Harvesting in a Periodic Habitat with Allee Effect

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Abstract—The dynamics of a harvested single species that goes extinct when rare is described by a nonlinear differential equation $\dot{N} = rN\left(1-\frac{N}{K}\right)\left(\frac{N}{K}-\frac{A}{K}\right)-hN$ **where *r, K, A* and *h* are positive continuous *T*-periodic functions. *N* is the population size, *A* (0 < *A* < *K*) is the threshold population size below which *N* < 0 due to Allee effect, *r* is the intrinsic growth rate and *K* is the carrying capacity of the environment and *h* is the harvesting. The aim of this paper is to study the existence of periodic solutions and their stability properties. We discuss the effect of harvesting on a single species population in a fluctuating environment whose dynamics is described by a nonlinear differential equation.**

***Index Terms*—Nonlinear differential equation; Allee Effect; periodic solutions; stability; existence; harvesting; positive solutions.**

I. Introduction

Seasonal habitat fluctuations should be preferably taken into consideration in mathematical models due to the significant effect they have on the population density, even during brief periods when the physical and biological environments remain nearly constant.

In this paper, we investigate the effect of harvesting on the dynamics of population in a fluctuating environment described by a nonlinear differential equation

$$\dot{N}(t) = r(t)N(t)\left(1-\frac{N(t)}{K(t)}\right)\left(\frac{N(t)}{K(t)}-\frac{A(t)}{K(t)}\right)-h(t)N(t) \qquad (1)$$

With continuous, positive *T*- periodic functions *r(t)*, *K(t)* and a continuous *T*-periodic function *A(t)* that is,
$r(t+T)=r(t), K(t+T)=K(t),$ and $A(t+T)=A(t)$ for all $t\in R$.
Eq. (1) describes the dynamics of a single species subject to Allee effect, cf. Amarasekare [4] where the case of constant coefficients is dealt with. *N* denotes the population size, *r(t)* denotes the maximum per capita population growth rate without Allee effect, *A(t)* is the Allee threshold for a strong Allee effect, that is, a critical population size or density below which the per capita population growth rate becomes negative, *K(t)* is the carrying capacity of the environment, *h(t)* is continuous function of harvesting (stocking). Allee effect increases the dynamical stability of populations (see, for instance, Scheuring [5], Fowler and Ruxton [6]); it decreases the emergence of instabilities associated with the rise of the growth rate. When Allee effect does not help to stabilize the dynamics of the population, it does restrict the amplitude of oscillations. For more information on the wide variety of Allee effects in mathematical ecology, we refer to an excellent monograph by Courchamp et al. [7] and review papers by Boukal and Berec [8] or Berec et al. [9]. The dynamics of population in a fluctuating environment described by Eq. (1) or its particular cases in the presence of harvesting has been studied by many authors. We would like mention to interesting contributions by Baruer and Sánchez [3], Lazer [11], Lazer and Sánchez [12], [13], Padhi et al. [10, p. 2817].

In this paper, the discussed technique for Eq. (1) allows one to determine the exact number of periodic solutions, localize them and describe their stability properties characterizing completely the dynamics of the system.

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II. Existence of Periodic Solutions under Harvesting

In this section, we study the existence of periodic solutions to Eq. (1) and their stability properties. We consider separately cases of harvesting, stocking, weak Allee effect and strong Allee effect.

Theorem1. Eq. (1) with continuous, positive *T*-periodic functions *r(t), K(t)* and *A(t)* such that

$$\max_{t\in\square} A(t) < \min_{t\in\square} K(t)$$

has exactly three *T*-periodic solutions.

Let the minimal and maximal values for the growth rate, carrying capacity, Allee effect and harvesting effort,

$$r_{\max} = \max_{t\in[0,+\infty)} r(t), r_{\min} = \min_{t\in[0,+\infty)} r(t), K_{\max} = \max_{t\in[0,+\infty)} K(t), K_{\min} = \min_{t\in[0,+\infty)} K(t)$$

$$A_{\max} = \max_{t\in[0,+\infty)} A(t), A_{\min} = \min_{t\in[0,+\infty)} A(t), h_{\max} = \max_{t\in[0,+\infty)} h(t), h_{\min} = \min_{t\in[0,+\infty)} h(t)$$

If $A(t)>0$ then $A_{\min}\le A_{\max}<K_{\min}\le K_{\max}$. If $A(t)<0$ then $|A_{\max}|\le|A_{\min}|<K_{\min}\le K_{\max}$.

Consider the square on the right-hand side of Eq. (1) in the form

$$\dot{N}(t) = -\frac{r(t)}{K^2(t)}N(t)\left(N(t)-\left(\frac{K(t)+A(t)}{2}\right)^2\right).$$
$$-\frac{r(t)}{K^2(t)}N(t)\left(\frac{K^2(t)}{r(t)}h(t)-\left(\frac{K(t)-A(t)}{2}\right)^2\right)$$

The derivative $dN/dt = \dot{N}(t)$ is negative for all $t\in[0,+\infty)$, we provided that the harvesting *h(t)* satisfied the condition.

$$\left(\left(K^2(t)/r(t)\right)h(t)-\left(\left(K(t)-A(t)\right)/2\right)^2\right)>0 \qquad (2)$$

In which case the population density $N(t)\to 0$ as $t\to+\infty$, the Eq. (1) has zero solution and the population goes extinct. Supposing that the opposite of inequality (2) holds for all $t\in[0,+\infty)$, we obtain the existence of periodic solutions, $N_1(t)=(1/2)$

$$\left((K(t)+A(t))+\sqrt{(K(t)+A(t))^2-4\left(K(t)A(t)+\left(K^2(t)/r(t)\right)h(t)\right)}\right)>0$$

$$N_2(t)=(1/2)\left((K(t)+A(t))-\sqrt{(K(t)+A(t))^2-4\left(K(t)A(t)+\left(K^2(t)/r(t)\right)h(t)\right)}\right)$$

One of the solutions is positive; the other one is positive or negative, depending on the following.

If

$A(t)+\left(K(t)/r(t)\right)h(t)>0$ provided by $h_{\min}>-A_{\min}r_{\min}/K_{\max}$

then, $N_2(t)>0,\ \text{for}\ t\geq 0$.

If

$A(t)+\left(K(t)/r(t)\right)h(t)<0$ provided by $h_{\max}<-A_{\max}r_{\max}/K_{\min}$

then $N_2(t)<0,\ \text{for}\ t\geq 0$.

The following Theorems show the existence results for the periodic solutions of Eq. (1). The computer simulations in examples use Mathematica Software.

Theorem2. Let $A(t)>0$ and $h(t)>0$ and harvesting satisfies the condition

$$h_{\max}<\left(r_{\min}/K_{\max}^2\right)\left((K_{\min}-A_{\max})/2\right)^2 \qquad N_1(t)$$

then Eq. (1) has three periodic solutions; the asymptotically stable trivial solution $N_0(t)$ and two positive solutions, asymptotically stable solution $N_1(t)$ and an unstable solution $N_2(t)$, satisfying, for all $t\in[0,+\infty)$, $-A_{\min}<N_0(t)<A_{\min}<N_2(t)<(1/2)(A_{\max}+K_{\min})<N_1(t)<K_{\max}$.

Example 3. Consider the following differential equation with $A(t)>0$ and $h(t)>0$

$$\dot{N}=(40+\sin 2\pi t)N\left(1-\left(N/(50+\cos 2\pi t)\right)\right)$$
$$\left(\left(N/(50+\cos 2\pi)t\right)-\left((10+\sin 2\pi t)/(50+\cos 2\pi t)\right)\right)$$
$$-(2+\cos 2\pi t)N$$

where $r(t)=40+\sin 2\pi t,$ $A(t)=10+\sin 2\pi t,$ $h(t)=2+cos2\pi t,\ K(t)=50+\cos 2\pi t,$

Two positive periodic solutions are sketched in Fig. 1.

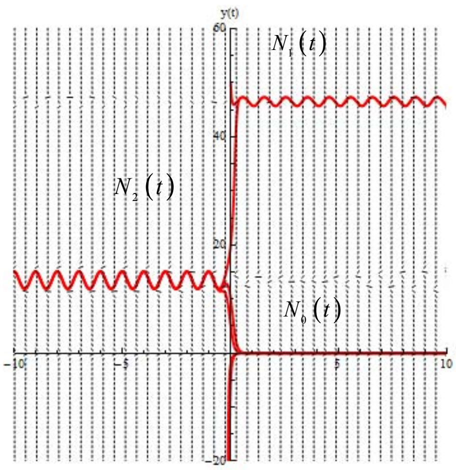

Fig. 1. Existence of one positive asymptotically stable and one positive unstable periodic solution in case of strong Allee effect and harvesting.

Theorem **4.** Let $A(t)>0$ and $h(t)<0$ then

$$D=(K(t)+A(t))^2-4\left(K(t)A(t)+\left(K^2(t)/r(t)\right)h(t)\right)$$
$$=4\left(\left((K(t)-A(t))/2\right)^2-\left(K^2(t)/r(t)\right)h(t)\right)>0$$

and harvesting satisfies the condition

(i) $h_{\min}>-(1/4)r_{\min}\left(A_{\min}/K_{\max}\right)^2$ then Eq. (1) has three periodic solutions; the asymptotically stable trivial solution $N_0(t)$ and two positive solutions, asymptotically stable solution $N_1(t)$ and an unstable solution $N_2(t)$, satisfying, for all $t\in[0,+\infty)$,

$-(1/2)A_{\min}<N_0(t)<(1/2)A_{\min}$.

$<N_2(t)<(1/2)(A_{\max}+K_{\min})<N_1(t)<A_{\min}+K_{\max}$

(ii) $-r_{\min}<h_{\min}<h_{\max}<-r_{\max}\left(\dfrac{A_{\max}+r_{\max}}{K_{\min}}\right)^2$ then Eq. (1) has three periodic solutions; the trivial solution $N_0(t)$ which is unstable a positive solutions $N_1(t)$ and a negative solution $N_2(t)$, both asymptotically stable, satisfying, for all $t\in[0,+\infty)$, $-K_{\max}<N_2(t)<-r_{\max}<N_0(t)<K_{\min}<N_1(t)<2K_{\max}$.

Example 5. Consider the following differential equation with $A(t)>0$ and $h(t)<0$

(i) $$\dot{N}=(10+\sin 2\pi t)N\left(1-\frac{N}{10+(1/4)\cos 2\pi t}\right)$$
$$\left(\frac{N}{10+(1/4)\cos 2\pi t}-\frac{8+\sin 2\pi t}{10+(1/4)\cos 2\pi t}\right)+\left(1+(1/100)\cos 2\pi t\right)N$$

where $r(t)=10+\sin 2\pi t, K(t)=10+(1/4)\cos 2\pi t,$ $A(t)=8+\sin 2\pi t, h(t)=-\left(1+(1/100)\cos 2\pi t\right)$.

(ii) $$\dot{N}=(8+\sin 2\pi t)N\left(1-\frac{N}{30+\cos 2\pi t}\right)$$
$$\left(\frac{N}{30+\cos 2\pi t}-\frac{2+\sin 2\pi t}{30+\cos 2\pi t}\right)+(5+\cos 2\pi t)N$$

where $r(t)=8+\sin 2\pi t, K(t)=30+\cos 2\pi t,$ $A(t)=2+\sin 2\pi t, h(t)=-(5+cos2\pi t)$. The solutions of this example are sketched in Fig. 2(a) and 2(b), respectively.

Theorem 6. Let $A(t)<0$ and $h(t)>0$ and harvesting satisfies the condition

(i) $\max_{t\geq 0}\left\{-A_{\min}r_{\min}/K_{\max},\left(r_{\max}/K_{\min}^2\right)K_{\max}\left((K_{\min}/K_{\max})-A_{\min}\right)\right\}$ $<h_{\min}<h_{\max}<(1/4)r_{\min}\left(K_{\min}/K_{\max}\right)^2$ then Eq. (1) has three periodic solutions; the asymptotically stable trivial solution $N_0(t)$ and two positive solutions, asymptotically stable solution $N_1(t)$ and an unstable solution $N_2(t)$, for all $t\in[0,+\infty)$, $2A_{\max}<N_0(t)<K_{\min}/K_{\max}<N_2(t)<K_{\min}/2<N_1(t)<K_{\max}$.

(ii) $h_{\max}<(3/4)r_{\min}\left(A_{\max}/K_{\max}\right)^2$ then Eq. (1) has three periodic solutions; the trivial solution $N_0(t)$ which is unstable a positive solutions $N_1(t)$ and a negative solution $N_2(t)$, both asymptotically stable, for all $t\in[0,+\infty)$, $2A_{\min}<N_2(t)<(1/2)A_{\max}<N_0(t)<-(1/2)A_{\max}<N_1(t)<K_{\max}$.

Example 7. Consider the following differential equation with $A(t)<0$ and $h(t)>0$

(i) $$\dot{N} = (45+\sin 2\pi t) N\left(1-\frac{N}{45+\cos 2\pi t}\right)\left(\frac{N}{45+\cos 2\pi t}+\frac{2+\sin 2\pi t}{45+\cos 2\pi t}\right)-(8+\cos 2\pi t)N$$

where $r(t)=45+\sin 2\pi t, K(t)=45+\cos 2\pi t$,

$A(t)=-(2+\sin 2\pi t), h(t)=8+\cos 2\pi t$

(ii) $$\dot{N} = (20+\sin 2\pi t) N\left(1-\frac{N}{30+\cos 2\pi t}\right)\left(\frac{N}{30+\cos 2\pi t}+\frac{17+\sin 2\pi t}{30+\cos 2\pi t}\right)-(2+\cos 2\pi t)N$$

where $r(t)=20+\sin 2\pi t, K(t)=30+\cos 2\pi t$,

$A(t)=-(17+\sin 2\pi t), h(t)=2+cos 2\pi t$ The solutions of this example are sketched in Fig. 3(a) and 3(b), respectively.

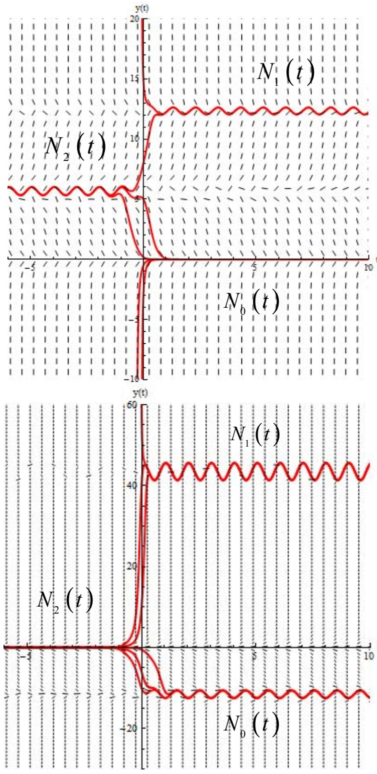

Fig. 2. (a) Existence of one positive asymptotically stable and one positive unstable periodic solution in case of strong Allee effect and stocking. (b) Existence of one positive asymptotically stable periodic solution in case of strong Allee effect and stocking.

Theorem 8. Let $A(t)<0$ and $h(t)<0$ then $D>0$ and harvesting satisfies the condition $h_{\max} > -r_{\min}$ then Eq. (1) has three periodic solutions; the trivial solution $N_0(t)$ which is unstable a positive solutions $N_1(t)$ and a negative solution $N_2(t)$, both asymptotically stable, for all $t \in [0,+\infty)$, $-K_{\min} < N_2(t) < (1/2)A_{\max} < N_0(t) < (1/2)K_{\min} < N_1(t) < 2K_{\max}$.

Example 9. Consider the following differential equation with $A(t)<0$ and $h(t)<0$,

$$\dot{N} = (5+\sin 2\pi t) N\left(1-\frac{N}{10+\cos 2\pi t}\right)\left(\frac{N}{10+\cos 2\pi t}+\frac{6+\sin 2\pi t}{10+\cos 2\pi t}\right)+(2+\cos 2\pi t)N$$

where $r(t)=5+\sin 2\pi t, K(t)=10+\cos 2\pi t$, $A(t)=-(6+\sin 2\pi t), h(t)=-(2+\cos 2\pi t)$. One positive solution of this example is sketched in Fig. 4.

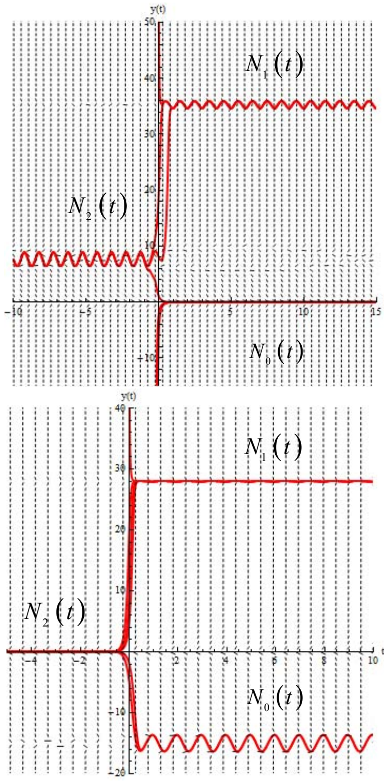

Fig. 3. (a) Existence of one positive asymptotically stable and one positive unstable periodic solution in case of weak Allee effect and harvesting. (b) Existence of one positive asymptotically stable in case of weak Allee effect and harvesting.

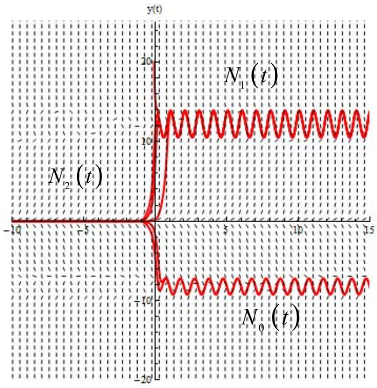

Fig. 4. Existence of one positive asymptotically stable periodic solution in case of weak Allee effect and stocking.

III. Conclusion

In this paper, we discuss the effect of harvesting (stocking) on a single species population in a fluctuating environment with Allee effect. We determined the conditions for existence of positive periodic solutions of the Eq. (1).

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