Semi-Analytical Solution of Heat Equation in Fuzzy Environment

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Abstract: In this paper, we develop and analyze the use of the Variational Iteration Method (VIM) to find the semi- analytical solution for an initial value problem involving the fuzzy heat parabolic equation. VIM allows for the solution of the partial differential equation to be calculated in the form of an infinite series in which the components can be easily computed. The VIM will be studied for fuzzy initial value problems involving partial parabolic differential equations. Also VIM will be constructed and formulated to obtain a semi-analytical solution of fuzzy heat equation by using the properties of fuzzy set theory. Numerical example involving fuzzy heat equation was solved to illustrate the capability of VIM in this regard. The numerical results that obtained by VIM were compared with the exact solution in the form of Table I-Table II and Fig. 1-Fig. 2.

Key words: Fuzzy numbers, fuzzy differential equations, fuzzy partial differential equations, variational iteration method, fuzzy heat equation.

1. Introduction

Many dynamical real life problems may be formulated as a mathematical model in the form of system of ordinary or partial differential equations. Differential equations have proved to be a successful modeling paradigm. Fuzzy set theory is a powerful tool for the modeling of vagueness, and for processing uncertainty or subjective information on mathematical models. For such mathematical modeling, the use of fuzzy differential equations (FDEs) may be necessary. FDEs appear when the modeling of these problems is imperfect and its nature is under uncertainty. FDEs are suitable mathematical models to model dynamical systems in which there exist uncertainties or vagueness. FDEs models have a wide range of applications in many branches of engineering and in the field of medicine. These models are used in various applications including population models [1]-[4], quantum optics gravity [5], control design [6] medicine [7]-[10] and other applications [11]. In recent years semi analytical methods such as Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) have been used to solve fuzzy problems involving ordinary differential equations. In [12], the HPM was used to solve first order linear fuzzy initial value problems involving ordinary differential equations. The ADM was employed in [13, 14] to solve first order linear and nonlinear FDEs. Furthermore, in [14] it was found that VIM is more effective than ADM and the convergence of VIM is much faster than ADM. Also, in [15] VIM was used to find the semi-analytical solution for fuzzy differential equations including nonlinear first order problem. Abbasbandy *et al*. in [16] used VIM to find the approximate solution for high order linear FIVP by converted it into first order system of fuzzy differential equations. The convergence of VIM for this system was also proved. Moreover, the use of ADM was introduced in [17] to solve fuzzy heat equations.

VIM is a semi-analytical method that was first proposed by He [18]-[22]. VIM has been applied to many physics and engineering problems [23]-[25].

Our mean motivation in this study is to analyze and develop the use of VIM to obtain a semi-analytical solution of fuzzy partial differential equation involving fuzzy heat equation. The structure of this paper is organized as follows. In Section 2, some basic definitions and notations are given which will be used in other sections. In Section 3, the structure of VIM is formulated for solving fuzzy partial differential equation involving fuzzy heat equation. In Section 4, the convergence analysis of VIM is presented and proved in detail. In Section 5, we employ the VIM on an example and finally, in Section VI, we give the conclusion of this study.

2. Preliminaries

Fuzzy numbers are a subset of the real numbers set, and represent uncertain values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. A fuzzy number [26] μ is called a triangular fuzzy number if defined by three numbers $\alpha < \beta < \gamma$ where the graph of $\mu(x)$ is a triangle with the base on the interval $[\alpha, \beta]$ and vertex at $x = \beta$ and its membership function has the following form:

$$\mu(x;\alpha,\beta,\gamma) = \begin{cases} 0, & if\ x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & if\ \alpha \le x \le \beta \\ \dfrac{\gamma-x}{\gamma-\beta}, & if\ \beta \le x \le y \\ 1, & if\ x > \gamma \end{cases}$$

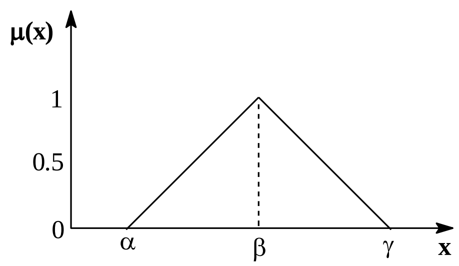

Fig. 1. Triangular fuzzy number.

and its r-level is: $[\mu]_r = [\alpha + r\,(\beta - \alpha), \gamma - r\,(\gamma - \beta)], r \in [0,1]$

In this paper the class of all fuzzy subsets of $\mathbb{R}$will be denoted byRFandsatisfy the following properties [26], [27]:

1) (t) is normal, i.e $\exists t_0 \in \mathbb{R}$ with $\mu(t_0) = 1$,
2) $\mu(t)$ Is convex fuzzy set, i. e. $\mu(\lambda t + (1-\lambda)s) \ge \min\{\mu(t), \mu(s)\}\ \forall t, s \in \mathbb{R}, \lambda \in [0,1]$,
3) μ upper semi-continuous on $\mathbb{R}$,
4) $\overline{\{t \in \mathbb{R}: \mu(t) > 0\}}$is compact.

RF is called the space of fuzzy numbers and $\mathbb{R}$ is a proper subset ofRF.

Define the r-level set $x \in \mathbb{R}$, $[\mu]_r = \{x \setminus \mu(x) \ge r\}$, $0 \le r \le 1$, where $[\mu]_0 = \{x \setminus \mu(x) > 0\}$ is compact [28] which is a closed bounded interval and denoted by $[\mu]_r = \left(\underline{\mu}(t), \overline{\mu}(t)\right)$. In the parametric form, a fuzzy number is represented by an ordered pair of functions $\left(\underline{\mu}(t), \overline{\mu}(t)\right)$, $r \in [0,1]$which satisfies [29]:

1) $\underline{\mu}(t)$is a bounded left continuous non-decreasing function over $[0,\ 1]$.
2) $\overline{\mu}(t)$is a bounded left continuous non-increasing function over $[0,\ 1]$.
3) $\underline{\mu}(t) \leq \overline{\mu}(t),\ r \in [0,1]$. A crisp number r is simply represented by $\underline{\mu}(r) = \overline{\mu}(r) = r,\ r \in [0,1]$.

Definition 2 [27]: A mapping $f\colon T \to E$ for some interval $T \subseteq E$is called a fuzzy process, and we denote r-level set by:

$$[\tilde{f}(t)]_r = \left[\underline{f}(t;r), \overline{f}(t;r)\right], t \in T, r \in [0,1]$$

The r-level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their r-level sets based on the resolution identity theorem [30].

Definition3 [27]: The fuzzy integral of fuzzy process,$\tilde{f}(t;r)$, $\int_a^b \tilde{f}(t;r)dt$, for $a, b \in T$and $r \in [0,1]$is defined by:

$$\int_a^b \tilde{f}(t;r)dt = \left[\int_a^b \underline{f}(t;r)dt, \int_a^b \overline{f}(t;r)dt\right]$$

Definition4 [30]: Each function $f\colon X \to Y$ induces another function $\tilde{f}\colon F(X) \to F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y) = \begin{cases} Sup_{x \in f^{-1}(y)} U(x), if\ y \in range\ (f) \\ 0 \qquad\qquad\qquad\quad , if\ y \in range\ (f) \end{cases}$$

This is called the Zadeh extension principle.

3. Fuzzy Heat Equation

Consider the fuzzy parabolic equation with the indicated initial conditions [17]

$$\frac{\partial \widetilde{U}(t,x)}{\partial t} = \frac{\partial^2 \widetilde{U}(t,x)}{\partial x^2} + \tilde{\theta} \qquad (1)$$

$$0 < x < 1, t > 0$$

where $\widetilde{U}(0,x) = \tilde{f}(x)$.

Here $\widetilde{U}(t,x)$ is the fuzzy function of the crisp variable t and x, $\frac{\partial \widetilde{U}(t,x)}{\partial t}, \frac{\partial^2 \widetilde{U}(t,x)}{\partial x^2}$ are fuzzy partial derivatives in [27], [31], $\tilde{\theta}$ is the fuzzy convex number as mentioned in section 2 and $\widetilde{U}(0,x)$ is the fuzzy initial condition with $\tilde{f}(x)$ is convex fuzzy number.

Now from Section II the defuzzification of Eq. (1) for all $r \in [0,1]$is as follows:

$$\left[\widetilde{U}(t,x)\right]_r = \underline{U}(t,x;r), \overline{U}(t,x;r) \qquad (2)$$

$$\left[\frac{\partial \tilde{U}(t,x)}{\partial t}\right]_r = \frac{\underline{U}(t,x;r)}{\partial t}, \frac{\partial \overline{U}(t,x;r)}{\partial t} \tag{3}$$

$$\left[\frac{\partial^2 \tilde{U}(t,x)}{\partial x^2}\right]_r = \frac{\partial^2 \underline{U}(t,x;r)}{\partial x^2}, \frac{\partial^2 \overline{U}(t,x;r)}{\partial x^2} \tag{4}$$

$$\left[\tilde{\theta}\right]_r = \underline{\theta}(r), \overline{\theta}(r) \tag{5}$$

$$\left[\tilde{U}(0,x)\right]_r = \underline{U}(0,x;r), \overline{U}(0,x;r) \tag{6}$$

$$\left[\tilde{f}(x)\right]_r = \underline{f}(x;r), \overline{f}(x;r) \tag{7}$$

where

$$\left[\tilde{f}(x)\right]_r = \underline{\alpha}(r)\beta(x), \overline{\alpha}(r)\beta(x) \tag{8}$$

such that $\underline{\alpha}(r), \overline{\alpha}(r)$ are convex fuzzy numbers as mentioned in Section 2 and $\beta(x)$ is a crisp function.Using the fuzzy extension principle as in Section 2, we can define the following membership function

$$\begin{cases} \underline{U}(t,x;r) = min\left\{\tilde{U}(t,\tilde{\mu}(r)) \middle| \tilde{\mu}(r) \in \tilde{U}(t,x;r)\right\} \\ \overline{U}(t,x;r) = max\left\{\tilde{U}(t,\tilde{\mu}(r)) \middle| \tilde{\mu}(r) \in \tilde{U}(t,x;r)\right\} \end{cases} \tag{9}$$

Now we can rewrite Eq. (1) for $0 < x \leq 1, t > 0$ and $r \in [0,1]$ as follows

$$\begin{cases} \frac{\partial \underline{U}(t,x;r)}{\partial t} = \frac{\partial^2 \underline{U}(t,x;r)}{\partial x^2} + \underline{\theta}(r) \\ \underline{U}(0,x;r) = \underline{\alpha}(r)\beta(x) \end{cases} \tag{10}$$

$$\begin{cases} \frac{\partial \overline{U}(t,x;r)}{\partial t} = \frac{\partial^2 \overline{U}(t,x;r)}{\partial x^2} + \overline{\theta}(r) \\ \overline{U}(0,x;r) = \overline{\alpha}(r)\beta(x) \end{cases} \tag{11}$$

4. Fuzzification and Defuzzification of VIM

The general structures of VIM for solving crisp PDE problems have been presented in [31]. In this section we present in details the structure of VIM for the approximate solution of FIVP. The VIM is applied to approximately solve fuzzy heat equation. According to VIM and Section 3, we can construct the following correction functional as follows

$$\underline{U}_{i+1}(t,x;r) = \underline{U}_i(t,x;r) + \int_0^t \lambda(t;\eta)\left\{\underline{U}_i(t,x;r)_\eta - \overbrace{\underline{U}_i(t,x;r)_{xx}} - \underline{\theta}(r)\right\}d\eta \tag{12}$$

$$\overline{U}_{i+1}(t,x;r) = \overline{U}_i(t,x;r) + \int_0^t \lambda(t;\eta)\left\{\overline{U}_i(t,x;r)_\eta - \overbrace{\overline{U}_i(t,x;r)_{xx}} - \overline{\theta}(r)\right\}d\eta \tag{13}$$

where $i = 0,1,2,\dots,\ r \in [0,1]$, $\lambda(t;\eta)$ is a general Lagrange multiplier which can be identified optimally via Variational theory. Now we let $\overbrace{\underline{U}_i(t,x;r)_{xx}}$ is considered as restricted variation. i.e. $\delta\overbrace{\underline{U}_i(t,x;r)_{xx}} = 0$. The general Lagrangian multiplier $\lambda(t;\eta)$ [21] related with Eq. (12) can be determined as:

$$\delta\underline{U}_{i+1}(t,x;r) = \delta\underline{U}_i(t,x;r) + \delta\int_0^t \lambda(t;\eta)\left\{\underline{U}_i(t,x;r)_\eta - \overbrace{\underline{U}_i(t,x;r)_{xx}} - \underline{\theta}(r)\right\}d\eta \tag{14}$$

$$\delta\underline{U}_{i+1}(t,x;r) = \delta\underline{U}_i(t,x;r) + \delta\int_0^t \lambda(t;\eta)\{\underline{U}_i(t,x;r)_\eta\}d\eta \tag{15}$$

Integrating by parts we obtain the followings:

$$\underline{U}_{i+1}(t,x;r) = \delta\underline{U}_i(t,x;r) + \lambda(t)\delta\underline{U}_i(t,x;r) + \int_0^t \lambda(t;\eta)'\,\delta\underline{U}_i(t,x;r)d\eta \tag{16}$$

Therefore, have the following stationary conditions:

$$\begin{cases} 1 - \lambda(\eta)|_{t=\eta} = 0 \\ \lambda(\eta)'|_{t=\eta} = 0 \end{cases} \tag{17}$$

Similarly we have the same Lagrangian multiplier for the upper bound of Eq. (1). From these conditions and according to the order of the Eq. (1) we can determine the Lagrangian multiplier

$$\lambda(t;\eta) = -1 \tag{18}$$

Here the initial guesses that satisfies the initial conditions in Eq. (1) are given by

$$\begin{cases} \underline{U}_0(t,x;r) = \underline{\alpha}(r)\beta(x) \\ \overline{U}_0(t,x;r) = \overline{\alpha}(r)\beta(x) \end{cases} \tag{19}$$

The successive approximations of VIM will be readily obtained by choosing all the above-mentioned parameters $\lambda(t;\eta)$and $\widetilde{U}_0(t,x;r)$. Consequently, for $i = 0,1,2,\dots$. the exact solution may be obtained by

$$\widetilde{U}(t,x;r) = \lim_{i\to\infty} \lceil\widetilde{U}_i(t,x;r) = \left\{[\widetilde{U}_i]_r\right\} \tag{20}$$

5. Numerical Example

Consider the fuzzy heat equation [17]

$$\frac{\partial\widetilde{U}(t,x)}{\partial t} = \frac{1}{2}x^2\frac{\partial^2\widetilde{U}(t,x)}{\partial x^2} \tag{21}$$

$$0 < x < 1, t > 0, \ \tilde{U}(0,x) = \tilde{\alpha}x^2$$

where $\tilde{\alpha}(r) = [r-1, 1-r]$ for all $r \in [0,1]$.The exact solution of Eq.(21) was given in [17] such that

$$\tilde{u}(t,x;r) = \tilde{\alpha}(r)x^2e^t \tag{22}$$

The initial approximation guesses of Eq. (1) are given by

$$\begin{cases} \underline{U}_0(0,x;r) = (r-1)x^2 \\ \overline{U}_0(0,x;r) = (1-r)x^2 \end{cases} \tag{23}$$

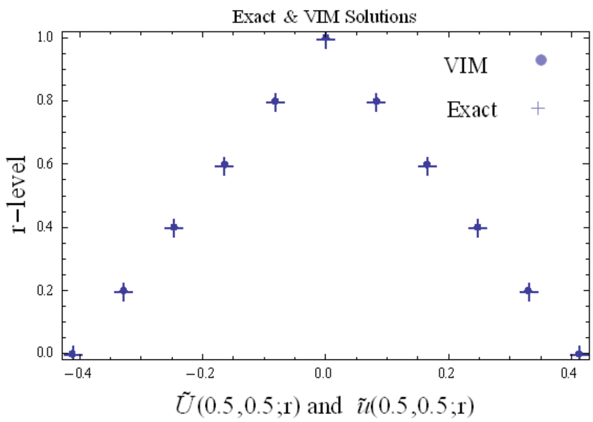

Fig. 2. Comparison between the exact solution and 10 terms VIM solution of Eq. (21).

Table 1. Lower Solution of Eq. (21) by 10 Terms of VIM at $t = 0.5, x = 0.5$ for All $r \in [0,1]$.

r-level	$\underline{U}(0.5,0.5;r)$	$[\underline{E}]_r$
0	−0.41218031767184143	$3.190614439319006 \times 10^{-12}$
0.2	−0.32974425413747316	$2.552513755915697 \times 10^{-12}$
0.4	−0.24730819060310483	$1.914385316936773 \times 10^{-12}$
0.6	−0.16487212706873658	$1.276256877957848 \times 10^{-12}$
0.8	−0.08243606353436826	$6.381284389789244 \times 10^{-13}$
1	$8.673418856537022 \times 10^{-19}$	$8.673418856537022 \times 10^{-19}$

Table 2. Upper Solution of Eq. (21) by 10 Terms of VIM at $t = 0.5, x = 0.5$ for All $r \in [0,1]$.

r-level	$\overline{U}(0.5,0.5;r)$	$[\overline{E}]_r$
0	0.41218031767184143	$3.190614439319006 \times 10^{-12}$
0.2	0.32974425413747316	$2.552513755915697 \times 10^{-12}$
0.4	0.24730819060310483	$1.914385316936773 \times 10^{-12}$
0.6	0.16487212706873658	$1.276256877957848 \times 10^{-12}$
0.8	0.08243606353436826	$6.381284389789244 \times 10^{-13}$
1	$8.673418856537022 \times 10^{-19}$	$8.673418856537022 \times 10^{-19}$

According to Section 4, the variational formula of Eq. (21) is given by

$$\widetilde{U}_{i+1}(t,x;r) = \widetilde{U}_i(t,x;r) - \int_0^t \left\{ \underline{U}_i(t,x;r)_\eta - \frac{1}{2}x^2 \underline{U}_i(t,x;r)_{xx} \right\} d\eta \quad (24)$$

Furthermore, the absolute error of the semi-analytical solution of Eq. (21) can be defined as

$$[\tilde{E}]_r = \left|\widetilde{U}(t,x;r) - \tilde{u}(t,x;r)\right| = \begin{cases} [\underline{E}]_r = \left|\underline{U}(t,x;r) - \underline{u}(t,x;r)\right| \\ [\overline{E}]_r = \left|\overline{U}(t,x;r) - \overline{u}(t,x;r)\right| \end{cases} \quad (25)$$

6. Conclusions

The main goal of this study has been to derive a semi-analytical solution for the fuzzy heat equation. We have achieved this goal by applying VIM. This method has a useful feature in that it provides the solution in a rapid convergent power series with elegantly computable convergence of the solution. Numerical example involving fuzzy heat equation shows that the VIM is a capable and accurate method for fuzzy partial differential equations. Also the obtained results by VIM are satisfying the properties of fuzzy numbers by taking triangular shape.

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