A Fixed Point Theorem on Four Complete Metric Spaces

Vishal Gupta, Naveen Mani, and Ashima Kanwar

***Abstract*—Our object in this paper to discuss about fixed point theorem in four metric spaces. Here we established a fixed point theorem in four complete metric spaces, which generalized many results of many authors [1]-[5].**

***Index Terms*—Complete metric space, fixed point.**

I. Introduction

Let $(X_1,d_1),(X_2,d_2),(X_3,d_3)$ and (X_4,d_4) be complete metric spaces.

If $A_1:X_1\to X_2, A_2:X_2\to X_3, A_3:X_3\to X_4$ and $A_4:X_4\to X_1$ are mapping than we denote

$$M_1(x^1,x^2)=\{d_1^p(x^1,A_4A_3A_2x^2),d_1^p(x^1,A_4A_3A_2A_1x^1),d_2^p(x^2,A_1x^1)\} \tag{1.1}$$

$$M_2(x^2,x^3)=\{d_2^p(x^2,A_1A_4A_3x^3),d_2^p(x^2,A_1A_4A_3A_2x^2),d_3^p(x^3,A_2x^2)\} \tag{1.2}$$

$$M_3(x^3,x^4)=\{d_3^p(x^3,A_2A_1A_4x^4),d_3^p(x^3,A_2A_1A_4A_3x^3),d_4^p(x^4,A_3x^3)\} \tag{1.3}$$

$$M_4(x^4,x^1)=\{d_4^p(x^4,A_3A_2A_1x^1),d_4^p(x^4,A_3A_2A_1A_4x^4),d_5^p(x^5,A_4x^4)\} \tag{1.4}$$

Let $F:[0,\infty]\to R^+$ be continuous mappings in 0 with $F(0)=0$

II. Main Result

Theorem

Let $(X_1,d_1),(X_2,d_2),(X_3,d_3),(X_4,d_4)$ and (X_5,d_5) be complete metric spaces where. If $A_1:X_1\to X_2, A_2:X_2\to X_3, A_3:X_3\to X_4$ and $A_4:X_4\to X_1$ are mapping satisfying the following

inequalities.

$$d_1^p(A_4A_3A_2x^2,A_4A_3A_2A_1x^1)\ \le\ c\max M_1(x^1,x^2)+F(\min M_1) \tag{1.5}$$

$$d_2^p(A_1A_4A_3x^3,A_1A_4A_3A_2x^2)\ \le\ c\max M_2(x^2,x^3)+F(\min M_2) \tag{1.6}$$

$$d_3^p(A_2A_1A_4x^4,A_2A_1A_4A_3x^3)\ \le\ c\max M_3(x^3,x^4)+F(\min M_3) \tag{1.7}$$

$$d_4^p(A_3A_2A_1x^1,A_3A_2A_1A_4x^4)\le\ c\max M_4(x^4,x^1)+F(\min M_4) \tag{1.8}$$

$\forall\, x^1\in X_1, x^2\in X_2, x^3\in X_3, x^4\in X_4$,where, $0\leqslant c<1$. Then $A_5A_4A_3A_2A_1$ has a unique fixed point $\beta_1\in X_1$; $A_1A_5A_4A_3A_2$ has a unique fixed point $\beta_2\in X_2$; $A_2A_1A_5A_4A_3$ has a unique fixed point $\beta_3\in X_3$; $A_3A_2A_1A_5A_4$ has a unique fixed point $\beta_4\in X_4$.

Further, $A_1(\beta_1)=\beta_2, A_2(\beta_2)=\beta_3, A_3(\beta_3)=\beta_4, A_4(\beta_4)=\beta_1$

Proof: Let x_0^1 be an arbitrary point in X_1 , let define sequence $\{x_m^1\}$, $\{x_m^2\},\{x_m^3\}$ and $\{x_m^4\}$ in X_1,X_2,X_3,X_4 respectively by $(A_4A_3A_2A_1)^m x_0^1=x_m^1$, $x_m^2=A_1(x_{m-1}^1)$, $x_m^3=A_2(x_m^2)$, $x_m^4=A_3(x_m^3)$ $x_m^1=A_4(x_m^4)$ for $m=1,2,3,\ldots$ We will assume that $x_m^1\ne x_{m+1}^1, x_m^2\ne x_{m+1}^2$ and so on $x_m^n\ne x_{m+1}^n$ for all m. Otherwise, if $x_m^1=x_{m+1}^1$ for some m, then $x_m^2=x_{m+1}^2$, $x_m^3=x_{m+1}^3$ and $x_m^4=x_{m+1}^4$,we could put $x_m^1=\beta_1$, $x_{m+1}^2=\beta_2$ and $x_{m+1}^4=\beta_4$ First, we prove the sequences $\{x_m^1\},\{x_m^2\},\{x_m^3\}$ and $\{x_m^4\}$ are cauchy sequences.Taking $x^1=x_m^1$, $x^2=x_m^2$ in (1.1) and (1.5) , we obtain

$$M_1(x_m^1,x_m^2)=\{d_1^p(x_m^1,A_4A_3A_2x_m^2),d_1^p(x_m^1,A_4A_3A_2A_1x_m^1),d_2^p(x_m^2,A_1x_m^1)\}$$

$$=\{d_1^p(x_m^1,x_m^1),d_1^p(x_m^1,x_{m+1}^1),d_2^p(x_m^2,x_{m+1}^2)\}$$
$$=\{0,d_1^p(x_m^1,x_{m+1}^1),d_2^p(x_m^2,x_{m+1}^2)\}$$

$$d_1^p(x_m^1,x_{m+1}^1)=$$
$$d_1^p(A_4A_3A_2x_m^2,A_4A_3A_2A_1x_m^1)$$
$$\le\ c\max M_1(x_m^1,x_m^2)+F(\min M_1(x_m^1,x_m^2))$$

$$=\ c\max\{0,d_1^p(x_m^1,x_{m+1}^1),d_2^p(x_m^2,x_{m+1}^2)\}+F(0)$$
$$=cd_2^p(x_m^2,x_{m+1}^2)$$

Since, if $\max M_1(x_m^1,x_m^2)=d_1^p(x_m^1,x_{m+1}^1)$,then $d_1^p(x_m^1,x_{m+1}^1)\le cd_1^p(x_m^1,x_{m+1}^1)$

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Vishal Gupta and Naveen Mani are with Department of Mathematics, M.M.University, Mullana, Ambala, India (e-mail address: vishal.gmn@gmail.com).

A. Kanwar is with Research Scholar, Department of Mathematics, M. M. University, Mullana, Ambala, India.

It follows $x_m^1 = x_{m+1}^1$, since $0 \le c < 1$, so $d_1^p(x_m^1, x_{m+1}^1) \le c d_2^p(x_m^2, x_{m+1}^2)$ (1.9)

Taking $x^2 = x_m^2, x^3 = x_{m-1}^3$ in (1.2) and (1.6), we get

$$M_2(x_m^2, x_{m-1}^3) = \{d_2^p(x_m^2, A_1A_4A_3x_{m-1}^3), d_2^p(x_m^2, A_1A_4A_3A_2x_m^2), d_3^p(x_{m-1}^3, A_2x_m^2)\}$$
$$= \{d_2^p(x_m^2, x_m^2), d_2^p(x_m^2, x_{m+1}^2), d_3^p(x_{m-1}^3, x_m^3)\}$$
$$= \{0, d_2^p(x_m^2, x_{m+1}^2), d_3^p(x_{m-1}^3, x_m^3)\}$$
$$d_2^p(x_m^2, x_{m+1}^2) = d_2^p(A_1A_4A_3x_{m-1}^3, A_1A_4A_3A_2x_m^2)$$
$$\le c \max M_2(x_m^2, x_{m-1}^3) + F(\min M_2(x_m^2, x_{m-1}^3))$$
$$= c \max\{0, d_2^p(x_m^2, x_{m+1}^2), d_3^p(x_{m-1}^3, x_m^3)\} + F(0)$$

Since $0 \le c < 1$, we get $d_2^p(x_m^2, x_{m+1}^2) \le c d_3^p(x_{m-1}^3, x_m^3)$ (1.10)

Taking $x^3 = x_m^3, x^4 = x_{m-1}^4$ in (1.3) and (1.7), we get

$$M_3(x_m^3, x_{m-1}^4) = \{d_3^p(x_m^3, A_2A_1A_4x_{m-1}^4), d_3^p(x_m^3, A_2A_1A_4A_3x_m^3), d_4^p(x_{m-1}^4, A_3x_m^3)\}$$
$$= \{d_3^p(x_m^3, x_m^3), d_3^p(x_m^3, x_{m+1}^3), d_4^p(x_{m-1}^4, x_m^4)\}$$
$$= \{0, d_3^p(x_m^3, x_{m+1}^3), d_4^p(x_{m-1}^4, x_m^4)\}$$
$$d_3^p(x_m^3, x_{m+1}^3) = d_3^p(A_2A_1A_5A_4x_{m-1}^4, A_2A_1A_5A_4A_3x_m^3)$$
$$\le c \max M_3(x_m^3, x_{m-1}^4) + F(\min M_3(x_m^3, x_{m-1}^4))$$
$$= c \max\{0, d_3^p(x_m^3, x_{m+1}^3), d_4^p(x_{m-1}^4, x_m^4)\} + F(0)$$
$$= c d_4^p(x_{m-1}^4, x_m^4)$$

Replacing m to m-1, we obtain

$$d_3^p(x_{m-1}^3, x_m^3) \le c d_4^p(x_{m-2}^4, x_{m-1}^4) \tag{1.11}$$

Taking $x^4 = x_m^4, x^1 = x_{m-1}^1$ in (1.4) and (1.8), we get

$$M_4(x_m^4, x_{m-1}^1) = \{d_4^p(x_m^4, A_3A_2A_1x_{m-1}^1), d_4^p(x_m^4, A_3A_2A_1A_4x_m^4), d_1^p(x_{m-1}^1, A_4x_m^4)\}$$
$$= \{d_4^p(x_m^4, x_m^4), d_4^p(x_m^4, x_{m+1}^4), d_1^p(x_{m-1}^1, x_m^1)\}$$
$$= \{0, d_4^p(x_m^4, x_{m+1}^4), d_1^p(x_{m-1}^1, x_m^1)\}$$
$$d_4^p(x_m^4, x_{m+1}^4) = d_4^p(A_3A_2A_1x_{m-1}^1, A_3A_2A_1A_4x_m^4)$$
$$\le c \max M_4(x_m^4, x_{m-1}^1) + F(\min M_4(x_m^4, x_{m-1}^1))$$
$$= c \max\{0, d_4^p(x_m^4, x_{m+1}^4), d_1^p(x_{m-1}^1, x_m^1)\} + F(0)$$
$$= c d_1^p(x_{m-1}^1, x_m^1)$$

Replacing m with m-2 we obtain

$$d_4^p(x_{m-2}^4, x_{m-1}^4) \le c d_1^p(x_{m-3}^1, x_{m-2}^1) \tag{1.12}$$

Using (1.9), (1.10), (1.11) and (1.12), we get

$$d_1^p(x_m^1, x_{m+1}^1) \le c d_2^p(x_m^2, x_{m+1}^2) \le c^2 d_3^p(x_{m-1}^3, x_m^3) \le c^3 d_4^p(x_{m-2}^4, x_{m-1}^4) \le c^4 d_1^p(x_{m-3}^1, x_{m-2}^1)$$
$$\le \ldots \le c^8 d_1^p(x_{m-6}^1, x_{m-5}^1)$$
$$\le \ldots \le \left\{\begin{matrix} c^{4k} d_1^p(x_1^1, x_2^1), , m = 3k+1 \\ c^{4k} d_1^p(x_0^1, x_1^1), , m = 3k \end{matrix}\right\}$$

Since $0 \le c < 1$, the sequences $\{x_m^1\}$, $\{x_m^1\}, \{x_m^2\}, \{x_m^3\}, \{x_m^4\}$ are cauchy sequences. Since (X_i, d_i) be complete metric spaces where $i = 1,2,3,4$

$\lim_{m\to\infty} x_m^1 = \beta_1 \in X_1, \lim_{m\to\infty} x_m^2 = \beta_2 \in X_2,$
$\lim_{m\to\infty} x_m^3 = \beta_3 \in X_3, \lim_{m\to\infty} x_m^4 = \beta_4 \in X_4$

Now taking $x^1 = x_m^1$ and $x^2 = \beta_2$ in the inequality (1.5) we obtain

$$d_1^p(A_4A_3A_2\beta_2, x_{m+1}^1) = d_1^p(A_4A_3A_2\beta_2, A_4A_3A_2A_1x_m^1)$$
$$\le c \max M_1(x_m^1, \beta_2) + F(\min M_1(x_m^1, \beta_2)) \tag{1.13}$$

Where $M_1(x_m^1, \beta_2) = \{d_1^p(x_m^1, A_4A_3A_2\beta_2), d_1^p(x_m^1, A_4A_3A_2A_1x_m^1), d_2^p(\beta_2, A_1x_m^1)\}$
$= \{d_1^p(x_m^1, A_5A_4A_3A_2\beta_2), d_1^p(x_m^1, x_{m+1}^1), d_2^p(\beta_2, x_{m+1}^2)\}$

As m tend to infinity in (1.13) and F is continuous at 0 we get $d_1^p(A_4A_3A_2\beta_2, \beta_1) \le c d_1^p(\beta_1, A_4A_3A_2\beta_2)$, So we get, $A_4A_3A_2\beta_2 = \beta_1$

In same way, we obtain, $A_1A_4A_3\beta_3 = \beta_2$, $A_2A_1A_4\beta_4 = \beta_3$, $A_3A_2A_1\beta_1 = \beta_4$

Using (1.5), taking $x^1 = \beta_1, x^2 = x_m^2$ we get $d_1^p(x_m^1, A_4A_3A_2A_1\beta_1) = d_1^p(A_4A_3A_2x_m^2, A_4A_3A_2A_1\beta_1)$
$\le c \max M_1(\beta_1, x_m^2) + F(\min M_1(\beta_1, x_m^2))$

Where, $M_1(\beta_1, x_m^2) = \{d_1^p(\beta_1, A_4A_3A_2x_m^2), d_1^p(\beta_1, A_4A_3A_2A_1\beta_1), d_2^p(x_m^2, A_1\beta_1)\}$
$= \{d_1^p(\beta_1, x_m^1), d_1^p(\beta_1, A_4A_3A_2A_1\beta_1), d_2^p(x_m^2, A_1\beta_1)\}$

Now letting m tend to infinity we get

$$d_1^p(\beta_1, A_4A_3A_2A_1\beta_1)$$
$$\le c \max\{d_1^p(\beta_1, \beta_1), d_1^p(\beta_1, A_4A_3A_2A_1\beta_1), d_2^p(\beta_2, A_1\beta_1)\}$$
$$= c \max\{d_1^p(\beta_1, A_4A_3A_2A_1\beta_1), d_2^p(\beta_2, A_1\beta_1)\}$$

From which it follows, or $d_1^p(\beta_1, A_4A_3A_2A_1\beta_1) \le c d_1^p(\beta_1, A_4A_3A_2\beta_1) \Leftrightarrow A_4A_3A_2A_1\beta_1 = \beta_1$ or $d_1^p(\beta_1, A_4A_3A_2A_1\beta_1) \le c d_2^p(\beta_2, A_1\beta_1)$. This can be also written in following form

$$d_1^p(\beta_1, A_4\beta_4) \le c d_2^p(\beta_2, A_1\beta_1) \tag{1.14}$$

Since, $A_4A_3A_2A_1\beta_1 = \beta_n$, Taking $x^3 = x_m^3, x^2 = \beta_2$ in inequality (1.6) in the same way as above, we obtain
$d_2^p(x_{m+1}^2, A_1A_4A_3A_2\beta_2) = d_2^p(A_1A_4A_3x_m^3, A_1A_4A_3A_2\beta_2)$
$\le c \max M_2(\beta_2, x_m^3) + F(\min M_2)$ where,
$M_2(\beta_2, x_m^3) =$
$\{d_2^p(\beta_2, A_1A_4A_3x_m^3), d_2^p(\beta_2, A_1A_4A_3A_2\beta_2), d_3^p(x_m^3, A_2\beta_2)\}$
$= \{d_2^p(\beta_2, x_{m+1}^3), d_2^p(\beta_2, A_1A_4A_3A_2\beta_2), d_3^p(x_m^3, A_2\beta_2)\}$

Now letting m tend to infinity we get

$$d_2^p(\beta_2, A_1A_4A_3A_2\beta_2) \le c \max\{d_2^p(\beta_2, \beta_2), d_2^p(\beta_2, A_1A_4A_3A_2$$

From which it follows,

$d_2^p(\beta_2, A_1A_4A_3A_2\beta_2) \le cd_2^p(\beta_2, A_1A_4A_3A_2\beta_2) \Leftrightarrow A_1A_4A_3A_2\beta_2 = \beta_2$ or $d_2^p(\beta_2, A_1\beta_1) \le cd_3^p(\beta_3, A_2\beta_2)$ (1.15)

Continuously like above, we get

$d_3^p(\beta_3, A_2\beta_2) \le cd_4^p(\beta_4, A_3\beta_3), d_4^p(\beta_4, A_3\beta_3) \le cd_1^p(\beta_1, A_4\beta_4)$,

By (1.14), (1.15), we obtain $d_1^p(\beta_1, A_5\beta_5) \le cd_2^p(\beta_2, A_1\beta_1)$

$\le c^2 d_3^p(\beta_3, A_2\beta_2) \le c^3 d_4^p(\beta_4, A_3\beta_3) \le c^4 d_1^p(\beta_1, A_4\beta_4)$

$\Rightarrow d_1^p(\beta_1, A_5\beta_5) \le c^4 d_1^p(\beta_1, A_4\beta_4)$

$\Leftrightarrow A_5\beta_5 = \beta_1$, since $0 \le c < 1$, Thus again $d_1^p(\beta_1, A_4A_3A_2A_1\beta_1) = d_1^p(\beta_1, A_4\beta_4) = 0$

$\Leftrightarrow A_4A_3A_2A_1\beta_1 = \beta_1$ So, we proved that β_1 is a fixed point of $A_4A_3A_2A_1$; $A_4A_3A_2A_1$ has fixed point $\beta_1 \in X_1$; $A_1A_4A_3A_2$ has fixed point $\beta_2 \in X_2$, $A_2A_1A_4A_3$ has fixed point $\beta_3 \in X_3$, and $A_3A_2A_1A_4$ has fixed point $\beta_4 \in X_4$. Further, we also showed that $A_1(\beta_1) = \beta_2, A_2(\beta_2) = \beta_3, A_3(\beta_3) = \beta_4$, $A_4(\beta_4) = \beta_1$

Now let assume now that $\beta_1^1 \in X_1$ is another fixed point of $A_4A_3A_2A_1$, different from β_1.

Using (1.5), if we take $x^1 = \beta_1, x^2 = A_1\beta_1^1$, we get

$d_1^p(\beta_1^1, \beta_1) = d_1^p(A_4A_3A_2A_1\beta_1^1, A_4A_3A_2A_1\beta_1)$

$\le c \max M_1(\beta_1, A_1\beta_1^1) + F(\min M_1(\beta_1, A_1\beta_1^1))$

Where

$M_1(\beta_1, A_1\beta_1^1) = \{d_1^p(\beta_1, A_4A_3A_2A_1\beta_1^1), d_1^p(\beta_1, A_4A_3A_2A_1\beta_1), d_2^p(A_1\beta_1^1, A_1\beta_1)\}$

$= \{d_1^p(\beta_1, \beta_1^1), d_1^p(\beta_1, \beta_1), d_2^p(A_1\beta_1^1, \beta_2)\}$

From which it follows

$$d_1^p(\beta_1^1, \beta_1) \le cd_2^p(A_1\beta_1^1, \beta_2) \qquad (1.16)$$

Taking $x^3 = \beta_3, x^2 = A_1\beta_1^1$ in inequality (1.6) we obtain

$d_2^p(\beta_2, A_1\beta_1^1) = d_2^p(A_1A_4A_3\beta_3, A_1A_3A_2A_1\beta_1^1)$

$\le c \max M_2(A_1\beta_1^1, \beta_3) + F(\min M_2(A_1\beta_1^1, \beta_3))$

Where,

$M_2(A_1\beta_1^1, \beta_3) = \{d_2^p(A_1\beta_1^1, A_1A_4A_3\beta_3), d_2^p(A_1\beta_1^1, A_1A_4A_3A_2A_1\beta_1^1), d_3^p(\beta_3, A_2A_1\beta_1^1)\}$

$= d_2^p(A_1\beta_1^1, \beta_2), d_2^p(A_1\beta_1^1, A_1\beta_1^1), d_3^p(\beta_3, A_2A_1\beta_1^1)\}$

Then,

$$d_2^p(\beta_2, A_1\beta_1^1) \le cd_3^p(\beta_3, A_2A_1\beta_1^1) \qquad (1.17)$$

Continuously like above, using (1.7), (1.8), we get

$$d_3^p(\beta_3, A_2A_1\beta_1^1) \le cd_4^p(\beta_4, A_3A_2A_1\beta_1^1), d_4^p(\beta_4, A_3A_2A_1\beta_1^1) \le cd_1^p(\beta_1^1, \beta_1) \qquad (1.18)$$

By using (1.16), (1.17) and (1.18)

$d_1^p(\beta_1^1, \beta_1) \le cd_2^p(A_1\beta_1^1, \beta_2)$

$\le c^2 d_3^p(\beta_3, A_2A_1\beta_1^1) \le c^3 d_4^p(A_3A_2A_1\beta_1^1, \beta_4) \le$

$c^4 d_1^p(\beta_1^1, \beta_1) \Leftrightarrow d_1^p(\beta_1^1, \beta_1) \le c^4 d_1^p(\beta_1^1, \beta_1)$ where $0 \le c < 1$. It follows that $\beta_1^1 = \beta_1$.

Thus we proved β_1 is the fixed point of $A_4A_3A_2A_1$. In the same way, it can be shown that $A_4A_3A_2A_1$ has a unique fixed point $\beta_1 \in X_1$, $A_1A_4A_3A_2$ has a unique fixed point $\beta_2 \in X_2$, $A_2A_1A_4A_3$ has a unique fixed point $\beta_3 \in X_3$ and $A_3A_2A_1A_4$ has a unique fixed point $\beta_4 \in X_4$.

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