. \tag{7}$$

Without loss of generality, suppose that $\Phi_1(s) \leq \Phi_2(t)$ for $s, t \geq 0$. By Lemma 2.1(3), we obtain

$$st \leq \Phi_1^{-1}(\Phi_1(s))\Phi_2^{-1}(\Phi_2(t)) \leq \Phi_1^{-1}(\Phi_2(t))\Phi_2^{-1}(\Phi_2(t)) \leq \Phi_3^{-1}(\Phi_2(t)).$$

Hence, we have

$$\Phi_3(st) \leq \Phi_3(\Phi_3^{-1}(\Phi_2(t))) \leq \Phi_2(t) \leq \Phi_2(t) + \Phi_1(s). \tag{8}$$

On the other hand, by using inequality (8), we obtain

$$\int_{\mathbb{R}^n} \Phi_3\left(\frac{|u_1(x) u_2(x) f_1(x) f_2(x)|}{\|f_1\|_{L_{\Phi_1}^{u_1}(\mathbb{R}^n)} \|f_2\|_{L_{\Phi_2}^{u_2}(\mathbb{R}^n)}}\right) dx \leq \int_{\mathbb{R}^n} \Phi_1\left(\frac{|u_1(x) f_1(x)|}{\|f_1\|_{L_{\Phi_1}^{u_1}(\mathbb{R}^n)}}\right) dx + \int_{\mathbb{R}^n} \Phi_2\left(\frac{|u_2(x) f_2(x)|}{\|f_2\|_{L_{\Phi_2}^{u_2}(\mathbb{R}^n)}}\right) dx \quad \leq 2, \tag{9}$$

whenever $f_1 \in L_{\Phi_1}^{u_1}(\mathbb{R}^n)$ and $f_2 \in L_{\Phi_2}^{u_2}(\mathbb{R}^n)$. From inequality (7) and (9) we have,

$$\int_{\mathbb{R}^n} \Phi_3\left(\frac{|u_3(x) f_1(x) f_2(x)|}{2 \| f_1 \|_{L_{\Phi_1}^{u_1}(\mathbb{R}^n)} \| f_2 \|_{L_{\Phi_2}^{u_2}(\mathbb{R}^n)}}\right) dx \leq 1.$$

By definition of $\|\cdot\|_{L_{\Phi_3}^{u_3}(\mathbb{R}^n)}$, we have $\| f_1 f_2 \|_{L_{\Phi_3}^{u_3}(\mathbb{R}^n)} \leq 2 \| f_1 \|_{L_{\Phi_1}^{u_1}(\mathbb{R}^n)} \| f_2 \|_{L_{\Phi_2}^{u_2}(\mathbb{R}^n)}$.

((2) $\Longrightarrow$ (1)). Assume that (2) holds. Take arbitrary $f_1 \in L_{\Phi_1}^{u_1}(\mathbb{R}^n)$ and $f_2 \in L_{\Phi_2}^{u_2}(\mathbb{R}^n)$. By Lemma 2.2, we have

$$\frac{u_3(x)}{C} \leq \| T_x f_1 T_x f_2 \|_{L^{u_3}_{\Phi_3}(\mathbb{R}^n)} \leq C \| T_x f_1 \|_{L^{u_1}_{\Phi_1}(\mathbb{R}^n)} \| T_x f_2 \|_{L^{u_2}_{\Phi_2}(\mathbb{R}^n)} \leq C u_1(x) u_2(x),$$

for every $x \in \mathbb{R}^n$. So, we obtain $u_3(x) \leq M u_1(x) u_2(x)$, for $M = C^2$.

Corollary 3.2. (Hölder's inequality in weighted Lebesgue spaces) *Let* $1 \leq p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ and $u_1, u_2, u_3 : \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Then the following statements are equivalent:

1) *There exists a constant $C > 0$ such that $u_3(x) \leq C u_1(x) u_2(x)$ for every $x \in \mathbb{R}^n$.*
2) *For $f_1 \in L^{u_1}_{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{u_2}_{p_2}(\mathbb{R}^n)$, there exists a constant $M > 0$ such that*

$$\| f_1 f_2 \|_{L^{u_3}_{p_3}(\mathbb{R}^n)} \leq M \| f_1 \|_{L^{u_1}_{p_1}(\mathbb{R}^n)} \| f_2 \|_{L^{u_2}_{p_2}(\mathbb{R}^n)}$$

for every $f_1 \in L^{u_1}_{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{u_2}_{p_2}(\mathbb{R}^n)$.

Proof.

Let $\Phi_1(t) := t^{p_1}, \Phi_2(t) := t^{p_2}$, $\Phi_3(t) := t^{p_3}$ for every $t \geq 0$. Since $1 \leq p_1, p_2, p_3 < \infty$, we have Φ_1, Φ_2, and Φ_3 are Young functions. Observe that, using the definition of Φ^{-1}, we also obtain

$$\Phi_1^{-1}(t) = t^{\frac{1}{p_1}},\ \Phi_2^{-1}(t) = t^{\frac{1}{p_2}}, \text{and}\ \Phi_3^{-1}(t) = t^{\frac{1}{p_3}}.$$

Moreover, $\Phi_1^{-1}(t)\Phi_2^{-1}(t) = t^{\frac{1}{p_1}} t^{\frac{1}{p_2}} = t^{\frac{1}{p_3}} = \Phi_3^{-1}(t)$. By using Theorem 3.1, we have (1) and (2) are equivalent.

Now we come to the sufficient and necessary conditions for Hölder's inequality in weighted weak Orlicz spaces as the following theorem.

Theorem 3.3 *Let Φ_1, Φ_2, Φ_3 be Young functions and $u_1, u_2, u_3 : \mathbb{R}^n \to \mathbb{R}$ be measurable functions such that $\Phi_1^{-1}(t)\Phi_2^{-1}(t) \leq \Phi_3^{-1}(t)$ for every $t > 0$. Then the following statements are equivalent:*

1) *There exists a constant $C > 0$ such that $u_3(x) \leq C u_1(x) u_2(x)$ for every $x \in \mathbb{R}^n$.*
2) *There exists a constant $M > 0$ such that*

$$\| f_1 f_2 \|_{wL^{u_3}_{\Phi_3}(\mathbb{R}^n)} \leq M \| f_1 \|_{wL^{u_1}_{\Phi_1}(\mathbb{R}^n)} \| f_2 \|_{wL^{u_2}_{\Phi_2}(\mathbb{R}^n)}$$

for every $f_1 \in wL^{u_2}_{\Phi_2}(\mathbb{R}^n)$ and $f_2 \in wL^{u_2}_{\Phi_2}(\mathbb{R}^n)$.

Proof.

((1) $\Longrightarrow$ (2)). The proof of (1) implies (2) can be found in [12] and it goes as follows. Suppose that (1) holds. Let f_i be elements of $wL^{u_i}_{\Phi_i}(\mathbb{R}^n)$, $i = 1,2$. By Lemma 2.3, for every $k \in \mathbb{N}$ we have

$$\Phi_1(t)\left|\left\{x \in \mathbb{R}^n : \frac{|u_1(x) f_1(x)|}{(1+\frac{1}{k})\|f_1\|_{wL^{u_1}_{\Phi_1}(\mathbb{R}^n)}} > t\right\}\right| \leq 1 \ \text{ and } \ \Phi_2(t)\left|\left\{x \in \mathbb{R}^n : \frac{|u_2(x) f_2(x)|}{(1+\frac{1}{k})\|f_2\|_{wL^{u_2}_{\Phi_2}(\mathbb{R}^n)}} > t\right\}\right| \leq 1$$

for every $t > 0$.

For each $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, let $\quad M(x,k) := \max\left\{\Phi_1\left(\frac{|u_1(x) f_1(x)|}{(1+\frac{1}{k})\|f_1\|_{wL^{u_1}_{\Phi_1}(\mathbb{R}^n)}}\right), \Phi_2\left(\frac{|u_2(x) f_2(x)|}{(1+\frac{1}{k})\|f_2\|_{wL^{u_2}_{\Phi_2}(\mathbb{R}^n)}}\right)\right\}.$

From $\Phi_i\left(\frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}\right) \le M(x,k)$ and Lemma 2.1 (3), we have

$$\frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}} \le \Phi_i^{-1}\left(\Phi_i\left(\frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}\right)\right) \le \Phi_i^{-1}(M(x,k)), \tag{10}$$

where $i = 1,2$.

Since inequality (9) is true for $i = 1,2$, we have

$$\prod_{i=1}^{2} \frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}} \le \Phi_1^{-1}\big(M(x,k)\big)\Phi_2^{-1}\big(M(x,k)\big) \le \Phi_3^{-1}\big(M(x,k)\big). \tag{11}$$

By using inequality (11) and Φ is increasing function, we obtain

$$\Phi_3\left(\prod_{i=1}^{2} \frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\parallel f_i \parallel_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}\right) \le \Phi_3(\Phi_3^{-1}(M(x,k))) \le M(x,k).$$

On the other hand, we have $M(x,k) \le \sum_{i=1}^{2} \Phi_i\left(\frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}\right)$.

Therefore

$$\Phi_3(t)\left|\left\{x \in \mathbb{R}^n: \prod_{i=1}^{2} \frac{|u_3(x)\, f_i(x)|}{(1+\frac{1}{k})\parallel f_i \parallel_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}} > t\right\}\right| = \Phi_3\left(\prod_{i=1}^{2} \frac{t_0|u_3(x)f_i(x)|}{(1+\frac{1}{k})\parallel f_i \parallel_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}\right)|\{x \in \mathbb{R}^n: 1 > t_0\}|$$

$$\le \Phi_3\left(\prod_{i=1}^{2} \frac{t_0|u_i(x)f_i(x)|}{(1+\frac{1}{k})\parallel f_i \parallel_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}\right)|\{x \in \mathbb{R}^n: 1 > t_0\}|$$

$$\le \sum_{i=1}^{2} \Phi_i\left(\frac{t_0|u_i(x)f_i(x)|}{(1+\frac{1}{k})\parallel f_i \parallel_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}\right)|\{x \in \mathbb{R}^n: 1 > t_0\}|$$

where $t_0 := \frac{t(1+\frac{1}{k})\|f_1\|_{wL^{u_1}_{\Phi_1}(\mathbb{R}^n)}\|f_2\|_{wL^{u_2}_{\Phi_2}(\mathbb{R}^n)}}{|u_3(x)f_1(x)f_2(x)|}$.

Next, we also have

$$\Phi_i\left(\frac{t_0|u_i(x)f_i(x)|}{\left(1+\frac{1}{k}\right)\parallel f_i \parallel_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}\right)|\{x \in \mathbb{R}^n: 1 > t_0\}| = \Phi_i(t_i)\left|\left\{x \in \mathbb{R}^n: \left(\frac{|u_i(x)f_i(x)|}{\left(1+\frac{1}{k}\right)\parallel f_i \parallel_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}\right) > t_i\right\}\right|$$
$$\le 1$$

where $t_i = \frac{t_0|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}}$, for $i = 1,2$. So, we obtain $\Phi_3(t)\left|\left\{x \in \mathbb{R}^n : \prod_{i=1}^{2} \frac{|u_3(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}} > t\right\}\right| \leq 2.$

On the other hand, we have

$$\Phi_3(t)\left|\left\{x \in \mathbb{R}^n : \prod_{i=1}^{2} \frac{|u_3(x)f_i(x)|}{\sqrt{2}(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}} > t\right\}\right| \leq \sup_{t>0}\Phi_3(t)\left|\left\{x \in \mathbb{R}^n : \prod_{i=1}^{2} \frac{|u_3(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}} > 2t\right\}\right|$$

$$= \sup_{s>0}\Phi_3\left(\frac{s}{2}\right)\left|\left\{x \in \mathbb{R}^n : \prod_{i=1}^{2} \frac{|u_3(x)f_i(x)|}{(1+\frac{1}{k})\parallel f_i \parallel_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}} > s\right\}\right|$$

$$\leq \sup_{s>0}\frac{1}{2}\Phi_3(s)\left|\left\{x \in \mathbb{R}^n : \prod_{i=1}^{2} \frac{|u_3(x)f_i(x)|}{(1+\frac{1}{k})\parallel f_i \parallel_{wL^{u_i}_{\Phi_i}(\mathbb{R}^n)}} > s\right\}\right| \leq 1.$$

Since $s > 0$ is an arbitrary positive real number, we get

$$\sup_{t>0}\Phi_3(t)\left|\left\{x \in \mathbb{R}^n : \frac{|u_3(x)f_1(x)f_2(x)|}{2(1+\frac{1}{k})\|f_1\|_{wL^{u_1}_{\Phi_1}(\mathbb{R}^n)}\|f_2\|_{wL^{u_2}_{\Phi_2}(\mathbb{R}^n)}} > t\right\}\right| \leq 1.$$

This shows that

$$\parallel f_1 f_2 \parallel_{wL^{u_3}_{\Phi_3}(\mathbb{R}^n)} \leq 2\left(1+\frac{1}{k}\right) \parallel f_1 \parallel_{wL^{u_1}_{\Phi_1}(\mathbb{R}^n)} \parallel f_2 \parallel_{wL^{u_2}_{\Phi_2}(\mathbb{R}^n)}$$

and this is true for every $k \in \mathbb{N}$. We can conclude that

$$\parallel f_1 f_2 \parallel_{wL^{u_3}_{\Phi_3}(\mathbb{R}^n)} \leq 2 \parallel f_1 \parallel_{wL^{u_1}_{\Phi_1}(\mathbb{R}^n)} \parallel f_2 \parallel_{wL^{u_2}_{\Phi_2}(\mathbb{R}^n)}.$$

((2) $\Longrightarrow$ (1)). Assume that (2) holds. Take arbitrary $f_1 \in L^{u_1}_{\Phi_1}(\mathbb{R}^n)$ and $f_2 \in L^{u_2}_{\Phi_2}(\mathbb{R}^n)$. By Lemma 2.2, we have

$$\frac{u_3(x)}{C} \leq \parallel T_x f_1 T_x f_2 \parallel_{wL^{u_3}_{\Phi_3}(\mathbb{R}^n)} \leq C \parallel T_x f_1 \parallel_{wL^{u_1}_{\Phi_1}(\mathbb{R}^n)} \parallel T_x f_2 \parallel_{wL^{u_2}_{\Phi_2}(\mathbb{R}^n)} \leq C u_1(x) u_2(x),$$

for every $x \in \mathbb{R}^n$. So, we obtain $u_3(x) \leq Cu_1(x)u_2(x)$, for every $x \in \mathbb{R}^n$.

Corollary 3.4. (Hölder's inequality in weighted weak Lebesgue spaces) *Let* $1 \leq p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ and $u_1, u_2, u_3 : \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Then the following statements are equivalent:

1) *There exists a constant* $C > 0$ *such that* $u_3(x) \leq Cu_1(x)u_2(x)$ *for every* $x \in \mathbb{R}^n$.
2) *For* $f_1 \in wL^{u_1}_{p_1}(\mathbb{R}^n)$ *and* $f_2 \in wL^{u_2}_{p_2}(\mathbb{R}^n)$, *there exists a constant* $M > 0$ *such that*

$$\parallel f_1 f_2 \parallel_{L^{u_3}_{p_3}(\mathbb{R}^n)} \leq M \parallel f_1 \parallel_{wL^{u_1}_{p_1}(\mathbb{R}^n)} \parallel f_2 \parallel_{wL^{u_2}_{p_2}(\mathbb{R}^n)}$$

for every $f_1 \in wL^{u_1}_{p_1}(\mathbb{R}^n)$ and $f_2 \in wL^{u_2}_{p_2}(\mathbb{R}^n)$.

Proof.

Let $\Phi_1(t) := t^{p_1}, \Phi_2(t) := t^{p_2}$, $\Phi_3(t) := t^{p_3}$ for every $t \geq 0$. Since $1 \leq p_1, p_2, p_3 < \infty$, we have Φ_1, Φ_2, and Φ_3 are Young functions. Observe that, using the definition of Φ^{-1}, we also obtain

$$\Phi_1^{-1}(t) = t^{\frac{1}{p_1}},\ \Phi_2^{-1}(t) = t^{\frac{1}{p_2}}, and\ \Phi_3^{-1}(t) = t^{\frac{1}{p_3}}.$$

Moreover, $\Phi_1^{-1}(t)\Phi_2^{-1}(t) = t^{\frac{1}{p_1}}t^{\frac{1}{p_2}} = t^{\frac{1}{p_3}} = \Phi_3^{-1}(t)$. By using Theorem 3.3, we have (1) and (2) are equivalent.

4. Conclusions

We have shown the sufficient and necessary conditions for generalized Hölder's inequality in $L^u_\Phi(\mathbb{R}^n)$ space and in $wL^u_\Phi(\mathbb{R}^n)$ space. From Theorems 3.1 and 3.3, we see that both Hölder's inequality in weighted Orlicz spaces and in weighted weak Orlicz spaces are equivalent to the same condition, namely $u_3(x) \leq Cu_1(x)u_2(x)$.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

Al A. Masta and Ifronika conceived the discussed idea and proved the theorem. S. Fatimah wrote the paper and supervised the results of the work. All of authors had approved the final results and contributed to the final manuscript.

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Al Azhary Masta is an assistant professor in the Department of Mathematics Educations, Universitas Pendidikan Indonesia, Indonesia. He was born in Sungai Liat, Indonesia. He got his first degree at Universitas Pendidikan Indonesia, Indonesia in 2011, the master degree at Institut Teknologi Bandung in 2013, and Ph.D degree from the Department of Mathematics, Institut Teknologi Bandung, Indonesia in 2018. His area of interests are mathematical analysis and functional analysis.

Ifronika is an assistant professor in the Department of Mathematics, Institut Teknologi Bandung. She was born in Lirik, Indonesia. She got her first degree at Universitas Riau, Indonesia in 2013 and the master degree at Institut Teknologi Bandung, Indonesia in 2015, both in mathematics. Her area of interests are mathematical analysis and functional analysis.

Siti Fatimah is an associate professor in the Department of Mathematics educations, Universitas Pendidikan Indonesia, Indonesia. She was born in Yogyakarta, Indonesia. She obtained her first degree from the Department of Mathematics Education, Universitas Pendidikan Indonesia, Indonesia in 1992 and the Ph.D degree at Utrecht University, Netherlands in 2002. Her major research interests are in the area of differential equations and mathematics educations.