A Higher-Order Numerical Method on the Shishkin Mesh for Time-Dependent Problems with Boundary Layers

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Abstract: In this paper, a higher-order numerical method for time-dependent singularly perturbed problems is constructed on the Shishkin mesh. The method consists of Crank-Nicolson method for the time discretization and a hybrid difference scheme that combines the midpoint upwind difference scheme on the coarse mesh and the central difference scheme on the fine mesh for the spatial discretization. We prove that the method is uniformly convergent with respect to the singular perturbation parameter, having order near two in space and order two in time. Finally, numerical results support the convergence behavior.

Key words: Time-dependent problems, Crank-Nicolson method, Hybrid finite difference method, Shishkin mesh, uniform higher-order error estimate.

1. Introduction

Consider the singularly perturbed initial-boundary value problem

$$\begin{cases} L_\varepsilon u \equiv u_t - \varepsilon u_{xx}(x,t) + a(x)u_x(x,t) + b(x,t)u(x,t) = f(x,t), (x,t) \in \Omega, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \ \ 0 \le t \le T, \end{cases} \tag{1}$$

where $\Omega = (0,1) \times [0,T]$, $0 < \varepsilon \le 1$ is a small perturbation parameter, functions $a(x), b(x,t), u_0(x)$ and $f(x,t)$ are sufficiently smooth satisfying $a(x) > \alpha > 0$ and $b = b(x,t) \ge 0$, where α is a constant. Under these conditions and some corner compatibility conditions, the problem (1) has a unique solution with a boundary layer at $x = 1$ (see [1], [2]). Time-dependent problems arise in various fields of engineering and science, for example elasticity, fluid dynamics, hydrodynamics, etc. Many authors have discussed fitted mesh finite difference methods to solve these problems (see [1]-[11]).

In this paper, we construct a fully discrete scheme to solve (1), using the Crank-Nicolson method to discretize in time and the hybrid difference on the Shishkin mesh in space. Furthermore, we establish the important discrete maximum principle and obtain the uniform higher-order convergence. Finally, the convergence behaviors are confirmed by numerical experiments.

Throughout the paper, C is a generic positive constant, dependent of the perturbation parameter ε and mesh parameters N and M, the norm $\|\cdot\|$ (sometimes subscripted) is the maximum norm.

2. The Fully Discrete Scheme and Its Uniform Higher-Order Convergence

2.1. Mesh Generation in Time and Space

The time interval $[0,T]$ is divided into M equal subintervals as $t_j = \frac{T}{M} j,\ j = 0,1,\ldots,M$. Denote $\tau = t_j - t_{j-1}$.

Let N be a positive even integer and $\sigma = \min\{\frac{1}{2}, \frac{4\varepsilon}{\alpha}\ln N\}$. We generally take $\sigma = \frac{4\varepsilon}{\alpha}\ln N \le \frac{1}{2}$. Choose $1-\sigma$ be the transition point. Divide the space interval $[0,1-\sigma]$ and $[1-\sigma,1]$ uniformly into $N/2$ subintervals, respectively. Then the Shishkin mesh is

$$x_i = \begin{cases} \frac{2(1-\sigma)}{N} i,\ 0 \le i \le N/2, \\ 1 - 2\sigma(1 - \frac{i}{N}),\ N/2 \le i \le N. \end{cases} \tag{2}$$

Denote $h_i = x_i - x_{i-1},\ i = 1,2,\ldots,N.$

Lemma 2.1 $N^{-1} \le h_i = \frac{2(1-\sigma)}{N} \le 2N^{-1}$ and $h_{N/2+i} = \frac{8\varepsilon}{\alpha} N^{-1}\ln N \le N^{-1}$ for $i = 1,2,\ldots,N/2$.

Similarly, the Bakhvalov-Shishkin mesh replaces (2) with $1 + \frac{4\varepsilon}{\alpha}\ln[1 - 2(1-N^{-1})(1-\frac{i}{N})],\ N/2 \le i \le N$. Then $h_{N/2+i} < \frac{8\varepsilon}{i\alpha} \le CN^{-1},\ i = 1,2,\ldots,N/2.$ in Lemma 2.1.

2.2. Fully Discrete Scheme

Consider using the Crank-Nicolson method for the time discretization and the hybrid difference scheme that combines the midpoint upwind difference scheme on the coarse mesh and the central difference scheme on the fine mesh for the spatial discretization:

$$\begin{aligned} & L_\varepsilon^{M,N} U_i^{j+1/2} = \tilde{f}_i^{j+1/2},\ \ 0 < i < N \text{ and } 0 \le j < M, \\ & U_i^0 = u_0(x_i),\ \ i = 0,\ldots,N, \\ & U_0^j = U_N^j = 0,\ \ j = 0,1,\ldots,M, \end{aligned} \tag{3}$$

where

$$L_\varepsilon^{M,N} U_i^{j+1/2} = \begin{cases} \delta_t\left(\frac{U_i^{j+1/2} + U_{i-1}^{j+1/2}}{2}\right) - \varepsilon\delta_x^2 U_i^{j+1/2} + a_{i-1/2} D_x^- U_i^{j+1/2} + b_{i-1/2}^{j+1/2} \frac{(U_{i-1}^{j+1/2} + U_i^{j+1/2})}{2}, 0 < i \le N/2, \\ \delta_t U_i^{j+1/2} - \varepsilon\delta_x^2 U_i^{j+1/2} + a_i D_x^0 U_i^{j+1/2} + b_i^{j+1/2} U_i^{j+1/2}, \quad N/2 < i < N, \end{cases}$$

$$\tilde{f}_i^{j+1/2} = \begin{cases} f(x_{i-1/2}, t_{j+1/2}),\ 0 < i \le N/2, \\ f(x_i, t_{j+1/2}), \quad N/2 < i < N \end{cases}, U_i^{j+1/2} = \frac{U_i^{j+1} + U_i^j}{2},\ \delta_t U_i^{j+1/2} = \frac{U_i^{j+1} - U_i^j}{\tau},$$

$$D_x^+ U_i^j = \frac{U_{i+1}^j - U_i^j}{h_{i+1}}, D_x^- U_i^j = \frac{U_i^j - U_{i-1}^j}{h_i}, D_x^0 U_i^j = \frac{U_{i+1}^j - U_{i-1}^j}{h_{i+1} + h_i}, \delta_x^2 U_i^j = \frac{2\left(D_x^+ U_i^j - D_x^- U_i^j\right)}{h_{i+1} + h_i},$$

$$a_{i-1/2}=a(x_{i-1/2})=a(\frac{x_i+x_{i-1}}{2}) \quad \text{and} \quad b_{i-1/2}^{j+1/2}=b(x_{i-1/2},t_{j+1/2})=b(\frac{x_i+x_{i-1}}{2},\frac{t_{j+1}+t_j}{2}).$$

That is

$$L_\varepsilon^{M,N}U_i^{j+1/2}=(\frac{\varepsilon}{h_ih_{i+1}}+\frac{a_{i-1/2}}{2h_i}+\frac{b_{i-1/2}^{j+1/2}}{4}+\frac{1}{2\tau})U_i^{j+1}-\frac{\varepsilon}{h_{i+1}(h_i+h_{i+1})}U_{i+1}^{j+1}-(\frac{\varepsilon}{h_i(h_i+h_{i+1})}+\frac{a_{i-1/2}}{2h_i}-\frac{b_{i-1/2}^{j+1/2}}{4}-\frac{1}{2\tau})U_{i-1}^{j+1}$$
$$+(\frac{\varepsilon}{h_ih_{i+1}}+\frac{a_{i-1/2}}{2h_i}+\frac{b_{i-1/2}^{j+1/2}}{4}-\frac{1}{2\tau})U_i^{j}-\frac{\varepsilon}{h_{i+1}(h_i+h_{i+1})}U_{i+1}^{j}-(\frac{\varepsilon}{h_i(h_i+h_{i+1})}+\frac{a_{i-1/2}}{2h_i}-\frac{b_{i-1/2}^{j+1/2}}{4}+\frac{1}{2\tau})U_{i-1}^{j},$$

for $0<i\le N/2,$ and

$$L_\varepsilon^{M,N}U_i^{j+1/2}=(\frac{\varepsilon}{h_ih_{i+1}}+\frac{b_i^{j+1/2}}{2}+\frac{1}{\tau})U_i^{j+1}-(\frac{\varepsilon}{h_{i+1}(h_i+h_{i+1})}-\frac{a_i}{2(h_i+h_{i+1})})U_{i+1}^{j+1}-(\frac{\varepsilon}{h_i(h_i+h_{i+1})}+\frac{a_i}{2(h_i+h_{i+1})})U_{i-1}^{j+1}$$
$$+(\frac{\varepsilon}{h_ih_{i+1}}+\frac{b_i^{j+1/2}}{2}-\frac{1}{\tau})U_i^{j}-(\frac{\varepsilon}{h_{i+1}(h_i+h_{i+1})}-\frac{a_i}{2(h_i+h_{i+1})})U_{i+1}^{j}-(\frac{\varepsilon}{h_i(h_i+h_{i+1})}+\frac{a_i}{2(h_i+h_{i+1})})U_{i-1}^{j},$$

for $N/2<i<N.$

Lemma 2.2 (Discrete Maximum Principle) If $\frac{\varepsilon}{h_ih_{i+1}}+\frac{a_{i-1/2}}{2h_i}+\frac{b_{i-1/2}^{j+1/2}}{4}-\frac{1}{2\tau}\le 0$ for $0<i\le\frac{N}{2}$, $\frac{\varepsilon}{h_ih_{i+1}}+\frac{b_i^{j+1/2}}{2}-\frac{1}{\tau}\le 0$ and $-\frac{\varepsilon}{h_{i+1}}+\frac{a_i}{2}\le 0$ for $\frac{N}{2}<i<N$, and the mesh function ψ_i^j satisfies $\psi_i^0\ge 0$ for $i=0,\dots,N$, $\psi_0^j\ge 0$ and $\psi_N^j\ge 0$ for $j=0,1,\dots,M$, then $L_\varepsilon^{M,N}\psi_i^{j+1/2}\ge 0$ for $0<i<N$ and $0\le j<M$ implies that $\psi_i^j\ge 0$ for $0\le i\le N$ and $0\le j\le M$.

Proof. Using mathematical induction to prove this theorem, if $\psi_i^j\ge 0$ for $0\le i\le N$ and $j\le q$, Assumed that there exist a point $(p,q+1)$ such that $\psi_p^{q+1}=\min_{0<i<N}\psi_i^{q+1}<0$, by $\frac{\varepsilon}{h_ph_{p+1}}+\frac{a_{p-1/2}}{2h_p}+\frac{b_{p-1/2}^{q+1/2}}{4}-\frac{1}{2\tau}\le 0$, we have $L_\varepsilon^{M,N}\psi_p^{q+1/2}<0$ for $0<p\le\frac{N}{2}$, and by $-\frac{\varepsilon}{h_{p+1}}+\frac{a_p}{2}\le 0$ and $\frac{\varepsilon}{h_ph_{p+1}}+\frac{b_p^{q+1/2}}{2}-\frac{1}{\tau}\le 0$, we have $L_\varepsilon^{M,N}\psi_p^{q+1/2}<0$ for $\frac{N}{2}<p<N$, which is a contradiction as $L_\varepsilon^{M,N}\psi_i^{j+1/2}\ge 0$ for $0<i<N$ and $0\le j<M$. Therefore, $\psi_i^j\ge 0$ for $0\le i\le N$ and $0\le j\le M$.

Corollary 2.3 When the first three conditions in Lemma 2.2 are replaced by $\frac{N}{\ln N}\ge\frac{2\|a\|_\infty}{\alpha}$ and $\frac{N^2\alpha}{2\sigma\ln N}+\frac{\|b\|_\infty}{2}\le\frac{1}{\tau}$ on the Shishkin mesh (2), the conclusion of Lemma 2.2 is effective.

Theorem 2.4 Supposed that $\frac{N}{\ln N}\ge\frac{2\|a\|_\infty}{\alpha}$ and $\frac{N^2\alpha}{2\sigma\ln N}+\frac{\|b\|_\infty}{2}\le\frac{1}{\tau}$, let u be the solution of the problem (1) and u_i^j be the solution of the problem (3) on (2), then the following error estimate exists:

$$\left|u_i^j-U_i^j\right|\le\begin{cases}C(N^{-2}+M^{-2}),\ 0\le i\le N/2,\ 0\le j\le M,\\ C(N^{-2}\ln^2 N+M^{-2}),\ N/2\le i\le N,\ 0\le j\le M.\end{cases}$$

Proof. The solution $u(x,t)$ of (1) and the solution U_i^j of (3) can be split into the smooth component and the layer component, respectively: $u(x,t)=v(x,t)+w(x,t)$, and $U_i^j=V_i^j+W_i^j$. Therefore, $\left|u_i^j-U_i^j\right|\le\left|v_i^j-V_i^j\right|+\left|w_i^j-W_i^j\right|$. As for the smooth component v_i^j and V_i^j, by using the techniques in [9], we have $\left|v_i^j-V_i^j\right|\le C((N^{-1}\ln N)^2+M^{-2})$ for $0\le i\le N$ and $0\le j\le M$. Similarly, the layer component satisfies $\left|w_i^j-W_i^j\right|\le CN^{-2}$ for $0\le i\le N/2$ and $0\le j\le M$. Furthermore, the techniques in [9] for the time-dependent singularly perturbed problem and in [1] for the two-point boundary value problem are extended to prove that $\left|w_i^j-W_i^j\right|\le C(N^{-2}\ln^2 N+M^{-2})$ for $N/2\le i\le N$ and $0\le j\le M$.

3. Numerical Results

The numerical results of the fully discrete scheme (3) are shown in Tables 1-4 and Figures 1-2. The maximum errors are given by $\mathrm{e}_L^{M,N}=\max\limits_{0\le i\le N/2}\left|u(x_i,t_M)-U_i^M\right|$ and $\mathrm{e}_R^{M,N}=\max\limits_{N/2<i\le N}\left|u(x_i,t_M)-U_i^M\right|$ for the coarse part and the fine part on the Shishkin mesh respectively, where $u(x_i,t_j)$ and U_i^j denote the exact and the numerical solutions with N mesh intervals in the spatial direction and M mesh intervals in the time direction. And the numerical convergence orders and the numerical convergence constants in the time direction are calculated by $\log_2(\frac{\mathrm{e}_L^{M,N}}{\mathrm{e}_L^{2M,N}}), \frac{\mathrm{e}_L^{M,N}}{M^{-2}}, \log_2(\frac{\mathrm{e}_R^{M,N}}{\mathrm{e}_R^{2M,N}}), \frac{\mathrm{e}_R^{M,N}}{M^{-2}}$ respectively. Similarly, the numerical convergence orders and the numerical convergence constants in the spatial direction are calculated by $\log_2(\frac{\mathrm{e}_L^{M,N}}{\mathrm{e}_L^{M,2N}}), \frac{\mathrm{e}_L^{M,N}}{N^{-2}}, \log_2(\frac{\mathrm{e}_R^{M,N}}{\mathrm{e}_R^{M,2N}}), \frac{\mathrm{e}_R^{M,N}}{N^{-2}\ln^2 N}$ respectively.

Problem 1

$$\begin{cases} u_t-\varepsilon u_{xx}+u_x+u=e^{-t}(1-e^{-1/\varepsilon})(\frac{\pi^2\varepsilon}{4}\sin\frac{\pi}{2}x+\frac{\pi}{2}\cos\frac{\pi}{2}x), \quad (x,t)\in(0,1)\times(0,1], \\ u(x,0)=u_0(x)=e^{-1/\varepsilon}+(1-e^{-1/\varepsilon})\sin\frac{\pi}{2}x-e^{-(1-x)/\varepsilon}, 0\le x\le 1, \\ u(0,t)=u(1,t)=0, \ 0\le t\le 1, \end{cases}$$

where the exact solution is $u(x,t)=e^{-t}(e^{-1/\varepsilon}+(1-e^{-1/\varepsilon})\sin\frac{\pi}{2}x-e^{-(1-x)/\varepsilon})$.

The numerical results of (3) for Problem 1 in time and space are shown in Tables 1 and 2, respectively.

Table 1. The Numerical Results of the Scheme (3) When *N*=3200 for Problem 1

M	$\mathrm{e}_L^{M,N}$	order	const	$\mathrm{e}_R^{M,N}$	order	const
2	4.290001e-02	2.9002	0.1716	4.290118e-02	2.9001	0.1716
4	5.746627e-03	2.0103	0.0919	5.746965e-03	2.0695	0.0920
8	1.426459e-03	1.9524	0.0913	1.369173e-03	1.8930	0.0876
16	3.685773e-04	1.9661	0.0944	3.686375e-04	1.8992	0.0944
32	9.433464e-05	2.0061	0.0966	9.882873e-05	1.3777	0.1012
64	2.348368e-05		0.0962	3.803356e-05		0.1558

Table 2. The Numerical Results of the Scheme (3) When M=3200 for Problem 1

N	$e_L^{M,N}$	order	const	$e_R^{M,N}$	order	const
10	3.630137e-03	4.6587	0.3630	5.710833e-02	0.2593	5.7108
20	1.437149e-04	2.0473	0.0575	4.771354e-02	0.9012	11.2753
40	3.476930e-05	2.0017	0.0556	2.554737e-02	1.3071	15.9261
80	8.681844e-06	2.0055	0.0556	1.032471e-02	1.6940	18.2448
160	2.162245e-06	2.0204	0.0554	3.191016e-03	1.6731	16.8151
320	5.329666e-07		0.0546	1.000629e-03		16.3270

The $\log 2-\log 2$ graphs of errors illustrate the convergence orders in time for the scheme (3) on $[0,1-\sigma]$ and $[1-\sigma]$ on the Shishkin mesh and the Crank-Nicolson & Simple method on $[0,1-\sigma]$ and $[1-\sigma]$ on the Bakhvalov-Shishkin mesh in Fig. 1 (a) and Fig. 1 (b). The convergence orders in space for both schemes are shown in Fig. 2 (a) and Fig. 2 (b).

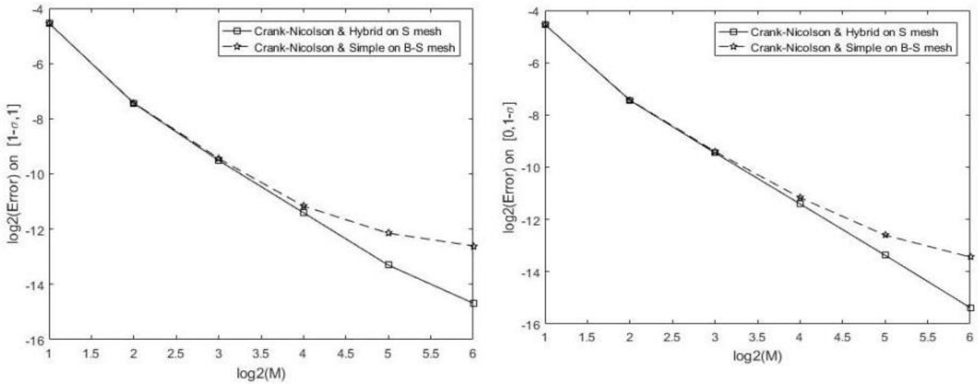

Fig. 1. The $\log 2-\log 2$ graphs of errors of (3) on S-mesh and the C-N simple scheme on BS-mesh in time.

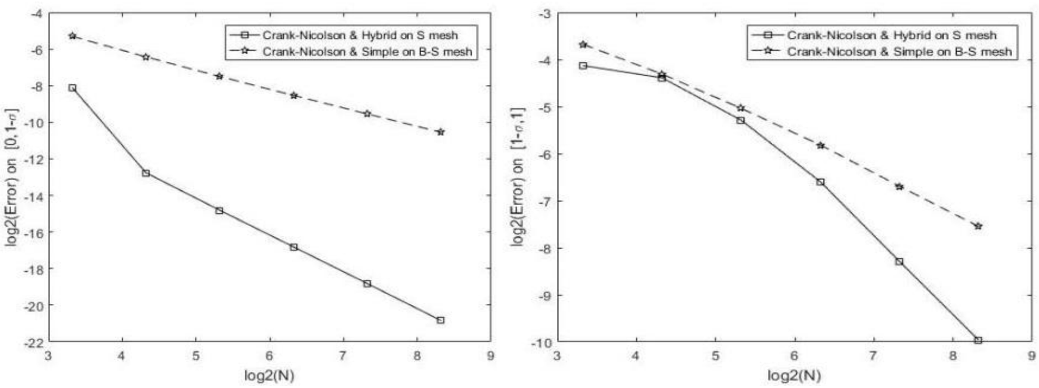

Fig. 2. The $\log 2-\log 2$ graphs of errors of (3) on S-mesh and the C-N simple scheme on BS-mesh in space.

Problem 2

$$\begin{cases} u_t - \varepsilon u_{xx} + u_x + u = e^{-t}\left(-\dfrac{\pi^2\varepsilon}{4}\cos\dfrac{\pi}{2}x + \dfrac{\pi}{2}\cos\dfrac{\pi}{2}x\right), \quad (x,t)\in(0,1)\times(0,2], \\ u(x,0) = u_0(x) = \dfrac{1-e^{-(1-x)/\varepsilon}}{1-e^{-1/\varepsilon}} - \cos\dfrac{\pi}{2}x,\ 0\le x\le 1, \\ u(0,t) = u(1,t) = 0,\ \ 0\le t\le 1, \end{cases}$$

where the exact solution is $u(x,t)=e^{-t}(\frac{1-e^{-(1-x)/\varepsilon}}{1-e^{-1/\varepsilon}}-\cos\frac{\pi}{2}x)$. Numerical results for Problem 2 are illustrated in Tables 3 and 4.

Table 3. The Numerical Results of the Scheme (3) When N=3200 for Problem 2

M	$e_L^{M,N}$	order	const	$e_R^{M,N}$	order	const
4	4.427881e-03	1.7481	0.0708	4.428742e-03	1.7481	0.0709
8	1.318121e-03	2.0138	0.0844	1.318449e-03	1.9942	0.0844
16	3.263841e-04	1.9914	0.0836	3.309387e-04	1.9791	0.0847
32	8.208650e-05	2.0006	0.0841	8.393926e-05	1.8365	0.0860
64	2.051310e-05	2.0012	0.0840	2.350329e-05	1.6240	0.0963
128	5.124163e-06		0.0840	7.625356e-06		0.1249

Table 4. The Numerical Results of the Scheme (3) When M=3200 for Problem 2

N	$e_L^{M,N}$	order	const	$e_R^{M,N}$	order	const
20	1.437069e-04	2.0473	0.0575	4.771353e-02	0.9012	11.2753
40	3.476770e-05	2.0028	0.0556	2.554737e-02	1.3071	15.9261
80	8.675223e-06	2.0111	0.0555	1.032471e-02	1.6940	18.2448
160	2.152211e-06	2.0426	0.0551	3.191016e-03	1.6731	16.8151
320	5.224028e-07	2.0391	0.0535	1.000629e-03	1.6900	16.3270
640	1.271043e-07		0.0521	3.101286e-04		16.1314

Tables 1 and 3 show that the numerical results of the scheme (3) in time is second-order convergent and Tables 2 and 4 show that the numerical results of the scheme (3) in space is second-order convergent on the coarse part and almost second-order on the fine part, which verify Theorem 2.4. And the tables and figures demonstrate the higher-order convergence and the effectiveness of the proposed scheme (3).

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

Prof. Zheng determined the topic and content of the paper, including the crucial discrete comparison principle. Miss Jin wrote the paper and made the numerical results.

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