An Inverse Transient Thermoelastic Problem of a solid Sphere Due to Partially distributed Heat Supply

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***Abstract*—As we know, thermal behavior of structures must be considered in many situation such as study of thermal effect on thermal strains, stresses, displacement. There is a practical requirement of solid sphere in various modern project. In this task, we endeavour to solve the differential equation of heat conduction, by applying heat flux to solid sphere of radius 'a' which is free from traction, when interior temperature is known. The initial temperature of the sphere is same as that of surrounding temperature, which is zero. The sphere is subjected to transient heat supply, angular symmetric i.e. along radial direction, at the outer surface. In this article, an attempt is being made to solve the differential equation of heat conduction. The result is obtained in a series form of Bessel function. The result is illustrated numerically and graphically. The obtained result may be useful in solving engineering problem, particularly for industrial problem, machines subjected to heating and cooling.**

***Index Terms*—Inverse transient heat conduction temperature, stains, stresses, displacement.**

I. Introduction

We consider a solid sphere of radius r where $0 \le r \le a$ $0 \le \theta \le 2\pi$ $0 \le \phi \le 2\pi$ .Initial temperature of the sphere is same as that of surrounding medium, which is kept constant as zero. Then sphere is subjected to heat supply along radial direction only i.e. angular symmetric by a heat flux $f(t)$. The lateral surface of the sphere is insulated. The material of the sphere is isotropic, homogeneous and all properties are assumed to be constant. The transient heat conduction in homogeneous solid sphere with constant thermal diffusivity k and no heat source generated is,

$$\frac{\partial^2}{\partial r^2}T(r,t)+\frac{2}{r}\frac{\partial}{\partial r}T(r,t)=\frac{1}{k}\frac{\partial}{\partial t}T(r,t) \tag{1}$$

With the boundary condition

$$T(r,t)=0 \tag{2}$$

$$T(a,t)=g(t) \quad \text{unknown} \tag{3}$$

$$T(\xi,t)=f(t) \quad \text{known} \quad 0<\xi<a \tag{4}$$

Equations (1) to (4) constitute mathematical formulation of the problem.

II. Solution

Taking Laplace transform of equation (1), (2), (3), (4),applying initial and boundary condition to it and then taking their inverse Laplace as in [1] which finally yields to the solution,

$$T(r,t)=\frac{2k}{\sqrt{r\xi}}\sum_{n=1}^{\infty}\frac{\xi_n J_{\frac{1}{2}}(\zeta_n r)}{J_{\frac{3}{2}}(\zeta_n \xi)}\int_0^t f(u)e^{-k\zeta_n{}^2(t-u)}du \tag{5}$$

ζ_n is a root of transcendental equation $J_{\frac{1}{2}}(\zeta_n \xi)=0$ (6)

Unknown temperature is given as

$$g(t)=\frac{2k}{\sqrt{a\xi}}\sum_{n=1}^{\infty}\frac{\zeta_n J_{\frac{1}{2}}(\zeta_n a)}{J_{\frac{3}{2}}(\zeta_n \xi)}\int_0^t f(u)e^{-k\zeta_n{}^2(t-u)}du \tag{7}$$

III. Thermoelastic Problem:

A. Stress-Strain-Displacement Relationship:

Consider a sphere of radius a in which the temperature is a function of only r. The displacement in this case as in [9] is,

$$u(r)=\frac{\alpha}{1-v}\left[(1+v)\frac{1}{r^2}\int_0^r T(r'^2)dr'+2(1-2v)\frac{r}{a^3}\int_0^a T(r)r^2dr\right] \tag{8}$$

$$\varepsilon_r=\frac{\partial u}{\partial r} \tag{9}$$

$$\varepsilon_t=\frac{u}{r} \tag{10}$$

Which must satisfy the equilibrium equation as in [7] and [8]

$$\varepsilon_r=\frac{d}{dr}(r\varepsilon_t) \tag{11}$$

$$\sigma_r=\frac{2\alpha E}{1-v}\left[\frac{1}{a^3}\int_0^a Tr^2dr-\frac{1}{r^3}\int_0^r T(r')r'^2dr'\right] \tag{12}$$

$$\sigma_t=\frac{\alpha E}{1-v}\left[\frac{2}{a^3}\int_0^a Tr^2dr+\frac{1}{r^3}\int_0^r T(r')r'^2dr'-T\right] \tag{13}$$

The differential mechanical equilibrium equation as in [7] and [8] is

$$\frac{d}{dr}\sigma_r+\frac{2}{r}(\sigma_r-\sigma_t)=0 \tag{14}$$

The strain-stress relations are,

$$\varepsilon_r=\alpha T+\frac{1}{E}(\sigma_r-2v\sigma_t) \tag{15}$$

$$\varepsilon_t=\alpha T+\frac{1}{E}[\sigma_t-v(\sigma_r+\sigma_t)] \tag{16}$$

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α -Thermal expansion coefficient, v -Poisson's ratio, E -Young's modulus, u -Radial displacement

ε_r -Radial strain, ε_t -Tangential strain, σ_r -radial stress, σ_t -Tangential stress

IV. Solution of Thermoelastic Problem

To solve thermoelastic problem we solve necessary integral

$$\int T(r) r^2 dr$$

$$= \frac{2k}{\sqrt{\xi}} \sum_{n=1}^{\infty} \frac{r^{3/2} J_{\frac{3}{2}}(\zeta_n r)}{J_{\frac{3}{2}}(\zeta_n \xi)} \int_0^t f(u) e^{-k\zeta_n^2 (t-u)} du \quad (17)$$

Radial displacement is given from equation (5) and (8) which yields as

$$u(r) = \frac{2k\alpha}{(1-v)\sqrt{\xi}} \sum_{n=1}^{\infty} \left\{ (1+v) \left[\frac{J_{\frac{3}{2}}(\zeta_n r)}{r^{1/2} J_{\frac{3}{2}}(\zeta_n \xi)} \right] + 2(1-2v) \left[\frac{r J_{\frac{3}{2}}(\zeta_n a)}{a^{3/2} J_{\frac{3}{2}}(\zeta_n \xi)} \right] \right\} \int_0^t f(u) e^{-k\zeta_n^2 (t-u)} du \quad (18)$$

Radial strain is given from equation (9) and (18) which yields as

$$\varepsilon_r = \frac{2k\alpha}{(1-v)\sqrt{\xi}} \sum_{n=1}^{\infty} \left\{ 1+v \left[\frac{\zeta_n J_{\frac{1}{2}}(\zeta_n r)}{r^{1/2} J_{\frac{3}{2}}(\zeta_n \xi)} - \frac{2 J_{\frac{3}{2}}(\zeta_n r)}{r^{3/2} J_{\frac{3}{2}}(\zeta_n \xi)} \right] + 2(1-2v) \left[\frac{J_{\frac{3}{2}}(\zeta_n a)}{a^{3/2} J_{\frac{3}{2}}(\zeta_n \xi)} \right] \right\}$$

$$\int_0^t f(u) e^{-k\zeta_n^2 (t-u)} du \quad (19)$$

Tangential strain is given from equation (10) and (18) which yields as

$$\varepsilon_t = \frac{2k\alpha}{(1-v)\sqrt{\xi}} \sum_{n=1}^{\infty} \left\{ (1+v) \left[\frac{J_{\frac{3}{2}}(\zeta_n r)}{r^{3/2} J_{\frac{3}{2}}(\zeta_n \xi)} \right] + 2(1-2v) \left[\frac{J_{\frac{3}{2}}(\zeta_n a)}{a^{3/2} J_{\frac{3}{2}}(\zeta_n \xi)} \right] \right\} \int_0^t f(u) e^{-k\zeta_n^2 (t-u)} du \quad (20)$$

From (19) and (20) equation (11) is satisfied

Radial stress is given from equation (5) and (12) which yields

$$\sigma_r = \frac{4k\alpha E}{(1-v)\sqrt{\xi}} \sum_{n=1}^{\infty} \left[\frac{J_{\frac{3}{2}}(\zeta_n a)}{a^{3/2} J_{\frac{3}{2}}(\zeta_n \xi)} - \frac{J_{\frac{3}{2}}(\zeta_n r)}{r^{3/2} J_{\frac{3}{2}}(\zeta_n \xi)} \right] \int_0^t f(u) e^{-k\zeta_n^2 (t-u)} du \quad (21)$$

Tangential stress is given from equation (5) and (13) which yields as

$$\sigma_t = \frac{2k\alpha E}{(1-v)\sqrt{\xi}} \sum_{n=1}^{\infty} \left[\frac{2 J_{\frac{3}{2}}(\zeta_n a)}{a^{3/2} J_{\frac{3}{2}}(\zeta_n \xi)} + \frac{J_{\frac{3}{2}}(\zeta_n r)}{r^{3/2} J_{\frac{3}{2}}(\zeta_n \xi)} - \frac{\zeta_n J_{\frac{1}{2}}(\zeta_n r)}{r^{1/2} J_{\frac{3}{2}}(\zeta_n \xi)} \right] \int_0^t f(u) e^{-k\zeta_n^2 (t-u)} du \quad (22)$$

From equation (21) and (22) equation (14) is satisfied

V. Numerical Calculation

Set $f(t) = e^t$ Let a=1m $\xi = 0.5m$

$$\int_0^t f(u) e^{-k\zeta_n^2 (t-u)} du = \left[\frac{e^t - e^{-k\zeta_n^2 t}}{1 + k\zeta_n^2} \right] \quad (23)$$

VI. Material Properties

Numerical calculation are carried out for steel sphere (SN 50C)

Set $k = 15.9 \times 10^{-6} m^2 s^{-1}$ $\alpha = 11.6 \times 10^{-6} k^{-1}$

$$E = 215 GPa \qquad v = .281$$

$$A = \frac{2k}{\sqrt{\xi}} \qquad B = \frac{2k\alpha}{(1-v)\sqrt{\xi}} \qquad C = \frac{4k\alpha E}{(1+v)\sqrt{\xi}}$$

$$D = \frac{2k\alpha E}{(1+v)\sqrt{\xi}}$$

Using equation (23) the above equations (5), (18), (19), (20) (21), (22) yields

$$\frac{T}{A} = \sum_{n=1}^{\infty} \frac{\zeta_n J_{\frac{1}{2}}(\zeta_n r)}{r^{1/2} J_{\frac{3}{2}}(0.5\zeta_n)} \left[\frac{e^t - e^{-k\zeta_n^2 t}}{1 + k\zeta_n^2} \right] \quad (24)$$

$$\frac{u(r)}{B} = \sum_{n=1}^{\infty} \left\{ (1+v) \left[\frac{J_{\frac{3}{2}}(\zeta_n r)}{r^{1/2} J_{\frac{3}{2}}(0.5\zeta_n)} \right] + 2(1-2v) \left[\frac{r J_{\frac{3}{2}}(\zeta_n)}{J_{\frac{3}{2}}(0.5\zeta_n)} \right] \right\} \left[\frac{e^t - e^{-k\zeta_n^2 t}}{1 + k\zeta_n^2} \right] \quad (25)$$

$$\frac{\varepsilon_r}{B} = \sum_{n=1}^{\infty} \left\{ 1+v \left[\frac{\zeta_n J_{\frac{1}{2}}(\zeta_n r)}{r^{1/2} J_{\frac{3}{2}}(0.5\zeta_n)} - \frac{2 J_{\frac{3}{2}}(\zeta_n r)}{r^{3/2} J_{\frac{3}{2}}(0.5\zeta_n)} \right] + 2(1-2v) \left[\frac{J_{\frac{3}{2}}(\zeta_n)}{J_{\frac{3}{2}}(0.5\zeta_n)} \right] \right\} \left[\frac{e^t - e^{-k\zeta_n^2 t}}{1 + k\zeta_n^2} \right] \quad (26)$$

$$\frac{\varepsilon_t}{B}=\sum_{n=1}^{\infty}\left\{(1+v)\left[\frac{J_{\frac{3}{2}}(\zeta_n r)}{r^{3/2}J_{\frac{3}{2}}(0.5\zeta_n)}\right]+2(1-2v)\left[\frac{J_{\frac{3}{2}}(\zeta_n)}{J_{\frac{3}{2}}(0.5\zeta_n)}\right]\right\}$$

$$\left[\frac{e^t-e^{-k\zeta_n{}^2 t}}{1+k\zeta_n{}^2}\right] \tag{27}$$

$$\frac{\sigma_r}{C}=\sum_{n=1}^{\infty}\left[\frac{J_{\frac{3}{2}}(\zeta_n)}{J_{\frac{3}{2}}(0.5\zeta_n)}-\frac{J_{\frac{3}{2}}(\zeta_n r)}{r^{3/2}J_{\frac{3}{2}}(0.5\zeta_n)}\right]$$

$$\left[\frac{et-e^{-k\zeta_n{}^2 t}}{1+k\zeta_n{}^2}\right] \tag{28}$$

$$\frac{\sigma_t}{D}=\sum_{n=1}^{\infty}\left[\frac{2J_{\frac{3}{2}}(\zeta_n)}{J_{\frac{3}{2}}(0.5\zeta_n)}+\frac{J_{\frac{3}{2}}(\zeta_n r)}{r^{3/2}J_{\frac{3}{2}}(0.5\zeta_n)}-\frac{\zeta_n J_{\frac{1}{2}}(\zeta_n r)}{r^{1/2}J_{\frac{3}{2}}(0.5\zeta_n)}\right]$$

$$\left[\frac{et-e^{-k\zeta_n{}^2 t}}{1+k\zeta_n{}^2}\right] \tag{29}$$

VII. Conclusion

- Figureno.1shows the temperature increases up to r=.8 then decreases. It is maximum at r=.8
- Figure no.2 shows the displacement is zero at r=.4 and r=.6. It is maximum at r=.5
- Figure no.3 shows the radial strain increases up to r=.8 then decreases. It is maximum at r=.8
- Figure no.4shows the tangential strain decreases up to r=.3.and maximum at r=.5
- Figure no.5 shows radial stress is negative. It is least lat r=.5 and vanishes at r=.5
- Figure no.6 shows tangential stress decreases continuously up to r=.8 and then increases. It is least at r=.8

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