# Fixed Points for Subsequential Continuous Mappings in Fuzzy Metric Space

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**Abstract:** In the present paper, first we introduce the notion of subsequential continuous mappings in the framework of fuzzy metric space and show that this concept is more general than continuous mappings as well as reciprocal continuous mappings. Also, we cited an example in support of this. Secondly, we introduce the concept of occasionally weakly compatible mappings which is more general than weakly compatible mappings in fixed point theory. At the end, we prove an interesting common fixed point theorem in fuzzy metric space for four self mappings by employing the notion of subsequential continuity, occasionally weakly compatible mappings and semi-compatible mappings.

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**Key words:** Common fixed point, compatible maps, continuous t-norm, fuzzy metric space, subsequential continuity, semi-compatible maps and occasionally weakly compatible maps.

## 1. Introduction

After Zadeh [1] introduced the concept of fuzzy sets in 1965, many authors have extensively developed the theory of fuzzy sets and its applications. Specially to mention, fuzzy metric spaces were introduced by Deng [2], Erceg [3], Kaleva and Seikkala [4], Kramosil and Michalek [5]. In this paper we use the concept of fuzzy metric space introduced by Kramosil and Michalek [5] and modified by George and Veeramani [6] to obtain Hausdorff topology for this kind of fuzzy metric space which has very important applications in quantum particle physics, particularly in connection with both string and $e^{\infty}$ theory (see, [7]–[9]). Fuzzy set theory also has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors viz Grabiec [10], Cho [11], [12], Subrahmanyam [13] and Vasuki [14]. In 2002, Aamri and El-Moutawakil [15] defined the notion of (E.A.) property for self mappings which contained the class of non-compatible mappings in metric spaces. It was pointed out that (E.A.) property allows replacing the completeness requirement of the space with a more natural condition of closedness of the range as well as relaxes the compleness of the whole space, continuity of one or more mappings and containment of the range of one mapping into the range of other which is utilized to construct the sequence of joint iterates. Many authors have proved common fixed point theorems in fuzzy metric spaces for different contractive conditions. Recently, Grabiec [10] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [16] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Jungck *et al.* [17] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Cho [11], [18] introduced the concept of compatible maps of type ($\alpha$) and compatible maps of type ($\beta$) in fuzzy metric space. In 2011, using the concept of compatible maps of type (A) and type ($\beta$), Singh *et al.* [19], [20] proved fixed point theorems in a fuzzy metric space. Recently, Sintunavarat and Kumam [21] defined the notion of (CLRg) property in fuzzy metric spaces and improved the results of Mihet [22] without any requirement of the closedness of the subspace. Recently in 2012, Jain *et al*. [23], [24] and Sharma *et al*. [25] proved various fixed point theorems using the concepts of semi-compatible mappings, property (E.A.) and absorbing mappings.

Recently Singh *et al*. [26] introduced the notion of semi-compatible maps in fuzzy metric space and compared this notion with the notion of compatible map, compatible map of type ($\alpha$), compatible map of type ($\beta$) and obtain some fixed point theorems in complete fuzzy metric space in the sense of Grabiec [10].

In the present paper, we prove fixed point theorems in complete fuzzy metric space by replacing continuity condition with a weaker condition called subsequential continuity.

## 2. Preliminaries

In this section we recall some definitions and known results in fuzzy metric space.

**Definition 2.1.** In ref. [27], a binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b \le c \times d$ whenever $a \le c$ and $b \le d$ for $a, b, c, d \in [0, 1]$.

Examples of *t*-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

**Definition 2.2.** In ref. [27], the 3-tuple $(X, M, *)$ is said to be a *Fuzzy metric space* if $X$ is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :
for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

**Example 2.1.** In ref. [6], let $(X, d)$ be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$. Then $(X, M, *)$ is a Fuzzy metric space. It is called the Fuzzy metric space induced by $d$.

**Definition 2.3.** In ref. [10], a sequence $\{x_n\}$ in a Fuzzy metric space $(X, M, *)$ is said to be a *Cauchy sequence.* If and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \ge n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point $x$ in $X$ if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \ge n_0$.

A Fuzzy metric space $(X, M, *)$ is said to be *complete* if every Cauchy sequence in it converges to a point in it.

**Definition 2.4.** In ref. [16], self mappings $A$ and $S$ of a Fuzzy metric space $(X, M, *)$ are said to be *compatible.* If and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $Sx_n, Ax_n \to p$ for some $p$ in $X$ as $n \to \infty$.

**Definition 2.5.** In ref. [27], two self maps $A$ and $B$ of a fuzzy metric space $(X, M, *)$ are said to be weak compatible if they commute at their coincidence points, i.e. $Ax = Bx$ implies $ABx = BAx$.

**Definition 2.6.** Self maps $A$ and $S$ of a Fuzzy metric space $(X, M, *)$ are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is a coincidence point of $A$ and $S$ at which $A$ and $S$ commute.

**Definition 2.7.** In ref. [26], suppose $A$ and $S$ be two maps from a Fuzzy metric space $(X, M, *)$ into itself. Then they are said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

**Definition 2.8.** Suppose $A$ and $S$ be two maps from a Fuzzy metric space $(X, M, *)$ into itself. Then they are said to be subsequential continuous if and only if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$ and satisfy $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$.

If $A$ and $S$ are both continuous then they are obviously subsequential continuous but the converse need not be true as seen in the following example.

**Example 2.2.** Let $X = R$, endowed with metric $d$ and $M_d(x, y) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$, $t > 0$. Define the self maps $A$, $S$ as

$$Ax = \begin{cases} 2, & x < 3 \\ 3, & x \geq 3 \end{cases}$$

and

$$Sx = \begin{cases} 2x - 4, & x \leq 3 \\ 3, & x > 3 \end{cases}.$$

Consider a sequence $x_n = 3 + \frac{1}{n}$ then

$$Ax_n = A\left(3 + \frac{1}{n}\right) \to 3$$

and

$$SAx_n = S\left(3 + \frac{1}{n}\right) = 3 \neq S(3) = 2 \text{ as } n \to \infty.$$

Thus $A$ and $S$ are not reciprocally continuous and also not continuous but, if we consider a sequence

$$\{x_n\} = \left(3 - \frac{1}{n}\right),$$

then

$$Ax_n = 2, \quad Sx_n = 2\left(3 - \frac{1}{n}\right) - 4 = 2,$$

$$ASx_n = A\left(2 - \frac{2}{n}\right) = 2 = A(2), \quad SAx_n = S(2) = 0 = S(2) \text{ as } n \to \infty.$$

Therefore $A$ and $S$ are sub-sequentially continuous.

**Lemma 2.1.** In ref. [10], let $(X, M, *)$ be a fuzzy metric space. Then for all $x, y \in X$, $M(x, y)$ is a non-decreasing function.

**Lemma 2.2.** Ref. [27] Let $(X, M, *)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$ $M(x, y, kt) \geq M(x, y, t)\ \forall\ t > 0$, then $x = y$.

**Lemma 2.3.** [15] Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. If there exists a number $k \in (0, 1)$ such that

$$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)\ \forall\ t > 0 \text{ and } n \in N.$$

Then $\{x_n\}$ is a Cauchy sequence in $X$.

Singh and Jain [26] proved the following result:

**Theorem 2.1.** Let $A, B, S$ and $T$ be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying

$$A(X) \subset T(X),\ B(X) \subset S(X) \tag{1}$$

$$\text{one of } A \text{ and } B \text{ is continuous,} \tag{2}$$

$$(A, S) \text{ is semi-compatible and } (B, T) \text{ is weak-compatible,} \tag{3}$$

for all $x, y \in X$ and $t > 0$

$$M(Ax, By, t) \geq r(M(Sx, Ty, t)) \tag{4}$$

where $r$: $[0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$. Then $A, B, S$ and $T$ have a unique common fixed point.

## 3. Main Results

In the following theorem we replace the continuity condition by weaker notion of subsequential continuity to get more general form of result of [26].

**Theorem 3.1.** Let $A$, $B$, $S$ and $T$ be self maps on a complete fuzzy metric space $(X, M, *)$ where $*$ is a continuous t-norm defined by $a * b = \min\{a, b\}$ satisfying :

$$A(X) \subseteq T(X),\ B(X) \subseteq S(X),$$
$(B, T)$ is occasionally weak compatible,

for all $x, y \in X$ and $t > 0$,

$$M(Ax, By, t) \geq \Phi(M(Sx, Ty, t)),$$

where $\Phi$: $[0, 1] \to [0, 1]$ is a continuous function such that $\Phi(1) = 1$, $\Phi(0) = 0$ and $\Phi(a) > a$ for each $0 < a < 1$.

If $(A, S)$ is semi-compatible pair of sub-sequential continuous maps then $A$, $B$, $S$ and $T$ have a unique common fixed point.

**Proof.** Let $x_0 \in X$ be any arbitrary point. Then for which there exists $x_1, x_2 \in$ X such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Thus we can construct a sequences $\{y_n\}$ and $\{x_n\}$ in $X$ such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, 3, \ldots$.

By contractive condition, we get

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &(M(Sx_{2n}, Tx_{2n+1}, t)) \\ &= (M(y_{2n}, y_{2n+1}, t)) \\ &> M(y_{2n}, y_{2n+1}, t). \end{aligned}$$

Similarly, we get

$$M(y_{2n+2}, y_{2n+3}, t) > M(y_{2n+1}, y_{2n+2}, t).$$

In general,

$$M(y_{n+1}, y_n, t) \geq \Phi(M(y_n, y_{n-1}, t)) > M(y_n, y_{n-1}, t).$$

Therefore $\{M(y_{n+1}, y_n, t)\}$ is an increasing sequence of positive real numbers in $[0, 1]$ and tends to limit $l \leq 1$. We claim that $l = 1$.

If $l < 1$ then $M(y_{n+1}, y_n, t) \geq M(y_n, y_{n+1}, t)$.

On letting $n \to \infty$, we get

$$\lim_{n\to\infty} M(y_{n+1}, y_n, t) \geq \Phi(\lim_{n\to\infty} M(y_n, y_{n-1}, t))$$

i.e. $l \geq \Phi(l) = l$, a contradiction. Now for any positive integer $p$,

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots * M(y_{n+p-1}, y_{n+p}, t/p).$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * 1 * \ldots * 1 = 1.$$

Thus,

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1.$$

Thus $\{y_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, $\{y_n\}$ converges to a point $z$ in $X$. Hence the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to $z$.

Now since $A$ and $S$ are subsequential continuous and semi-compatible then we have

$$\lim_{n\to\infty} ASx_{2n} = Az,\ \lim_{n\to\infty} SAx_{2n} = Sz \text{ and } \lim_{n\to\infty} M(ASx_{2n}, Sz, t) = 1.$$

Therefore we get $Az = Sz$.

Now we will show $Az = z$. For this suppose $Az \neq z$. Then by contractive condition, we get

$$M(Az, Bx_{2n+1}, t) \geq \Phi(M(Sz, Tx_{2n+1}, t)).$$

Letting $n \to \infty$, we get

$$M(Az, z, t) \geq \Phi(M(Az, z, t)) > M(Az, z, t),$$

a contradiction, thus $z = Az = Sz.$

Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Az = Tu.$

Putting $x = x_{2n}$ and $y = u$ in (3.3) we get,

$$M(Ax_{2n}, Bu, t) \geq \Phi(M(Sx_{2n}, Tu, t)).$$

Letting $n \to \infty$, we get

$$M(z, Bu, t) \geq \Phi(M(z, z, t)) = \Phi(1) = 1,$$

i.e. $z = Bu = Tu$ and the occasionally weak-compatibility of $(B, T)$ gives $TBu = BTu$, i.e.

$$Tz = Bz.$$

Again by contractive condition and assuming $Az \neq Bz$, we get $Az = Bz = z$.

Hence finally, we get

$z = Az = Bz = Sz = Tz,$ i.e. $z$ is a common fixed point of $A$, $B$, $S$ and $T$. The uniqueness follows from contractive condition. This completes the proof.

## 4. Conclusion

Our result is a generalization of the result of Singh and Jain [26] in the sense that the condition of weak compatibility has been replaced by occasionally weak compatibility. Moreover, we replace the continuity condition with a weaker condition called subsequential continuity, which is more general than continuity condition.

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